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QD-Type Algorithms for the Nonnormal Newton–Padé Approximation Table

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Abstract. It is well known that solutions of the rational interpolation problem or Newton–Padé approximation problem can be represented with the help of continued fractions if certain normality assumptions are satisfied. By comparing two interpolating continued fractions, one obtains a recursive QD-type scheme for computing the required coefficients. In this paper a uniform approach is given for two different interpolating continued fractions of ascending and descending type, generalizing ideas of Rutishauser, Gragg, Claessens, and others.

In the nonnormal case some of the interpolants are equal yielding so-called singular blocks. By appropriate "skips" in the Newton–Padé table modified interpolating continued fractions are derived which involve polynomials known from the Kronecker algorithm and from the Werner–Gutknecht algorithm as well as from the modification of the cross-rule proposed recently by the authors. A corresponding QD-type algorithm for the nonnormal Newton–Padé table is presented. Finally, the particular case of Padé approximation is discussed where—as in Cordellier's modified cross-rule—the given recurrence relations become simpler.

1. Introduction

Throughout this paper we will represent rational interpolants with the help of *continued fractions*

$$B_0 + \frac{A_1 \rfloor}{\lceil B_1} + \frac{A_2 \rfloor}{\lceil B_2} + \frac{A_3 \rfloor}{\lceil B_3} + \cdots, \tag{1.1}$$

where $A_1, A_2, \ldots$ and $B_0, B_1, \ldots$ are complex-valued polynomials. It is well known that the numerator p_j and the denominator q_j of the jth *convergent*

$$\frac{p_j \rfloor}{\lceil q_j} = B_0 + \frac{A_1 \rfloor}{\lceil B_1} + \frac{A_2 \rfloor}{\lceil B_2} + \frac{A_3 \rfloor}{\lceil B_3} + \cdots + \frac{A_j \rfloor}{\lceil B_j}$$

satisfy the three-term recurrence relation (for $j = 1, 2, 3, \ldots$)

$$\begin{aligned} p_j &= B_j \cdot p_{j-1} + A_j \cdot p_{j-2}, \\ q_j &= B_j \cdot q_{j-1} + A_j \cdot q_{j-2}, \end{aligned}$$

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together with the initializations $p_{-1} = 1$, $p_0 = B_0$, $q_{-1} = 0$, and $q_0 = 1$ (see, e.g., [13, Chapter 1]). For these basic recurrence relations we will use the shorter notation

$$(1.2) \qquad \begin{pmatrix} p_j \\ q_j \end{pmatrix} = B_j \cdot \begin{pmatrix} p_{j-1} \\ q_{j-1} \end{pmatrix} + A_j \cdot \begin{pmatrix} p_{j-2} \\ q_{j-2} \end{pmatrix}.$$

Note that the numerator and denominator of a convergent are only unique up to *scaling*, i.e., up to multiplication with a common factor. Switching to a different scaling corresponds to an *equivalence transformation* of the continued fraction. Obviously, for a continued fraction to be nonterminating we have to suppose that $A_j \neq 0$ for all j or, equivalently, that two succeeding convergents have to be distinct.

For Padé approximation (i.e., rational interpolation at a single confluent knot), Rutishauser proposed to consider the following *interpolating continued fraction*

$$(1.3) \qquad c_0 + \cdots + c_k x^k + \frac{c_{k+1}x^{k+1} \rfloor}{\lfloor 1} - \frac{q_1^{(k+1)}x \rfloor}{\lfloor 1} - \frac{e_1^{(k+1)}x \rfloor}{\lfloor 1} - \cdots,$$

where $c_0, c_1, c_2, \ldots$ are the coefficients of a given power series. The quantities $q_j^{(k)}$ and $e_j^{(k)}$ are usually displayed in the so-called QD-table

$$\begin{array}{cccccccc}
e_0^{(1)} & & & & & & & \\
 & q_1^{(1)} & & & & & & \\
e_0^{(2)} & & e_1^{(1)} & & & & & \\
 & q_1^{(2)} & & q_2^{(1)} & & & & \\
e_0^{(3)} & & e_1^{(2)} & & e_2^{(1)} & & \cdot & \\
 & q_1^{(3)} & & q_2^{(2)} & & & \ddots & \\
e_0^{(4)} & \vdots & e_1^{(3)} & \vdots & e_2^{(2)} & & & \\
\vdots & & \vdots & & \vdots & & &
\end{array}$$

