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h-adaptive boundary element schemes

C. Carstensen, D. Estep, P. Stephan

Abstract We consider an *h*-adaptive algorithm for the boundary element method applied to two integral equations of the first kind on polygons, namely Symm's integral equation and the hypersingular equation with the normal derivative of the double layer potential. Our reliable and operative algorithm is based on new *a posteriori* estimates for the boundary element Galerkin solution. Numerical experiments show the effectivity of the algorithm.

1 Introduction

In this paper an adaptive *h*-version algorithm is presented and analyzed for a boundary element method for linear strongly elliptic problems. The algorithm is proven to be (i) *reliable* in the sense that the error in the energy norm is guaranteed to be below a given tolerance and (ii) *operative* in the sense that the error indicator (an a posteriori bounding quantity) can be made less than any given tolerance by suitable mesh refinement. The adaptive algorithm is based on a posteriori error estimates Carstensen and Stephan (1993a, 1994a,b) which yield (i).

The problem of constructing an adaptive boundary element method is of very high practical importance. For FEM we refer to the pionnering work of Babuška and Miller (1981) and Eriksson and Johnson (1988, 1991).

Whereas an almost complete approach to adaptive algorithms for FEM is available (see e.g. Babuška and Miller 1981; Eriksson and Johnsson 1988, 1991; Johnsson and Hansbo 1992), comparably little is known for BEM (see e.g. Postell and Stephan 1990; Rank 1987; Wendland and Yu 1988), for the *h*-adaptive coupling of FEM and BEM see Carstensen and Stephan (1993b). An adaptive $h-p$ version for the Dirichlet problem of the Laplacian based on Symm's integral equation is implemented in Ervin, Heuer and Stephan (1993). In this paper, we show that the a posteriori error analysis Eriksson and Johnson (1988, 1991) made for the finite element method of the Dirichlet problem for the Laplacian carries over to the adaptive *h*-version for two boundary integral equations, namely Symm's integral equation and a hypersingular integral equation which handle the Dirichlet and Neumann problems of the Laplacian, respectively. This analysis leads directly to a *rigorous* global error control algorithm.

We let Ω denote a bounded two dimensional domain with polygonal boundary Γ.

As it is well known, we can convert the Dirichlet problem

$$\Delta u = 0 \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma$$

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C. Carstensen
Department of Mathematics, Heriot-Watt University,
Edinburgh EH4 14AS,UK.

D. Estey
School of Mathematics, Georgia Institute of Technology,
Atlanta, Ga 30332

E. P. Stephan
Institut fur Angewandte Mathematik, UNIHannover,
Welfengarten 1, D-30167 Hannover, FRG.

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by use of the single layer potential into Symm's integral equation

$$V\psi(x) := -\frac{1}{\pi}\int_\Gamma \psi(y)\ln|x-y|ds_y = (I+K)g(x), \quad x\in\Gamma, \tag{1}$$

for the unknown density ψ. On the other hand making use of a double layer ansatz for the solution we can convert the Neumann problem

$$\Delta u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = f \quad \text{on } \Gamma$$

into a hypersingular integral equation

$$Dv(x) := -\frac{1}{\pi}\frac{\partial}{\partial n_x}\int_\Gamma v(y)\frac{\partial}{\partial n_y}\ln|x-y|ds_y = (I-K^*)f(x), \quad x\in\Gamma, \tag{2}$$

for the unknown density v. The operators V and D are strongly elliptic pseudodifferential operators of orders minus one and plus one, respectively (see Costabel 1988; Costabel and Stephan 1983, 1985), and the Galerkin schemes for (1) and (2), respectively, are:

Find $\psi_h \in S_h^0(\Gamma)$ such that for all $\phi_h \in S_h^0(\Gamma)$

$$\langle V\psi_h, \phi_h\rangle = \langle g, \phi_h\rangle. \tag{3}$$

Find $v_h \in S_h^1(\Gamma)$ such that for all $w_h \in S_h^1(\Gamma)$

$$\langle Dv_h, w_h\rangle = \langle f, w_h\rangle. \tag{4}$$

Here $\langle\cdot,\cdot\rangle$ denotes the L^2-duality of Γ given by $\int_\Gamma f(x)g(x)dx$ for smooth functions f, g. $S_h^0(\Gamma)$ consists of piecewise constant functions and $S_h^1(\Gamma)$ consists of continuous, piecewise linear functions.

The operators D and V define continuous, positive definite (provided cap $(\Gamma) < 1$, sufficient for that is, e.g., that the domain is included in a circle with radius < 1 which can always be arranged by scaling) and symmetric bilinear forms $a(v, w) = \langle Dv, w\rangle$ and $a(\psi, \phi) = \langle V\psi, \phi\rangle$ for $v, w \in \tilde{H}^{1/2}(\Gamma) := \{v \in H^{1/2}(\Gamma): \int_\Gamma v ds = 0\}$ and $\psi, \phi \in H^{-1/2}(\Gamma)$ (see Costabel 1988; Costabel and Stephan 1983, 1985). (For the definition of the Sobolev spaces see Lions and Magenes 1972.) Therefore, the Galerkin solutions of (3) and (4) converge quasioptimally in the energy norm towards the exact solutions of the integral Eqs. (1) and (2). Form Costabel (1988), Costabel and Stephan (1983, 1985) we know that, if cap $(\Gamma) \neq 1$, there exists exactly one solution $\psi \in H^{1/2}(\Gamma)$ of (1) for $g \in H^{1/2}(\Gamma)$, and that there exists exactly one solution $v \in \tilde{H}^{1/2}(\Gamma)$ of (2) for $f \in H^{-1/2}(\Gamma)$ with $\int_\Gamma f ds = 0$.

