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SOME IMPROVED INCLUSION METHODS FOR POLYNOMIAL ROOTS WITH WEIERSTRASS' CORRECTIONS

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Abstract—One decade ago, the third order method without derivatives for the simultaneous inclusion of simple zeros of a polynomial was proposed in [1]. Following Nourein's idea [2], some modifications of this method with the increased convergence are proposed. The acceleration of convergence is attained by using Weierstrass' corrections without additional calculations, which provides a high computational efficiency of the modified methods. It is proved that their R-orders of convergence are asymptotically greater than 3.5. The presented interval methods are realized in circular complex arithmetic.

1. INTRODUCTION

Let P be a monic complex polynomial of degree $n \geq 3$ with simple zeros $\zeta_1, \dots, \zeta_n$ and let $z_1, \dots, z_n$ be distinct approximations of these zeros. Then an arbitrary zero can be expressed by

$$\zeta_j = z_j - \frac{W_j}{1 - \sum\limits_{k=1, k\neq j}^{n} \frac{W_k}{z_k - \zeta_j}}, \qquad (j = 1, \dots, n) \tag{1}$$

(see [3,4]), where

$$W_j = \frac{P(z_j)}{\prod\limits_{k=1, k\neq j}^{n} (z_j - z_k)} \tag{2}$$

is the so-called Weierstrass' correction [5].

Suppose that we have found disjoint disks $Z_1, \dots, Z_n$ in the complex plane such that $\zeta_j \in Z_j$ for any $j \in \{1, \dots, n\}$. Starting from the fixed-point relation (1) and using circular complex arithmetic Petković stated in [1] the third order method for the simultaneous inclusion of all zeros of P,

$$\hat{Z}_j = z_j - \frac{W_j}{1 - \sum\limits_{k=1, k\neq j}^{n} \frac{W_k}{z_k - Z_j}}, \qquad (i = 1, \dots, n), \tag{3}$$

where $\hat{Z}_j$ denotes the new circular approximation for ζ_j.

Considering the fixed-point relation (1), we observe that the exact zero ζ_j on the right-hand side can be substituted by the Weierstrass' approximation $z_j - W_j$. In this way, we obtain the fourth order method in ordinary complex arithmetic,

$$\hat{z}_j = z_j - \frac{W_j}{1 - \sum\limits_{k=1, k\neq j}^{n} \frac{W_k}{z_k - z_j + W_j}}, \qquad (j = 1, \dots, n) \tag{4}$$

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as it was proposed by Nourein in [2]. The purpose of this paper is to improve the cubic interval method (3) following the mentioned Nourein's idea, that is, applying Weierstrass' correction. The increased convergence rate (R-order greater than 3.5) is attained without additional calculations which provides a great computational efficiency.

Let $Z = \{z, r\} = \{w : |w - z| \leq r\}$ be a disk with the center $z = \mathrm{mid}(Z)$ and the radius $r = \mathrm{rad}(Z)$. To construct new algorithms we will use three type of inversion of a disk Z which does not contain the origin,

$$\{z,r\}^{-1} := \left\{ \frac{1}{z(1 - r^2/|z|^2)}, \frac{r}{|z|^2 - r^2} \right\},$$
$$\{z,r\}^{I_1} := \left\{ \frac{1}{z}, \frac{r}{|z|(|z| - r)} \right\},$$
$$\{z,r\}^{I_2} := \left\{ \frac{1}{z}, \frac{2r}{|z|^2 - r^2} \right\}.$$

It is easy to prove that $Z^{-1} \subseteq Z^{I_1} \subseteq Z^{I_2}$. The inversion Z^{I_1} has the centered form (see [6]), while Z^{I_2} is introduced because Z^{I_1} needs the calculation of $|z| = |x + iy| = \sqrt{x^2 + y^2}$, which is very costly. In the sequel, $INV(Z)$ will denote one of the three inversions Z^{-1}, Z^{I_1}, Z^{I_2}. We will use the following estimates

$$|\mathrm{mid}(INV(Z))| \leq \frac{|z|}{|z|^2 - r^2}, \qquad \mathrm{rad}(INV(Z)) \leq \frac{2r}{|z|^2 - r^2}. \tag{5}$$