These coefficients can be computed simultaneously for all indices k according to $e_0^{(k)} = 0$, $q_1^{(k)} = c_{k+1}/c_k$,

$$e_\ell^{(k)} - e_{\ell-1}^{(k+1)} = q_\ell^{(k+1)} - q_\ell^{(k)} \qquad \text{and} \qquad \frac{q_{\ell+1}^{(k)}}{q_\ell^{(k+\ell)}} = \frac{e_\ell^{(k+1)}}{e_\ell^{(k)}}$$

$(k, \ell = 1, 2, 3, \ldots)$ which justifies Rutishauser's notion "Quotienten-Differenzen-Algorithms" or, simply, QD-algorithm [21]. This method is well established in numerical analysis, and we may refer to standard literature for proofs and further interesting properties of the algorithm. In particular, the convergents of (1.3) are equal to the entries

$$(1.4) \qquad T_k^{\text{desc}}\colon\ r_{k,0}, r_{k+1,0}, r_{k+1,1}, r_{k+2,1}, r_{k+2,2}, r_{k+3,2}, \ldots \qquad (k = 0, 1, 2, \ldots)$$

of the Padé table, which lie on a *descending staircase*.

Generalizations of the QD-algorithm to the Newton–Padé case (rational interpolation with arbitrary knots) using various scalings were given in [7], [17], and [24]. An ascending QD-type algorithm was stated in [16] and was generalized to Newton–Padé approximation in [8]. It yields rational interpolants on an *ascending staircase*

$$(1.5) \qquad T_k^{\text{asc}}\colon\ r_{k,0}, r_{k-1,0}, r_{k-1,1}, r_{k-2,1}, r_{k-2,2}, r_{k-3,2}, \ldots, r_{0,k} \qquad (k = 1, 2, \ldots).$$

In the first part of the paper we briefly review both types of interpolating continued fractions using a uniform approach. It is shown that the entries of a so-called b-table, which is computed by a QD-type algorithm, enable us to construct simultaneously all T_k^{desc}, $k \geq 0$, and T_k^{asc}, $k > 0$. The resulting interpolating continued fractions include those from [8] and [16], but we use a scaling different from that in (1.3) (and its generalizations [7], [17], [24]), which is the same as in the descending interpolating continued fraction due to Thiele.

All QD-type algorithms mentioned above obviously require the regularity assumption that any two *neighboring* rational interpolants (i.e., successive convergents in any T_k^{desc} or T_k^{asc}) are distinct. This is called the *normal case*. In the second part of this paper we consider the more general nonnormal case where *singular blocks* might occur, which leads to various difficulties. In [2] and [3] the authors generalized Claessens' cross-rule to the nonnormal Newton–Padé approximation table so that the question "How to modify the above continued fractions then?" appears naturally and is solved in this paper.

An appropriate modification for interpolating continued fractions seems to be straightforward. The main idea here is to consider a (maximal) subsequence of interpolants from T_k^{desc} or T_k^{asc} such that two successive approximants are distinct, i.e., we "skip" singular blocks along diagonals and antidiagonals, respectively. Descending continued fractions modified in this way are obtained by, e.g., the method of Werner [23] and its generalizations due to Gutknecht [18], or by the Viskovatov method (see, e.g., [13, p. 89] and [1, I, p. 129 ff]).

The skips on ascending staircases are implicit in the recurrence relation of the modified Kronecker algorithm [1, Part II]. However, for these methods we fix k and consider only one continued fraction, hence they are not efficient if one is interested in more than one interpolating continued fraction.

As far as we know, for the nonnormal case, QD-type schemes connecting neighboring staircases T_k^{desc} and T_{k+1}^{desc} have only been given for Padé approximation [9], [14], [19], and [20]. In [9], horizontal and vertical skips are suggested leading to quite complicated recurrence relations. Moreover, Draux mentioned [14, p. 167] that in addition to the proposed three basic relations one has to apply 14 other rules in order to cover all singular cases.