Furthermore, for smoother given data g and f we known from Costabel and Stephan (1983, 1985) that ψ and v allow decompositions into so-called corner singularities plus "smooth" remainders where the lack of regularity of ψ and v near the corner stems from the geometry of the polygonal domain Ω. Those corner singularities lead to poor solutions of the numerical schemes unless they are incorporated directly into the scheme or a suitable mesho refinement is used.

In practical problems such information is missing, and this is why one would like to have an adaptive feedback algorithm where the algorithm itself decides when and where to refine the mesh in order to improve the computed Galerkin solution.

2
Adaptive algorithms

First consider (1). There holds the following a posteriori error estimate.

Theorem 1 (Carstensen and Stephan 1994a,b)
There exists a constant $C > 0$ independent of h such that if $g \in H^1(\Gamma)$ and ψ and ψ_h are the solutions of (1) and (3), respectively, then for $R = g - V\psi_h$, $R' := \frac{\partial R}{\partial s}$,

$$\|\psi - \psi_h\|_{H^{-1/2}(\Gamma)} \leqq C \cdot \|R'\|^{1/2}_{L^2(\Gamma)} \cdot \|h \cdot R'\|^{1/2}_{L^2(\Gamma)} \tag{5}$$

and

$$\|\psi-\psi_h\|_{H^{-1/2}(\Gamma)} \leqq C\cdot\sum_{j=1}^{N} h_j^{1/2}\cdot\|R'\|_{L^2(\Gamma_j)} \tag{6}$$

Next consider (2). There holds the following a posteriori error estimate.

Theorem 2 (Carstensen and Stephan 1994a,b)
There exists a constant $C>0$ independent of h such that if $f\in L^2(\Gamma)$ and v and v_h are the solutions of (2) and (4), respectively, then for $r=f-Dv_h$

$$\|v-v_h\|_{H^{1/2}(\Gamma)} \leqq C\cdot\|r\|^{1/2}_{L^2(\Gamma)}\cdot\|h\cdot r\|^{1/2}_{L^2(\Gamma)} \tag{7}$$

$$\|v-v_h\|_{H^{1/2}(\Gamma)} C\cdot \leqq \sum_{j=1}^{N} h_j^{1/2}\cdot\|r\|_{L^2(\Gamma_j)}. \tag{8}$$

The integral equations (1) and (2) and the corresponding a posteriori estimates (Theorems 1 and 2) can be looked at in a general frame work.

Let $A:\mathscr{H}\to\mathscr{H}^*$ with $\mathscr{H}\subseteq H^m(\Gamma)$, $m\in\mathbb{R}$, $\mathscr{H}^*$ the dual of $\mathscr{H}$, be some pseudodifferential operator which is bounded, linear and positive definite on the closed subspace $\mathscr{H}$ of $H^m(\Gamma)$, i.e. we have a constant $\alpha>0$ with

$$\langle Au,u\rangle \geq \alpha\|u\|^2_{H^m(\Gamma)} \quad \text{for all} \quad u\in\mathscr{H}\subseteq H^m(\Gamma).$$

Here, $\langle,\rangle$ extends the $L^2(\Gamma)$ scalar product to the duality between the Sobolev spaces $\mathscr{H}$ and $\mathscr{H}^*$. Due to the Lax-Milgram lemma we then have a unique solution $u\in\mathscr{H}\subseteq H^m(\Gamma)$ of

$$Au=f \tag{9}$$

for any given right hand side $f\in H^{-m}(\Gamma)\subseteq\mathscr{H}^*$.

For the numerical approximation of u let $\Gamma=\overline{\cup_{j=1}^{N}\Gamma_j}$ be partitioned into N pairwise disjoint elements $\Gamma_1,\ldots,\Gamma_N$ and let S_h^k denote a finite dimensional subspace of $\mathscr{H}\subseteq H^m(\Gamma)$ of piecewise polynomials, i.e. $v_{h|\Gamma_j}$ is a polynomial of degree k at most for any $v_h\in S_h^k$ and $j=1,\ldots,N$. Then, the Galerkin equations

$$\langle Au_h,v_h\rangle=\langle f,v_h\rangle \quad \text{for all} \quad v_h\in S_h^k \tag{10}$$

have a unique solution $u_h\in S_h^k$. Due to Cea's lemma we have

$$\|u-u_h\|_{H^m(\Gamma)}\leq C\cdot\inf_{v_h\in S_h^k}\|u-v_h\|_{H^m(\Gamma)} \tag{11}$$

where C denotes a generic positive constant which is independent of the data f and S_h^k.