2. THE IMPROVED INTERVAL METHODS

Assume that we have found n disjoint disks $Z_1, \ldots, Z_n$ containing the zeros $\zeta_1, \ldots, \zeta_n$ of a given polynomial P. One step of the new interval methods with Weierstrass' correction reads $(Z_1, \ldots, Z_n) \to (\hat{Z}_1, \ldots, \hat{Z}_n)$ with

$$\hat{Z}_j := z_j - W_j \cdot \left[1 - \sum_{k=1, k \neq j}^{n} W_k \cdot INV(z_k - Z_j + W_j) \right]^{-1}, \qquad (j = 1, \ldots, n), \tag{6}$$

where $z_j = \mathrm{mid}(Z_j)$ and W_j is Weierstrass' correction given by (4). INV in (6) denotes inversions of a disk defined in Section 1, that is, $INV \in \{(\ )^{-1}, (\ )^{I_1}, (\ )^{I_2}\}$. We note that the inversions $(\ )^{I_1}$ and $(\ )^{I_2}$ can be also applied to the disk in bracket (instead of $(\ )^{-1}$), but such approach does not improve the convergence rate of (6). Besides, although $(\ )^{-1}$ defines the exact operation, that is, $Z^{-1} = \{z^{-1} : z \in Z\}$, only using $(\ )^{I_1}$ or $(\ )^{I_2}$ in (6) will provide the fourth order of convergence for the centers of $\hat{Z}_1, \ldots, \hat{Z}_n$; namely, these centers behave like complex approximations defined by Nourein's formula (4).

3. CONVERGENCE RESULTS

In the following, we will show that the interval methods (6) have the R-order of convergence equal to $\frac{3+\sqrt{17}}{2} \cong 3.562$ or 4 (in an asymptotical sense), depending on the choice of $INV \in \{(\ )^{-1}, (\ )^{I_1}, (\ )^{I_2}\}$. Before establishing the convergence results, we will prove several auxiliary assertions.

For any $j,k \in \{1,\dots,n\}$ we introduce the following abbreviations

$$\begin{aligned}
z_j &:= \mathrm{mid}(Z_j);\\
r_j &:= \mathrm{rad}(Z_j);\\
\epsilon_j &:= z_j - \zeta_j;\\
\epsilon &:= \max_{j=1,\dots,n} |\epsilon_j|;\\
r &:= \max_{j=1,\dots,n} \mathrm{rad}(Z_j);\\
d &:= \min_{\substack{i,j=1,\dots,n\\ i\neq j}} |\mathrm{mid}(Z_i) - \mathrm{mid}(Z_j)|;\\
v_{kj} &:= z_k - z_j + W_j;\\
H_j &:= 1 - \sum_{k=1,k\neq j}^{n} W_k \cdot INV(z_k - Z_j + W_j) =: \{u_j, \rho_j\}.
\end{aligned}$$

In this paper, we will consider circular complex arithmetic. For definitions and properties of this kind of interval arithmetic see, e.g., [7, Chapters 5 and 6].

LEMMA 1. *Under the condition*

$$d \geq 4(n-1)\, r \tag{7}$$

the inequality

$$|W_j| < \alpha|\epsilon_j| \leq \alpha r \tag{8}$$

holds, where $\alpha = e^{1/4} \cong 1.284$.

PROOF. The sequence $(a(k))$, defined by $a(k) = \left(1 + \frac{1}{4k}\right)^k$, is bounded and monotonically increasing so that

$$a(k) < \lim_{k\to\infty} a(k) = e^{1/4} = \alpha,$$

for each $k \in IN$. According to this, for any $j \in \{1,\dots,n\}$, we have

$$\begin{aligned}
|W_j| &= \frac{|P(z_j)|}{\prod\limits_{k\neq j} |z_j - z_k|} = |z_j - \zeta_j| \cdot \prod_{k\neq j} \left|\frac{z_j - \zeta_k}{z_j - z_k}\right|\\
&\leq |z_j - \zeta_j| \cdot \prod_{k\neq j} \frac{|z_j - z_k| + r_k}{|z_j - z_k|} \leq |\epsilon_j| \left(1 + \frac{r}{d}\right)^{n-1}\\
&= |\epsilon_j| \left(1 + \frac{1}{4(n-1)}\right)^{n-1} < \alpha|\epsilon_j| \leq \alpha r.
\end{aligned}$$

∎

LEMMA 2. *If (7) holds, then the implication*

$$\zeta_j \in Z_j \Rightarrow \zeta_j \in Z_j - W_j \tag{9}$$

is valid for any $j \in \{1,\dots,n\}$.