The paper is organized as follows. In Section 2 we fix the notation and give *Froebenius* identities of type (1.2). These yield the interpolating continued fractions T_k^{desc} and T_k^{asc}, $k \geq 0$, for the normal case discussed in Section 3. All required scalar coefficients $a_{\mu,\nu} = b_{\mu,\nu}$ are displayed in the b-table introduced in Section 4, where we propose a new QD-type algorithm closely connected to Claessens' cross-rule [6].

For the convenience of the reader, we give a description of singular blocks in the nonnormal Newton–Padé table and review some results from [2], [3], [5], and [18] in Section 5. Then, in Section 6, the computation of the nonnormal b-table is discussed. Using some technical lemmas proved in Section 7, we treat the modification of the ascending continued fraction in Section 8. Here—similar to the reliable modification of Claessens' cross-rule [2], [3]—the *Kronecker* polynomials α are essential. The modification of the descending interpolating Thiele fraction in Section 9 leads to different polynomials β related to *Werner* polynomials [23], as already studied by Gutknecht [18]. In Section 10 we derive recurrence relations for these polynomials where again the coefficients $b_{\mu,\nu}$ are required. Connections between the Kronecker and the Werner polynomials are pointed out in Section 11. In Section 12 we study the particular case of nonnormal Padé approximation where many formulas simplify. Some conclusions and a summary of the results of this paper are given in Section 13.

2. Notation

Let $(z_0, z_1, z_2, \ldots)$ be a sequence of (not necessarily distinct) knots in the complex plane. Let f be a function, sufficiently smooth in a neighborhood of these knots. For integers $m, n \geq 0$ the *rational interpolation problem* consists in finding polynomials $p_{m,n}$ and $q_{m,n}$ with degrees bounded by m and n, respectively, such that $f - p_{m,n}/q_{m,n}$ has at least the zeros $z_0, \ldots, z_{m+n}$, counting multiplicities. We consider the *Newton–Padé approximation problem* which is the *linearized rational interpolation problem*, i.e., we demand that $q_{m,n} \cdot f - p_{m,n}$ has the above zeros. Claessens [4] showed that there exist unique polynomials $p^*_{m,n}$ and $q^*_{m,n}$ of *minimal degree* $\deg p^*_{m,n} \leq m$, $\deg q^*_{m,n} \leq n$, $q^*_{m,n}$ being a monic polynomial, such that $q^*_{m,n} \cdot f - p^*_{m,n}$ has the zeros $z_0, \ldots, z_{m+n}$, counting multiplicities, and any other solution of this linearized rational approximation problem is of the form $s \cdot p^*_{m,n}$ and $s \cdot q^*_{m,n}$ with an arbitrary polynoimal s of suitable degree. In particular, for any solution $p_{m,n}$ and $q_{m,n}$ of the linearized rational approximation problem the meromorphic function

$$r_{m,n} := \frac{p_{m,n}}{q_{m,n}} = \frac{p^*_{m,n}}{q^*_{m,n}}$$

is the same. It is known (see, e.g., [1], [4], and [13]), that, in general, the minimal polynomials $p^*_{m,n}$ and $q^*_{m,n}$ might have common, possibly multiple zeros called unattainable points that are members of $(z_0, \ldots, z_{m+n})$. Moreover, the rational interpolation problem is solvable (with the unique solution $r_{m,n}$) iff there are no such unattainable points.

In this paper we consider the computation of the table $(r_{m,n}\colon\ m, n = 0, 1, 2, \ldots)$, called the *Newton–Padé table*, or some finite part of it. This problem is well studied if the table is *normal*, i.e., if two neighbors are distinct:

$$r_{m,n+1} \neq r_{m,n} \neq r_{m+1,n} \qquad (m, n = 0, 1, 2, \ldots).$$

It is well known (see, e.g., [3]) that in the normal case the degrees of the numerator and denominator are maximal, $\deg p^*_{m,n} = m$ and $\deg q^*_{m,n} = n$.