In Carstensen and Stephan (1994a) we derive a posteriori error estimates of the form

$$\|u-u_h\|_{H^m(\Gamma)}\leq C\cdot\left(\sum_{j=1}^{N} a_j^2\right)^{r/2}\cdot\left(\sum_{j=1}^{N} h_j^2\cdot a_j^2\right)^{(1-r)/2} \tag{12}$$

where r is some real some real number (dependin on A), $h_j:=|\Gamma_j|$ is the length of the element Γ_j, and a_j is the local contribution of the residual $R:=g-Au_h$ assuming $R\in H^{1-k}(\Gamma)$, i.e.

$$a_j:=\left\|\left(\frac{\partial}{\partial s}\right)^{1-k}R\right\|_{L^2(\Gamma_j)} \tag{13}$$

where prime or $(\partial/\partial s)$ denotes differentiation with respect to the are length. Inspections show that (12) corresponds to (5) and (7) with $r=1/2$. In (12) any element contributes two terms to the right hand

side, namely a_j and $a_j h_j$. Usually, an adaptive steering has the aim to equidistribute all members of the sum (which bounds the error) in order to yield some optimal mesh. Since we have two sums in (12) it is not obvious which of them and how should be equidistributed (cf. Remark 3 below).

In Carstensen and Stephan (1994b) we prove improved a posteriori error estimates of the form

$$\|u - u_h\|_{H^m(\Gamma)} \leq C \cdot \sum_{j=1}^{N} a_j \cdot h_j^{1-r} \tag{14}$$

via "local interpolation". The estimate (14) correspondents to (6) and (8) with $r = 1/2$. The new upper bound in (14) is more convenient for an adaptive feedback algorithm steering the mesh-refinements than (12) since any element Γ_j contributes only to one summand.

From (14), an adaptive algorithm for automatic mesh refinement is easily derived following Eriksson and Johnson. In this way, we consider five adaptive algorithms, four of them based on (12) and (14), and compare the resulting convergence behavior in numerical examples. Following the works of Eriksson and Johnson (1988, 1991) and Johnson and Hansbo (1992) (and the literature quoted therein) we use the above a posteriori estimates to steer the mesh refinement within a successively refining h-version of a Galerkin procedure.

Given a tolerance $TOL > 0$ we cannot decide whether or not the norm of the error is smaller than TOL as far as the constants in (12) and (14) are unknown. Without computing upper bounds for these constants we have only some relative error control. Hence, neglecting the constants we can compute $a_j := \|R'\|_{L^2(\Gamma_j)}$ for Symm's integral equation or $a_j := \|r\|_{L^2(\Gamma_j)}$ for the hypersingular integral equation (at least by numerical approximations-see Carstensen and Stephan (1994b) for details). We consider the right hand sides

$$B_1 := B_1(a_1, \ldots, a_N, h_1, \ldots, h_N) := \left(\sum_{j=1}^{N} a_j^2 \right)^{r/2} \cdot \left(\sum_{j=1}^{N} h_j^2 \cdot a_j^2 \right)^{(1-r)/2}$$

and

$$B_2 := B_2(a_1, \ldots, a_N, h_1, \ldots, h_N) := \sum_{j=1}^{N} a_j \cdot h_j^{1-r}$$

related to (12) and (14), respectively, where $0 \leq r \leq 1$ is fixed. Then, the problem of finding *optimal meshes with respect to the above bound* $B_\ell (\ell = 1, 2)$ reads as follows (see Carstensen and Stephan 1994b):

Problem of optimal meshes. Given a threshold $TOL > 0$, find a polygon with the sides $\Gamma_1, \ldots, \Gamma_N$ which covers Γ and which minimizes the computational costs needed for assembling and solving the related Galerkin equations under the side condition that the corresponding terms $a_1, \ldots, a_N$, $h_1, \ldots, h_N) \leq TOL$.

Remark 1 The computational costs for assembling and solving the related Galerkin equations can be estimated approximately by the number of degrees of freedom that is the number of elements N, i.e. we have to minimize N under the side condition $B_\ell \leq TOL$.

Our adaptive algorithms have the following general structure and differ just by the refinement rule (Ak) in step (iii).

2.1
Adaptive algorithm (Ak)

Given a coarse initial mesh and then take the acutal partition $\Gamma_1, \ldots, \Gamma_N$ and do (i)–(iii) until termination.

(i) Solve the Galerkin equations with a trial space S_h^k, a space of piecewise polynomials with respect to the actual partition $\Gamma_1, \ldots, \Gamma_N$.

(ii) Compute $a_1, \ldots, a_N, h_1, \ldots, h_N$ and the related B_ℓ. If $B_\ell \leq TOL$ stop, else continue with (iii).

(iii) For $j = 1, \ldots, N$ refine Γ_j by the rule (Ak) and continue with the new mesh in (i).