PROOF. Since $z \in \{c, r\} \Leftrightarrow |z - c| \leq r$, it is sufficient to prove the implication

$$|z_j - \zeta_j| = |\epsilon_j| \leq r \Rightarrow |z_j - W_j - \zeta_j| \leq r_j.$$

Let

$$b_k^{(j)} := \frac{\epsilon_k}{z_j - z_k}, \qquad (k = 1,\dots,n; k \neq j)$$

and let

$$S_\mu := \sum_{k_1<k_2<\cdots<k_\mu} b_{k_1}^{(j)} b_{k_2}^{(j)} \cdots b_{k_\mu}^{(j)}, \qquad S_0 = 1$$

be the symmetric function relative to $b_k^{(j)}$. It is obvious

$$|S_\mu| \le \sum_{k_1<k_2<\cdots<k_\mu} |b_{k_1}^{(j)}|\cdot|b_{k_2}^{(j)}|\cdots|b_{k_\mu}^{(j)}| \le \binom{n-1}{\mu}\left(\frac{r}{d}\right)^\mu .$$

Then we have

$$\begin{aligned}
|z_j - W_j - \zeta_j| &= \left|\epsilon_j - \epsilon_j \cdot \prod_{k\ne j}\left(1+\frac{\epsilon_k}{z_j-z_k}\right)\right| = |\epsilon_j|\left|1-\sum_{\mu=0}^{n-1}S_\mu\right| \le |\epsilon_j|\sum_{\mu=1}^{n-1}|S_\mu| \\
&\le |\epsilon_j|\sum_{\mu=1}^{n-1}\binom{n-1}{\mu}\left(\frac{r}{d}\right)^\mu = |\epsilon_j|\left[\left(1+\frac{r}{d}\right)^{n-1}-1\right] \\
&\le |\epsilon_j|\left[\left(1+\frac{1}{4(n-1)}\right)^{n-1}-1\right] < |\epsilon_j|\left(e^{1/4}-1\right) \\
&< \frac{1}{3}|\epsilon_j| < r_j,
\end{aligned}$$

which proves (9). ∎

LEMMA 3. *If (7) holds and $\zeta_j \in Z_j$ for all $j \in \{1,\dots,n\}$, then the inversions in (6) exist, that is, $0 \notin z_k - Z_j + W_j$ and $0 \notin H_j$.*

PROOF. First, according to Lemma 1, we have

$$|v_{kj}| = |z_k - z_j + W_j| \ge |z_k - z_j| - |W_j| > d - \alpha r > d - 2r. \tag{10}$$

Therefore, $|v_{jk}| > d-2r > r \ge r_j$, whence $0 \notin \{v_{kj}, r_j\} = z_k - Z_j + W_j$.

The second assertion of Lemma 3 will be proved using the estimates for the center u_j and the radius ρ_j of the disk H_j. In view of (5), (10) and Lemma 1 we find

$$\begin{aligned}
\rho_j &= \sum_{k\ne j}|W_k|\,\mathrm{rad}(INV(z_k - Z_j + W_j)) \\
&\le \sum_{k\ne j}|W_k|\frac{2r_j}{|v_{kj}|^2 - r_j^2} < 2\alpha\epsilon\sum_{k\ne j}\frac{r_j}{(d-2r)^2-r^2} \\
&\le \frac{2(n-1)\alpha\epsilon r_j}{(d-r)(d-3r)} \le \frac{2(n-1)\alpha r^2}{(d-r)(d-3r)} < \frac{2}{13}
\end{aligned}$$

(by (7) and for $n \ge 3$), and

$$\begin{aligned}
|u_j| &\ge 1 - \sum_{k\ne j}|W_k|\cdot|\,\mathrm{mid}(INV(z_k - Z_j + W_j))| \\
&\ge 1 - \sum_{k\ne j}\frac{|W_k||v_{kj}|}{|v_{kj}|^2 - r_j^2} \ge 1 - \frac{\alpha(n-1)(d-2r)r}{(d-2r)^2-r^2} > \frac{2}{3}.
\end{aligned}$$

Therefore,

$$|u_j|^2 - \rho_j^2 > \left(\frac{2}{3}\right)^2 - \left(\frac{2}{13}\right)^2 > \frac{21}{50},$$

which means that $0 \notin \{u_j, \rho_j\} = H_j$. ∎

The convergence of the interval methods (6) is considered in the following.