In the context of QD-type algorithms the numbers

$$\text{(2.6)} \qquad a_{m,n} := \text{leading coefficient of } p^*_{m,n}, \qquad b_{m,n} := \lim_{z\to\infty} z^{n-m} \cdot r_{m,n}(z),$$

will play an important role. The quantity $a_{m,n} \in \mathbf{C}\backslash\{0\}$ is called the *leading coefficient* of $r_{m,n}$ since the leading coefficient of $q^*_{m,n}$ always has to be 1. The *asymptotic coefficient* $b_{m,n}$ takes one of the values $a_{m,n}$, 0, ∞ from the extended complex plane $\overline{\mathbf{C}}$; in particular $b_{m,n} = a_{m,n}$ if and only if $\deg p^*_{m,n} - \deg q^*_{m,n} = m - n$.

Note that the tables $(a_{m,n}\colon\ m, n = 0, 1, 2, \ldots)$ and $(b_{m,n}\colon\ m, n = 0, 1, 2, \ldots)$ (shortly called a-table and b-table, respectively) coincide exactly in the normal case.

For simplicity, we assume that there are no singularities at the border of the Newton–Padé table. Then $1/f$ has a Newton series, and we find the coefficients $a_{0,j}$ and $a_{j,0}$

in

$$f(z) = \sum_{j=0}^{\infty} a_{j,0} \cdot \omega_{0,k}(z) \tag{2.7}$$

or

$$\frac{1}{f(z)} = \sum_{j=0}^{\infty} \frac{1}{a_{0,j}} \cdot \omega_{0,j}(z) \tag{2.8}$$

using the polynomials $\omega_{i,i}(z) := 1$ and for $i < j$

$$\omega_{i,j}(z) := (z - z_i) \cdot (z - z_{i+1}) \cdot \cdots \cdot (z - z_{j-1}).$$

As mentioned in Section 1, a continued fraction is determined by the three-term recurrences (1.2). The recurrences corresponding to T_k^{desc} and T_k^{asc} are given in the next theorem. These *Froebenius identities* are well known from Padé approximation (see, e.g., [15, p. 359 f] or [1]) and have a natural generalization to the normal Newton–Padé case [22].

Theorem 2.1 (Froebenius Identities). *Assuming that the Newton–Padé table is normal, we have for $m, n \geq 0$*

$$\begin{pmatrix} p^*_{m,n+1} \\ q^*_{m,n+1} \end{pmatrix} = (z - z_{m+n+1}) \cdot \begin{pmatrix} p^*_{m,n} \\ q^*_{m,n} \end{pmatrix} - \frac{a_{m,n}}{a_{m+1,n}} \cdot \begin{pmatrix} p^*_{m+1,n} \\ q^*_{m+1,n} \end{pmatrix}, \tag{2.9}$$

$$\begin{pmatrix} p^*_{m,n+1} \\ q^*_{m,n+1} \end{pmatrix} = \begin{pmatrix} p^*_{m+1,n+1} \\ q^*_{m+1,n+1} \end{pmatrix} - \frac{a_m + 1, n + 1}{a_{m+1,n}} \cdot \begin{pmatrix} p^*_{m+1,n} \\ q^*_{m+1,n} \end{pmatrix}, \tag{2.10}$$

$$\begin{aligned} \begin{pmatrix} p^*_{m+1,n+1} \\ q^*_{m+1,n+1} \end{pmatrix} &= \frac{a_{m+1,n+1}}{a_{m,n}} \cdot (z - z_{m+n+1}) \cdot \begin{pmatrix} p^*_{m,n} \\ q^*_{m,n} \end{pmatrix} \\ &\quad + \left(1 - \frac{a_{m+1,n+1}}{a_{m,n}}\right) \cdot \begin{pmatrix} p^*_{m,n+1} \\ q^*_{m,n+1} \end{pmatrix}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} \begin{pmatrix} p^*_{m+1,n+1} \\ q^*_{m+1,n+1} \end{pmatrix} &= (z - z_{m+n+1}) \cdot \begin{pmatrix} p^*_{m,n} \\ q^*_{m,n} \end{pmatrix} \\ &\quad + \left(\frac{a_{m+1,n+1}}{a_{m+1,n}} - \frac{a_{m,n}}{a_{m+1,n}}\right) \cdot \begin{pmatrix} p^*_{m+1,n} \\ q^*_{m+1,n} \end{pmatrix}. \end{aligned} \tag{2.12}$$