Remark 2 The Algorithm (Ak) requires in step (iii) to create a new mesh which satisfies $B_\ell \leq TOL$ and leads to the minimal number of new elements. This means that we have to choose natural numbers $k_1, \ldots, k_N$ such that the new mesh is determined by dividing Γ_j in k_j new elements (e.g. of the same size) and $k_1 + \cdots + k_N = \hat{N}$ is minimal under the side condition that for the new mesh $\hat{\Gamma}_1, \ldots, \hat{\Gamma}_{\hat{N}}$ with related

coefficients $\hat{a}_1, \ldots, \hat{a}_{\hat{N}}$ and element lengths $\hat{h}_1, \ldots, \hat{h}_{\hat{N}}$ we have

$$B_\ell(\hat{a}_1, \ldots, \hat{a}_{\hat{N}}, \hat{h}_1, \ldots, \hat{h}_{\hat{N}}) \leq TOL. \tag{15}$$

Note that

$$(\hat{h}_1, \ldots, \hat{h}_{\hat{N}}) = \left(\underbrace{\frac{h_1}{k_1}, \ldots, \frac{h_1}{k_1}}_{k_1}, \underbrace{\frac{h_2}{k_2}, \ldots, \frac{h_2}{k_2}}_{k_2}, \ldots, \underbrace{\frac{h_N}{k_N}, \ldots, \frac{h_N}{k_N}}_{k_N}\right). \tag{16}$$

The first idea here is that the new element sizes can be controlled directly but the corresponding coefficients $\hat{a}_1, \ldots, \hat{a}_{\hat{N}}$ are more involved. If we assume that the residual does not change during this iteration step or, at least, will not become very large, we may assume in this step that $\hat{a}_1, \ldots, \hat{a}_{\hat{N}}$ can be replaced by $a_1, \ldots, a_N$—more precisely, we regard the meshsize h_j as a (piecewise constant) *weight function* of the fixed residual. Therefore, the side condition (15) is replaced by the side condition

$$B_\ell(a_1, \ldots, a_N, h_1/k_1, \ldots, h_N/k_N) \leq TOL. \tag{17}$$

Note that, e.g., $B_2(a_1, \ldots, a_N, h_1/k_1, \ldots, h_N/k_N) = \sum_{j=1}^N a_j \cdot (h_j/k_j)^{1-r}$.

The minimization of $k_1 + \cdots + k_N$ under the side condition (17) leads to our refinement rules below.

We use the following refinement rules (**Ak**), $k = 1, 2, 3, 4, 5$ (for Sobolev spaces in the energy norm, i.e. for $r = 1/2$) in the numerical examples where $\theta \in [0, 1]$ is some fixed parameter, and $TOL > 0$ is a given tolerance.

2.2
Refinement rules (Ak)

(A1) Halve Γ_j if and only if $a_j \cdot h_j \geq \theta \cdot \max_{k=1,\ldots,N} a_k \cdot h_k$.
(A2) Halve Γ_j if and only if $a_j \cdot h_j^{1-r} \geq \theta \cdot \max_{k=1,\ldots,N} a_k \cdot h_k^{1-r}$.
(A3) Divide Γ_j in k_j pieces of the same length where k_j is the smallest integer ≥ 1 with

$$h_j/k_j \leq \frac{TOL^{1/(1-r)}}{a_j \cdot \sqrt{N} \cdot a^{r/(1-r)}}, \quad a := \sqrt{\sum_{j=1}^N a_j^2}.$$

(A4) Divide Γ_j in k_j pieces of the same length where k_j is the smallest integer ≥ 1 with

$$h_j/k_j \leq \left(\frac{TOL}{Na_j}\right)^{1/(1-r)}$$

(A5) Divide Γ_j in k_j pieces of the same length where k_j is the smallest integer ≥ 1 with

$$h_j/k_j \leq \left(\frac{TOL}{Nb_j}\right)^{1/(1-r)}, \quad b_j := \|R\|_{H^1(\Gamma_j)}$$

We conclude this section with some remarks on the motivation and some simple properties of the adaptive algorithm.

Remark 3 (i) The last three rules are related to (B_ℓ) for $\ell = 1, 2$ and the notion to reach (17) in one step (i.e. $k_1, \ldots, k_N \in \{1, 2, \ldots\}$) under the side condition that the contributions of the new elements to the sum on the left hand side in (17) are nearly equal to TOL/N, i.e. they are nearly equal to each other and nearly equality in (17).

(ii) Note that (A1) does not depend on the parameter r. Hence the mesh created will be the same and independent of the norm chosen for minimizing the error.

(iii) For each step we have some guaranteed error estimate. In this sense, the adaptive algorithms are *reliable* (up to the constant factor C).

(iv) If the terms $a := \sqrt{\sum_{j=1}^N a_j^2}$ and b_j blow up, we may modify the algorithm and include steps of uniform refinements so that this term a becomes small enough. If we perform this additional control we achieve *operability* in the sense that the side condition (15) is satisfied within a finite number of steps.

(v) If we modify the algorithm as in (iv) and, in addition, reduce the tolerance TOL once (15) is reached (by a factor 1/2, say) we get a sequence of Galerkin solutions which converges to the exact solution, i.e. we have *convergence*.

(vi) The question of efficiency is tackled here just by numerical examples and a comparison of convergence rates and a listing of the respective effectivity indices.