THEOREM 1. *Let $(Z_1,\dots,Z_n) := (Z_1^{(0)},\dots,Z_n^{(0)})$ be initial disks such that $\zeta_j \in Z_j$ $(j=1,\dots,n)$ and let $(Z_j^{(m)})$ $(j=1,\dots,n)$ denote the sequence of disks produced by (6), where $m=0,1,2,\dots$ is the iteration index. If the condition*

$$d^{(0)} \ge 4(n-1)\,r^{(0)} \tag{11}$$

is satisfied, where

$$r^{(m)} := \max_{j=1,\dots,n} \operatorname{rad}(Z_j^{(m)}) \quad \textit{and} \quad d^{(m)} := \min_{i,j=1,\dots,n i\neq j} \left|\operatorname{mid}(Z_i^{(m)}) - \operatorname{mid}(Z_j^{(m)})\right|,$$

then for any $j \in \{1,\dots,n\}$ *and* $m = 0,1,2,\dots$ *there holds*

$$\zeta_j \in Z_j^{(m)}$$

and the sequences of radii $(\operatorname{rad}(Z_j^{(m)})\ (j = 1,\dots,n))$ *tend monotonically towards zero.*

PROOF. Theorem 1 will be proved by induction on m and we regard the typical step for $m = 0$ omitting the iteration index m. For example, we will write $Z_j, \hat{Z}_j, r_j, \hat{r}_j, z_j, \hat{z}_j$ instead of $Z_j^{(m)}, Z_j^{(m+1)}, r_j^{(m)}, r_j^{(m+1)}, z_j^{(m)}, z_j^{(m+1)}$. Thus $\hat{Z}_1,\dots,\hat{Z}_n$ are the improved disks given by (6) having the centers $\hat{z}_1,\dots,\hat{z}_n$ and the radii $\hat{r}_1,\dots,\hat{r}_n$. We define

$$\hat{r} := \max_{j=1,\dots,n} \hat{r}_j \quad \text{and} \quad \hat{d} := \min_{\substack{i,j=1,\dots,n \\ i\neq j}} |\hat{z}_i - \hat{z}_j|.$$

The iterative formula (6) may be written in the form

$$\hat{Z}_j = z_j - W_j H_j^{-1} = z_j - W_j\{u_j, \rho_j\}^{-1},$$

wherefrom

$$\hat{r}_j = \operatorname{rad}(\hat{Z}_j) = |W_j| \frac{\rho_j}{|u_j|^2 - \rho_j^2}$$

and

$$\hat{z}_j = \operatorname{mid}(\hat{Z}_j) = z_j - \frac{W_j \cdot \overline{u_j}}{|u_j|^2 - \rho_j^2}.$$

By virtue of Lemma 1 and the estimates of ρ_j and u_j from the proof of Lemma 3, we get

$$\hat{r}_j < \frac{2(n-1)\alpha^2\, \epsilon|\epsilon_j|\, r_j}{(|u_j|^2 - \rho_j^2)(d-r)(d-3r)} < \frac{2(n-1)\alpha^2}{21/50} \cdot \frac{\epsilon\, |\epsilon_j|\, r_j}{(d-r)(d-3r)}. \tag{12}$$

Hence,

$$\begin{aligned} \hat{r}_j &< \frac{100}{21}\alpha^2 \cdot r_j \cdot \frac{n-1}{\left(\frac{d}{r}-3\right)\left(\frac{d}{r}-1\right)} \leq r_j \cdot \frac{100\cdot\alpha^2\cdot(n-1)}{21\cdot(4n-7)(4n-5)} \\ &\leq \frac{40}{147}\alpha^2 r_j < \frac{9}{20}\cdot r_j \end{aligned} \tag{13}$$