Proof. As an example, we give the proof of (2.9). Let

$$\begin{pmatrix} p^* \\ q^* \end{pmatrix} = \begin{pmatrix} p^*_{m,n+1} \\ q^*_{m,n+1} \end{pmatrix} - (z - z_{m+n+1}) \cdot \begin{pmatrix} p^*_{m,n} \\ q^*_{m,n} \end{pmatrix},$$

then $\deg q^* \leq n$ (since the denominators are monic) and $\deg p^* = m + 1$. Moreover,

$$p^* - f \cdot q^* = (p^*_{m,n+1} - f \cdot q^*_{m,n+1}) - (z - z_{m+n+1})(p^*_{m,n} - f \cdot q^*_{m,n}),$$

and therefore $(p^* - f \cdot q^*)$ has the zeros $z_0, \ldots, z_{m+n+1}$. Consequently, (p^*, q^*) solves the linear rational approximation problem and, by definition of the minimal polynomials, is a scalar multiple of $(p^*_{m+1,n}, q^*_{m+1,n})$. The factor $-a_{m,n}/a_{m+1,n}$ is found by comparing the leading coefficients of p^* and $p^*_{m+1,n}$. ∎

A different way to prove Theorem 2.1 is based on so-called *Padé determinants*, which are also useful for nonnormal tables (see Section 7).

3. Continued Fractions in the Normal Newton–Padé Approximation Table

From Theorem 2.1 we immediately obtain the following interpolating continued fractions. In Theorem 3.1(a) we use—in contrast to [7], [17], [21], and [24]—the same scaling as for the Thiele continued fraction whereas the scaling in (b) coincides with that in [8] and [16].

Theorem 3.1. *Assume that the Newton–Padé table is normal. Then the following holds.*

(a) *For fixed $k \geq 0$, the $(2j)$ and $(2j+1)$th convergent of the continued fraction T_k^{desc}*

$$
\begin{aligned}
f(z) = {} & \sum_{j=0}^{k} a_{j,0} \cdot \omega_{0,j}(z) \\
& + \frac{\omega_{0,k+1}(z)\,\big|}{\big|\,1/a_{k+1,0}} + \frac{z - z_{k+1}\,\big|}{\big|\,a_{k+1,1} - a_{k,0}} + \frac{z - z_{k+2}\,\big|}{\big|\,1/a_{k+2,1} - 1/a_{k+1,0}} \\
& + \sum_{j=2}^{\infty} + \frac{z - z_{k+2j-1}\,\big|}{\big|\,a_{k+j,j} - a_{k+j-1,j-1}} + \frac{z - z_{k+2j}\,\big|}{\big|\,1/a_{k+j+1,j} - 1/a_{k+j,j-1}}
\end{aligned}
$$

is equal to $r_{k+j,j}$ and $r_{k+j+1,j}$, respectively, $j = 0, 1, 2, \ldots$.

(b) *For fixed $k \geq 1$, the $(2j)$ and $(2j+1)$th convergent of the continued fraction T_k^{asc}*

$$
\begin{aligned}
f(z) = {} & \sum_{j=0}^{k} a_{j,0} \cdot \omega_{0,j}(z) - \frac{a_{k,0} \cdot \omega_{0,k}(z)\,\big|}{\big|\,1} - \frac{a_{k-1,0}/a_{k,0}\,\big|}{\big|\,z - z_k} - \frac{a_{k-1,1}/a_{k-1,0}\,\big|}{\big|\,1} \\
& + \sum_{j=2}^{k} \left(- \frac{a_{k-j,j-1}/a_{k-j+1,j-1}\,\big|}{\big|\,z - z_k} - \frac{a_{k-j,j}/a_{k-j,j-1}\,\big|}{\big|\,1} \right) \\
& - \frac{a_{0,k-1}/a_{1,k-1}\,\big|}{\big|\,z - z_k}
\end{aligned}
$$

is equal to $r_{k-j,j}$ and $r_{k-j-1,j}$, respectively, $j = 0, 1, 2, \ldots$.