3
Numerical results

We present one example for *Symm's integral equation* and a second example for the *hypersingular integral equation* using the *h*-version of the Galerkin method. We take piecewise constant and continuous, piecewise linear trial functions in S_h^0 and S_h^1, respectively. Ω is the L-shaped domain with vertices (0, 0),

Table 1. Adaptive *h*-version for Symm's integral equation.

N	Uniform mesh e_N	α_N	γ_N^2
8	0.18803612		0.21747
16	0.11259966	0.740	0.21228
32	0.070981643	0.666	0.20157
64	0.044727288	0.666	0.19155
128	0.028239541	0.663	0.17982
256	0.017926080	0.656	0.16596
N	(A1) for $\theta = 0.5$ e_N	α_N	γ_N^1
8	0.18803612		0.41398
10	0.11480572	2.211	0.40524
12	0.07613727	2.253	0.39361
16	0.04892621	1.537	0.36506
20	0.03011395	2.175	0.36228
26	0.01789988	1.983	0.29450
38	0.01112847	1.277	0.29535
50	0.00433037	3.406	0.16346
68	0.00111284	4.419	0.06516
N	(A1) for $TOL = 3.0$ e_N	α_N	γ_N^2
8	0.18803612		0.32442
12	0.08942822	1.833	0.20543
18	0.04253587	1.833	0.12141
26	0.02802225	1.135	0.09706
36	0.01683484	1.566	0.06351
N	(A2) for $\theta = 0.5$ e_N	α_N	γ_N^2
8	0.188036093887699		0.21747
10	0.114805720083032	2.211	0.19245
12	0.076137268453540	2.253	0.16520
14	0.052517624082683	2.409	0.14085
16	0.039464335297708	2.140	0.12547
22	0.020652945322717	2.033	0.11332
24	0.015357936931417	3.404	0.09595
30	0.007360172681442	3.296	0.05919
36	0.005549105541033	1.549	0.05874
46	0.001934266012570	4.300	0.02694
N	(A3) for $TOL = 0.6$ e_N	α_N	γ_N^1
8	0.188036093887699		0.41398
14	0.074973926354012	1.643	0.39016
18	0.042167583016977	2.290	0.30699
22	0.029806401983176	1.729	0.24435
24	0.026973506611762	1.148	0.23345
26	0.025751660097479	0.579	0.23442
28	0.025251145343351	0.265	0.24132
30	0.025049590424076	0.116	0.25082

Table 1. (*Continued*)

N	(A4) for $TOL = 3.0$ e_N	α_N	γ_N^2
8	0.18803612		0.21747
10	0.11480572	2.211	0.19245
14	0.060639315	1.897	0.14958
20	0.034973051	1.543	0.12146
26	0.028897927	0.727	0.11970
36	0.019924578	1.143	0.10965

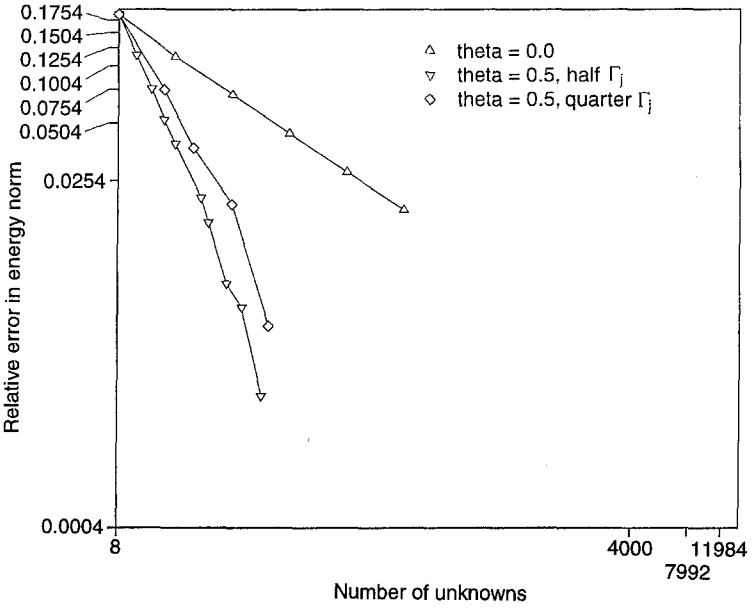

Fig. 1. Errors in the Galerkin method obtained in Algorithm (A2) for Symm's integral equation

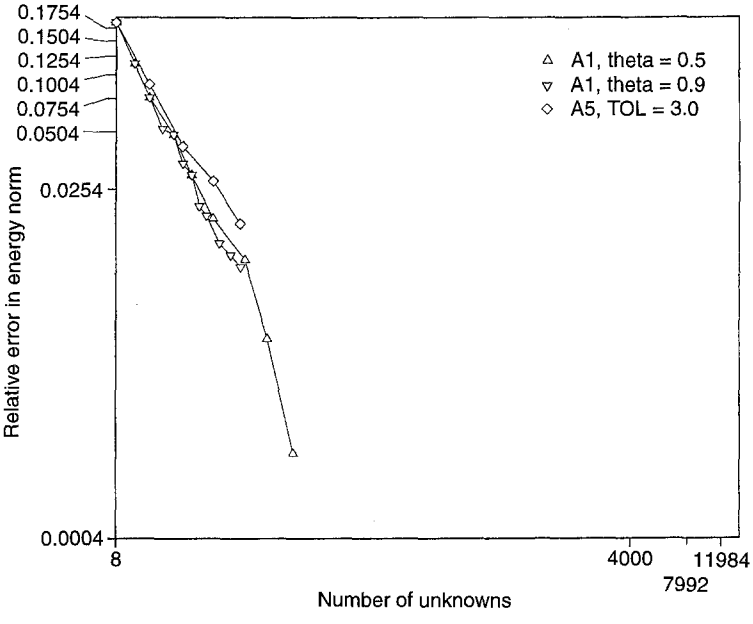

Fig. 2. Errors in the Galerkin method obtained in Algorithm (A1) and (A5) for Symm's integral equation