(since $\frac{d}{r} \geq 4(n-1), n \geq 3$). On the other hand, according to Lemma 2 and the inclusion property, from (1), we obtain

$$\zeta_j \in \hat{Z}_j, \qquad \text{that is } |\hat{z}_j - \zeta_j| < \hat{r}_j < \frac{9}{20}\cdot r.$$

Since $\zeta_j \in Z_j$, that is, $|z_j - \zeta_j| < r$, we have

$$|\hat{z}_j - z_j| < \frac{29}{20}\cdot r.$$

Now we have for some $j,k \in \{1,\dots,n\}, j \neq k$,

$$\hat{d} = |\hat{z}_j - \hat{z}_k| \geq |z_j - z_k| - |z_j - \hat{z}_j| - |z_k - \hat{z}_k| > d - \frac{29}{10}\cdot r.$$

Using the last inequality, (7) and (13) there follows

$$\frac{\hat{r}}{\hat{d}} < \frac{\frac{9}{20}r}{d - 2.9 \cdot r} = \frac{r}{d} \cdot \frac{9}{20 - 58 \cdot \frac{r}{d}} \le \frac{r}{d} \cdot \frac{3}{4}.$$

Hence, we conclude by induction that the initial condition (11) implies the inequality $d^{(m)} > 4(n-1)r^{(m)}$ for each $m = 0, 1, \ldots$. For this reason, the assertions of Lemma 1, 2 and 3 are valid for each $m = 0, 1, \ldots$. In regard to Lemma 2, we have $\zeta_k \in Z_k^{(m)} - W_k^{(m)}$ for any $k \in \{1, \ldots, n\}$. Then, by the inclusion property from (1), we obtain $\zeta_j \in Z_j^{(m+1)}$. Since $\zeta_j \in Z_j^{(0)}$ according to induction it follows $\zeta_j \in Z_j^{(m)}$ for any $j \in \{1, \ldots, n\}$ and $m = 0, 1, \ldots$

From Lemma 3, we see that the inversions in (6) are defined in each iterative step so that the methods (6) are feasible. Besides, since $d^{(m)} > 2r^{(m)}$ it follows that the disks $Z_1^{(m)}, \ldots, Z_n^{(m)}$ are pairwise disjoint. Finally, the inequality (13) shows that the sequences of radii $(r_j^{(m)})(j = 1, \ldots, n)$ converge monotonically towards zero. ∎

In order to determine the R-order of convergence of the interval methods defined by (6), we may assume, without loss of generality, that the condition (11) is satisfied, which yields $d^{(m)} > 4(n-1)r^{(m)}$ for all $m = 1, 2, \ldots$ Taking into account that the zeros $\zeta_1, \ldots, \zeta_n$ are fixed and included in the disks $Z_1^{(m)}, \ldots, Z_n^{(m)}$, various quantities appearing in Lemmas 1, 2 and 3 are bounded. Accordingly, we can use the Landau symbol $\mathcal{O}(\ )$ to suppress the bounds and stress the asymptotical behaviour. Moreover, such estimate procedure avoids tedius calculations with constants but it is sufficient to control the behaviour of the sequences $(\mathrm{mid}(Z_j^{(m)}) - \zeta_j)$ and $(\mathrm{rad}(Z_j^{(m)}))(j = 1, \ldots, n)$. For two expressions, $\mathrm{term}_1(j, m)$ and $\mathrm{term}_2(j, m)$ (which depend on j, m, P and the initial zero distribution), we define

$$\mathrm{term}_1(j,m) = \mathcal{O}(\mathrm{term}_2(j,m)) \qquad \text{iff} \max_{j=1,\ldots,n} \sup_{m=0,1,\ldots} \left| \frac{\mathrm{term}_1(j,m)}{\mathrm{term}_2(j,m)} \right| < +\infty.$$

This approach will be used for a qualitative analysis of the behaviour of the centers and radii of the circular approximations $Z_1^{(m)}, \ldots, Z_n^{(m)}$ in order to find the R-order of convergence. Therefore, define for any $m = 0, 1, \ldots$

$$\epsilon_j^{(m)} := z_j^{(m)} - \zeta_j, \quad \epsilon_m := \max_{j=1,\ldots,n} |\epsilon_j^{(m)}|, \quad r_m := \max_{j=1,\ldots,n} |r_j^{(m)}| = r^{(m)}.$$

If we omit the iteration index m and write ϵ instead of $\epsilon^{(m)}$, then $\hat{\epsilon}$ denotes $\epsilon^{(m+1)}$.