Table 2. Modified adaptive *h*-version for Symm's integral equation.

N	(A2) for $\theta = 0.5$ quater Γ_j e_N	α_N
8	0.1880360938	
14	0.0749739260	1.643
20	0.0375267700	1.940
32	0.0190906560	1.438
50	0.0044439687	3.266

Table 3. Numerical results for hypersingular integral equation.

	Uniform mesh	
N	e_N	α_N
8	0.47030	
16	0.42849	0.134
24	0.40787	0.122
32	0.39335	0.126
40	0.38247	0.126
48	0.37381	0.126
56	0.36664	0.126
64	0.36054	0.126
72	0.35525	0.126
80	0.35058	0.125

	(A2) for $\theta = 0.9$	
N	e_N	α_N
8	0.47030	
9	0.45175	0.342
10	0.43337	0.394
11	0.41063	0.566
12	0.38574	0.850
14	0.33837	0.850
16	0.29402	1.052
18	0.25285	1.281
20	0.21437	1.567
22	0.17805	1.948
24	0.13911	2.837
26	0.10289	3.768
28	0.062753	6.672
30	0.031020	10.212

	(A2) for $\theta = 0.5$	
N	e_N	α_N
8	0.47030	
10	0.42756	0.427
12	0.39221	0.473
14	0.35913	0.572
16	0.32900	0.656
18	0.30155	0.740
20	0.27053	1.030
22	0.24832	0.899
24	0.22625	1.070
26	0.20693	1.116
28	0.18854	1.256
30	0.17129	1.391
32	0.15624	1.425
34	0.14243	1.526
36	0.13009	1.585
38	0.11913	1.628
40	0.10951	1.641
42	0.10078	1.703
44	0.092817	1.770
46	0.085515	1.843
48	0.078794	1.923
52	0.071366	1.237
55	0.068790	0.655

$(1,0), (1,1), (-1,1), (-1,-1), (0,-1)$. For the implementation of our adaptive h-version of the boundary element method see Carstensen and Stephan (1994a,b). Symm's integral equation (1) is considered with given $g = Im\, z^{2/3}$ on the boundary Γ of the L-shaped domain Ω.

For the hypersingular integral equation (2) on the boundary Γ of the L-shaped domain the given data $f = (\partial u/\partial n)$ on Γ is computed from $u(z) = Im\, z^{1/7}$.

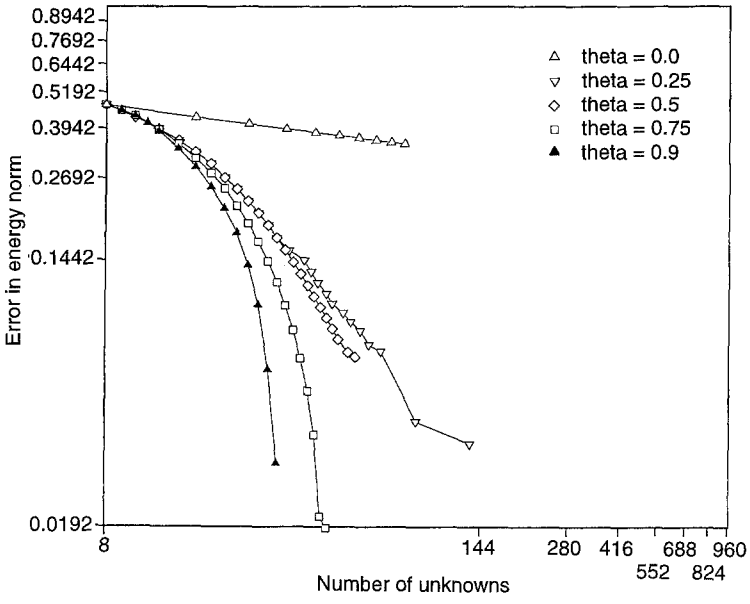

Fig. 3. Errors in the Galerkin method obtained in Algorithm (A2) for the hypersingular integral equation