LEMMA 4. *Let β be equal 1 if $INV = (\ )^{-1}$ and 0 otherwise. Then for all $j, k \in \{1, \ldots, n\}, j \neq k$*

(i) $r_{m+1} = \mathcal{O}(r_m \epsilon_m^2)$;

(ii) $\epsilon_{m+1} = \mathcal{O}(\epsilon_m^4) + \mathcal{O}(\epsilon_m^3 r_m^2) + \beta \mathcal{O}(\epsilon_m^2 r_m^2)$.

PROOF. For simplicity, we will omit the iteration index and use the notations introduced previously.

PROOF. Item (i). From (12), we obtain

$$\hat{r}_j \le \hat{r} < \frac{100(n-1)\,\alpha^2 \epsilon^2\, r}{21 d^2 \left(1 - \frac{r}{d}\right)\left(1 - \frac{3r}{d}\right)} \le \frac{100(n-1)\,\alpha^2 \epsilon^2\, r}{21 d^2 \left(1 - \frac{1}{4(n-1)}\right)\left(1 - \frac{3}{4(n-1)}\right)}$$
$$\le \frac{100(n-1)\,\alpha^2 \epsilon^2\, r}{21 d^2 \left(1 - \frac{1}{8}\right)\left(1 - \frac{3}{8}\right)} < \frac{15(n-1)\,\epsilon^2\, r}{d^2}.$$

Since d is bounded (actually, d tends to $\min_{i \neq j} |\zeta_i - \zeta_j|$), we obtain

$$\hat{r} = \mathcal{O}(r\epsilon^2),$$

which proves (i).

Proof. Item (ii). Let us define

$$s_j(\beta) := \sum_{k \neq j} \frac{W_k}{v_{kj}\left(1 - \frac{\beta r_j^2}{|v_{kj}|^2}\right)} = 1 - u_j, \qquad (\beta = 0 \text{ or } 1),$$

$$t_j := \frac{\rho_j^2}{|u_j|^2}.$$

Using relation

$$W_j = \epsilon_j \left(1 - \sum_{k \neq j} \frac{W_k}{z_k - \zeta_j}\right), \tag{14}$$

which follows from (1), we find from (6)

$$\hat{\epsilon_j} = \hat{z}_j - \zeta_j = z_j - \zeta_j - \frac{W_j}{u_j(1 - \rho_j^2/|u_j|^2)} = \epsilon_j - \frac{\epsilon_j\left(1 - \sum_{k \neq j} \frac{W_k}{z_k - \zeta_j}\right)}{(1 - s_j(\beta))(1 - t_j)}$$

$$= \frac{\epsilon_j \left[s_j(\beta) t_j - t_j + \sum_{k \neq j} \frac{W_k}{z_k - \zeta_j} - s_j(\beta)\right]}{(1 - s_j(\beta))(1 - t_j)}.$$

Further, we arrange the difference

$$\sum_{k \neq j} \frac{W_k}{z_k - \zeta_j} - s_j(\beta) = \sum_{k \neq j} W_k \left(\frac{1}{z_k - \zeta_j} - \frac{1}{v_{kj}\left(1 - \frac{\beta r_j^2}{|v_{kj}|^2}\right)}\right)$$

$$= \sum_{k \neq j} W_k \frac{\left(\zeta_j - z_j + W_j - \frac{\beta r_j^2}{|v_{kj}|^2} \cdot v_{kj}\right)}{v_{kj}(z_k - \zeta_j)\left(1 - \frac{\beta r_j^2}{|v_{kj}|^2}\right)}.$$

According to the proof of Lemma 3, there follows

$$u_j = \mathcal{O}(1), \quad v_{kj} = \mathcal{O}(1), \quad \rho_j = \mathcal{O}(\epsilon r),$$

whence

$$t_j = \mathcal{O}(\epsilon^2 r^2), \quad s_j(\beta) = \mathcal{O}(\epsilon), \quad (1 - s_j(\beta))(1 - t_j) = \mathcal{O}(1),$$

where we used (14) for $W_j = \mathcal{O}(\epsilon)$. In regard to the quadratic convergence of Weierstrass' method, we have

$$z_j - W_j - \zeta_j = \mathcal{O}(\epsilon^2),$$

(which is easily seen using (14)).

Taking into consideration the above estimates, we obtain

$$\sum_{k \neq j} \frac{W_k}{z_k - \zeta_j} - s_j(\beta) = \mathcal{O}(\epsilon^3) + \beta\mathcal{O}(r^2\epsilon)$$

so that

$$\hat{\epsilon}_j = \epsilon\left[\mathcal{O}(\epsilon^3 r^2) + \mathcal{O}(\epsilon^2 r^2) + \mathcal{O}(\epsilon^3) + \beta\mathcal{O}(r^2\epsilon)\right],$$

and finally,

$$\hat{\epsilon}_j = \mathcal{O}(\epsilon^4) + \mathcal{O}(\epsilon^3 r^2) + \beta\mathcal{O}(r^2\epsilon^2),$$

which completes the proof of (ii). ∎

In order to determine the R-order of convergence of the interval method (6) when $INV = (\ )^{-1}$, we will use the following result from the theory of iterative processes.

LEMMA 5. *Let (K_m) be a positive bounded sequence and let (e_m) be a sequence of positive numbers tending towards zero such that $e_{m+2} \le K_m e_{m+1}^p e_m^q$, where p and q being natural numbers. Then the R-order of (e_m) is at least $\left(p + \sqrt{p^2 + 4q}\right)/2$.*

The convergence rate of the interval methods (6) is considered in the following theorem.

THEOREM 2. *Let $O_R(6)$ denotes the R-order of convergence of the iterative interval methods (6), where $INV \in \{(\ )^{-1}, (\ )^{I_1}, (\ )^{I_2}\}$. Then*

$$O_R(6) \ge \begin{cases} \frac{3+\sqrt{17}}{2} & \text{if } INV = (\ )^{-1}, \\ 4 & \text{otherwise.} \end{cases}$$

PROOF. It is necessary to prove that the sequence (r_m) has the R-order $\frac{3+\sqrt{17}}{2} \cong 3.562$ if $INV = (\ )^{-1} (\beta = 1)$ and 4 if $INV \in \{(\ )^{I_1}, (\ )^{I_2}\} (\beta = 0)$. Since the assertions of Lemma 4 are the same with those presented in [8], the technique of the proof of Theorem 2 is the same as in the corresponding convergence theorem given in [8]. For this reason, we give only the sketch of the proof.

First, in the case $\beta = 0$, using Lemma 4, we may prove that the sequence (τ_m), defined by

$$\tau_m := \sqrt{\frac{r_{m+1}}{D_m r_m^4}}, \qquad (m = 0, 1, 2, \dots)$$

is bounded. Here (D_m) is a bounded sequence of positive real numbers. This means that the sequence (r_m) has the Q-order at least four. Then, according to [9, p. 296], we have $O_R(6) \ge 4$.

Applying Lemma 4 in the case $\beta = 1 (INV = (\ )^{-1})$, we are in the position to derive the following relation

$$r_{m+2} \le K_m r_{m+1}^3 r_m^2, \qquad (m = 0, 1, 2, \dots),$$

where K_m is a positive constant which depends on m. Then, applying Lemma 5 for $p = 3$ and $q = 2$, we prove that the sequence (r_m) has the R-order $\frac{3+\sqrt{17}}{2}$. ∎

4. NUMERICAL ASPECTS

Practical aspects of interval methods with corrections have been presented in [8]. For this reason, we give only some particular properties of the new methods.

The values of the R-order of convergence, presented in Theorem 2, should be regarded as the asymptotical ones. Practically, in the beginning of iterative procedure, these values are somewhat smaller, which can be drawn from Lemma 4 (i). In fact, the new algorithms are the most powerfull if at least three iterations are applied.

From Theorem 2, we see that the new methods possess the high order of convergence. The increase of the convergence speed is attained without additional calculations because we use already found values of Weierstrass' corrections. Consequently, we can expect a high computational efficiency of the proposed methods.