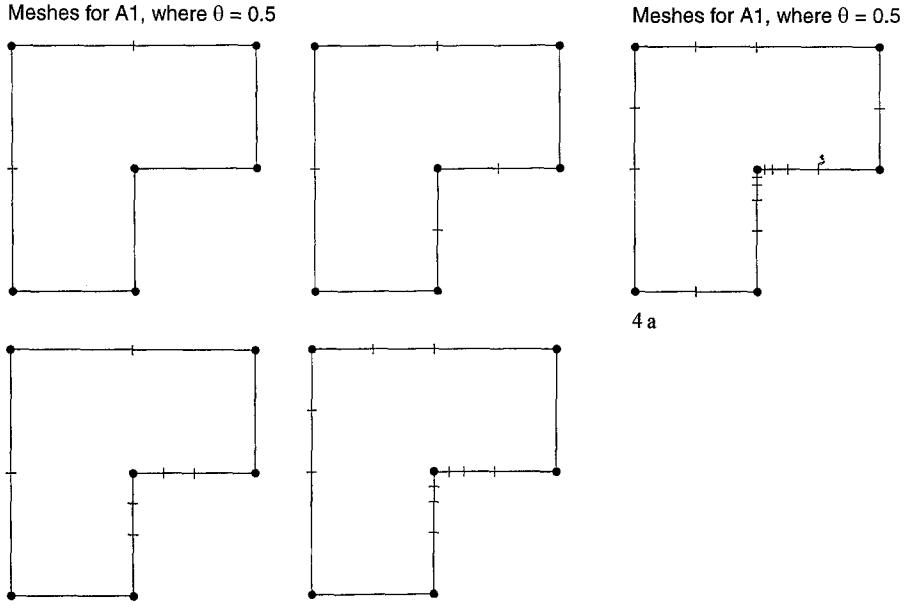

Fig. 4.

In Table 1 we list the error $e_N := \|\psi - \psi_N\|_E$ in the energy norm $\|\psi\|_E^2 = \langle V\psi, \psi\rangle$ for the h-adaptive Galerkin solution ψ_N of (3), α_N and γ_N. N denotes the number of degrees of freedom (chosen by the algorithm; a new row corresponds to a new refinement step in the adaptive algorithm). α_N is the experimental convergence rate computed as

$$\alpha_N = \frac{\log(e_N/e_{N'})}{\log(N'/N)}$$

where N' and $e_{N'}$ are the corresponding values of the previous row. Define the "effectivity index"

$$\gamma_n^I := \frac{\sqrt{\langle V(\psi_N - \psi), \psi_N - \psi\rangle}}{B_I(a_1, \ldots, a_N, h_1, \ldots, h_N)}$$

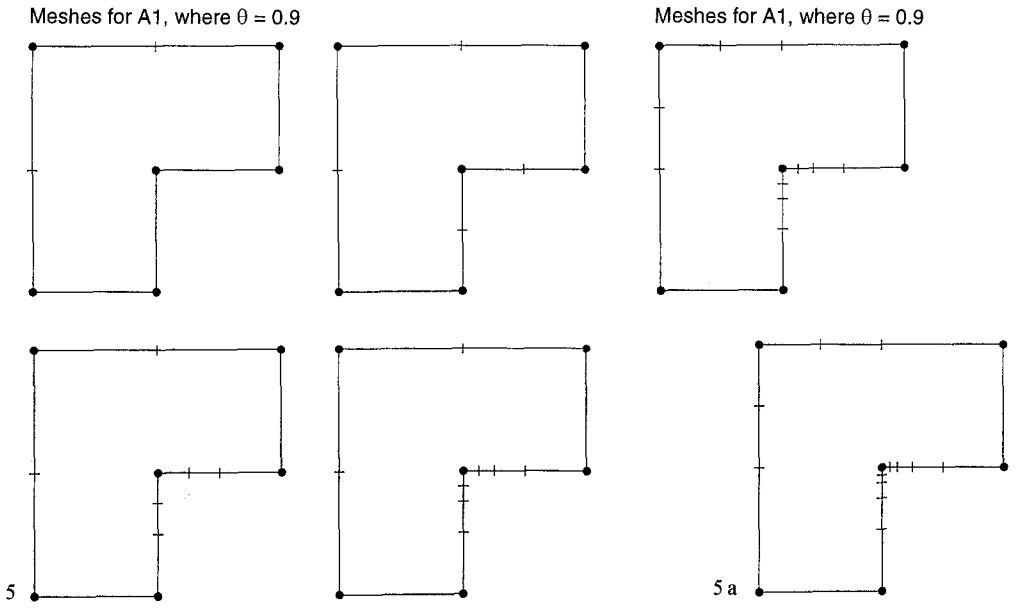

Fig. 5.