Let M_I, M_{I_1} and M_{I_2} denote the considered methods which use the corresponding inversions $(\ )^{-1}, (\ )^{I_1}$ and $(\ )^{I_2}$, respectively. The basic interval method (3) will be denoted by M_P. As it was mentioned in Section 1, the interval method M_{I_1} requires too much extra operations because the inversion $(\ )^{I_1}$ (which appears $n(n-1)$ times per iteration) needs the calculation of square root. This lack makes the method M_{I_1} to be less efficient than M_I and M_{I_2}. The interval methods M_I and M_{I_2} claim almost the same number of the basic arithmetic operations in an implementation on digital computers. Since M_{I_2} has the higher order of convergence than M_I, its computational efficiency is the greatest from a theoretical point of view. But, its order of convergence, almost four, can be achieved not earlier than after three or four iterative steps. In the starting iterations, the convergence speeds of M_I and M_{I_2} are very close so that these two methods are of the same efficiency. On the other side, numerical examples show that the use of the increased radius $2r/(|z|^2 - r^2)(= 2\mathrm{rad}\{z, r\}^{-1})$ applying M_{I_2} often produces larger inclusion disks compared to M_I. Therefore, we may say that, from a practical point of view, the method M_I is the best among three proposed methods with Weierstrass' corrections. Circular approximations generated by M_{I_2} can be somewhat improved if we use in the first iterative step a new type of inversion introduced in [8].

To demonstrate the proposed methods (and the basic method M_P for comparison purpose), we present the following example.

EXAMPLE. We considered the polynomial

$$P(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300$$

with simple zeros $\zeta_1 = -3, \zeta_2 = -1, \zeta_3 = 2i, \zeta_{4,5} = -2 \pm i, \zeta_{6,7} = 2 \pm i, \zeta_8 = 1$ and $\zeta_9 = -2i$. The following circular regions were taken to be the initial inclusion disks for these zeros

$$\begin{aligned} &Z_1^{(0)} = \{-3.2 + 0.2i; 0.35\}, && Z_2^{(0)} = \{-1.1 - 0.2i; 0.35\}, && Z_3^{(0)} = \{0.1 + 1.7i; 0.35\}, \\ &Z_4^{(0)} = \{-1.9 + 1.3i; 0.35\}, && Z_5^{(0)} = \{-1.8 - 0.8i; 0.35\}, && Z_6^{(0)} = \{2.3 + 1.1i; 0.35\}, \\ &Z_7^{(0)} = \{1.9 - 0.7i; 0.35\}, && Z_8^{(0)} = \{1.2 + 0.2i; 0.35\}, && Z_9^{(0)} = \{0.2 - 2.2i; 0.35\}. \end{aligned}$$

The programs were implemented on the computer VAX 3400 in quadruple precision arithmetic. The radii $r_j^{(3)}$ of the inclusion disks $Z_j^{(3)} (j = 1, \dots, 9)$, obtained by four methods M_P, M_I, M_{I_1} and M_{I_2}, are displayed in Table 1. The underlined value denote the maximal radius for each of the applied methods.

Table 1. The radii of inclusion disks obtained in the third iteration. $A(-h)$ means $A \times 10_h$.

	M_P	M_I	M_{I_1}	M_{I_2}
$r_1^{(3)}$	1.57(−17)	4.24(−20)	5.21(−21)	8.33(−17)
$r_2^{(3)}$	6.76(−17)	7.84(−20)	<u>1.35(−19)</u>	<u>2.12(−15)</u>
$r_3^{(3)}$	1.35(−15)	2.62(−21)	3.54(−22)	4.06(−18)
$r_4^{(3)}$	4.29(−17)	1.47(−19)	1.59(−20)	4.47(−16)
$r_5^{(3)}$	4.78(−16)	3.55(−21)	6.23(−20)	9.01(−16)
$r_6^{(3)}$	4.28(−15)	1.02(−19)	2.41(−21)	9.63(−18)
$r_7^{(3)}$	1.55(−14)	7.50(−21)	6.79(−23)	4.01(−18)
$r_8^{(3)}$	<u>2.54(−14)</u>	<u>2.17(−19)</u>	2.70(−20)	3.11(−16)
$r_9^{(3)}$	3.66(−17)	6.06(−22)	2.15(−22)	3.54(−18)

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