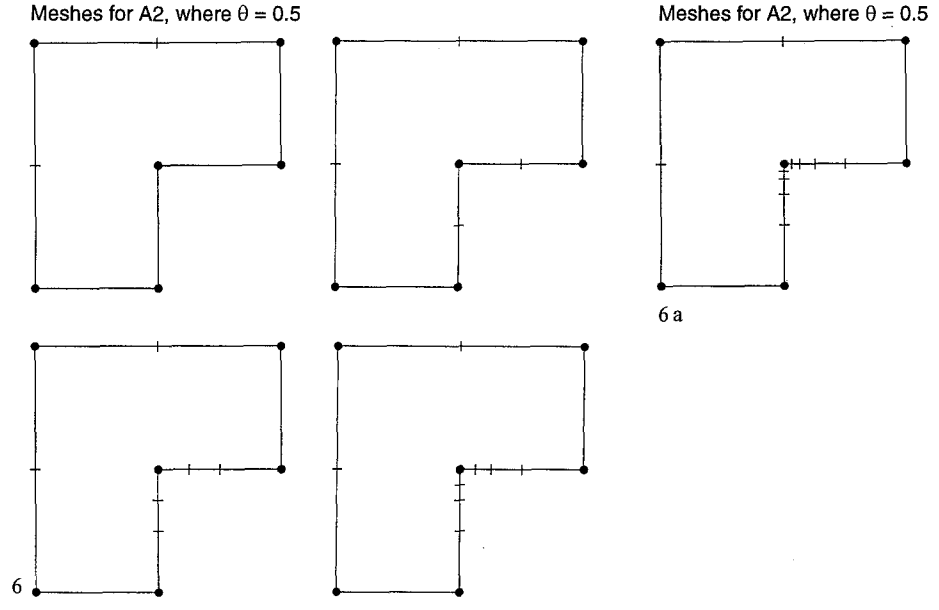

Fig. 6.

with B_l yielding the adaptive algorithm considered. For the algorithm (A5) we set $a_j := \|R\|_{H^1(I_j)}$. We present the respective results for the adaptive algorithms (A1)–(A5) and for the uniform mesh.

In our example the solution has typical corner signularity. Hence the convergence rate of an h-version boundary element scheme (3) with uniform mesh is reduced to 2/3 while the adaptive versions (compare Table 1 and Fig. 1 and 2) show improved convergence rates, the optimal is 3/2. Whereas (A1) and (A2) in Table 1 are performed via interval halving, the results of Table 2 are obtained with (A2) via dividing the intervals into 4 pieces (of equal length). Comparisons show that "quartering" is superior to interval "halving", since the error is already after a few steps drastically reduced. The effectivity index listed in Table 1 shows clearly the effectivity of our algorithms (A1)–(A5).

Table 3 we show the error $e_n := \|v - v_n\|_E$ in the energy norm $\|v\|_E^2 = \langle D\psi, \psi \rangle$ for the h-adaptive Galerkin solution v_n of (4). We give the respective results for the algorithm (A2). In Fig. 3. we plot the Galerkin error due to algorithm (A4).

Figures 4–8 show various meshes which are created via the algorithms (A1), (A2) and (A5) when solving Symm's integral equation.

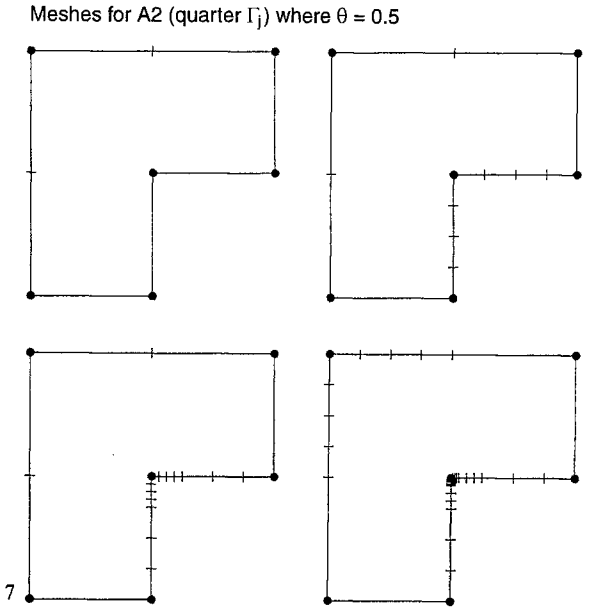

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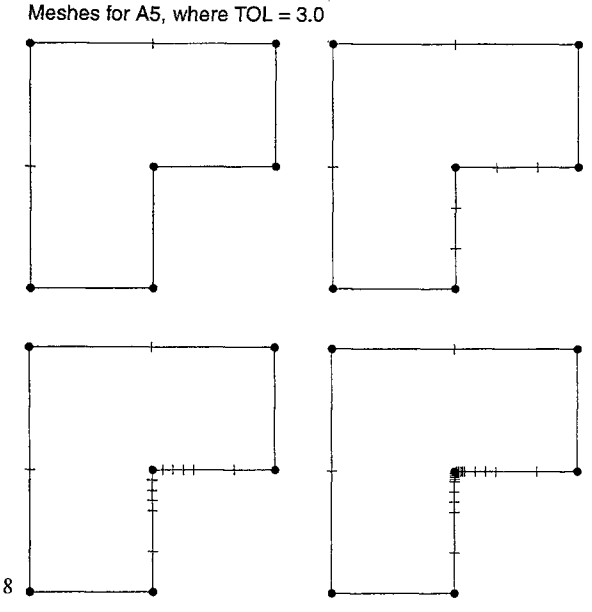
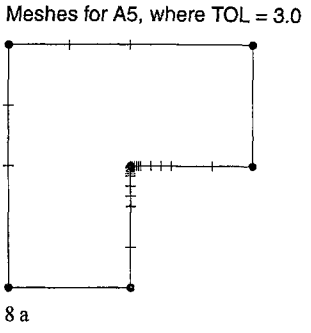
Fig. 7.





8 a

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Fig. 8.

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