Proof. To show (a) one can easily verify that (1.2) and the corresponding initial conditions yield the partial numerators and denominators

$$
\begin{pmatrix} p_{k+2j} \\ q_{k+2j} \end{pmatrix} := \begin{pmatrix} p^*_{k+j,j} \\ q^*_{k+j,j} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p_{k+2j+1} \\ q_{k+2j+1} \end{pmatrix} := \frac{1}{a_k + j + 1,\, j} \begin{pmatrix} p^*_{k+j+1,j} \\ q^*_{k+j+1,j} \end{pmatrix}.
$$

Here, the initialization is obvious and the three-term recurrence relations follow from (2.11) and (2.12).

Similarly, in (b) the partial numerators and denominators

$$
\begin{pmatrix} p_{k+2j} \\ q_{k+2j} \end{pmatrix} := \begin{pmatrix} p^*_{k-j,j} \\ q^*_{k-j,j} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p_{k+2j+1} \\ q_{k+2j+1} \end{pmatrix} := \begin{pmatrix} p^*_{k-j-1,j} \\ q^*_{k-j-1,j} \end{pmatrix}
$$

are determined using (2.9) and (2.10). ∎

Remark 1. The connections between the coefficients $b_{m,n} = a_{m,n}$ and the *reciprocal differences* are pointed out in [13, p. 144].

Remark 2. The polynomial identities of Theorem 2.1 are also valid in the upper half of the Newton–Padé table; therefore, descending continued fractions can also be given for values $k \leq -1$. This follows already from a more general duality principle: $1/r_{m,n}$ is the (n, m)-rational interpolant of $1/f$ with leading coefficent $1/a_{m,n}$ and asymptotic coefficient $1/b_{m,n}$.

4. The Normal b-Table

Claessens' ε-algorithm [6] allows us to compute the values of the normal Newton–Padé table with the cross-rule

$$\begin{aligned}(4.13)\qquad &\frac{1}{z - z_{m+n}}\left\{\frac{1}{r_{m,n-1}(z) - r_{m,n}(z)} - \frac{1}{r_{m-1,n}(z) - r_{m,n}(z)}\right\}\\ &= \frac{1}{z - z_{m+n+1}}\left\{\frac{1}{r_{m+1,n}(z) - r_{m,n}(z)} - \frac{1}{r_{m,n+1}(z) - r_{m,n}(z)}\right\}\end{aligned}$$

which yields a recursion for $r_{m,n+1}$. Supposing the normal case, we can use the cross-rule to compute the coefficients $a_{m,n} = b_{m,n}$ as described below. For initialization only the coefficients of the (formal) Newton series (2.7) are required. Note that (4.13) as well as the QD-type algorithm of Theorem 4.1 are invariant with respect to the duality principle of Remark 2.

Theorem 4.1. *Let $a_{m,n}$ be the leading coefficient of $p^*_{m,n}$, and let, in addition, $a_{-1,j} := 0$, $a_{j,-1} := \infty$ for $j = 0, 1, \dots$. Then*

$$(4.14)\qquad \frac{a_{m,n}}{a_{m,n-1}} + \frac{a_{m-1,n}}{a_{m,n}} = \frac{a_{m,n}}{a_{m+1,n}} + \frac{a_{m,n+1}}{a_{m,n}} + z_{m+n+1} - z_{m+n}.$$

Proof. The initialization is the same as for (4.13) in [6]. In order to prove (4.14) multiply (4.13) with $(z - z_{m+n}) \cdot r_{m,n}$, consider the asymptotic series of the negative powers of z for $z \to \infty$, and compare the coefficient of z^{-1}. First, note that

$$\begin{aligned}\frac{r_{m,n}(z)}{r_{m,n-1}(z) - r_{m,n}(z)} &= z^{-1}\frac{a_{m,n}}{a_{m,n-1}} + o(z^{-1}),\\ \frac{r_{m,n}(z)}{r_{m+1,n}(z) - r_{m,n}(z)} &= z^{-1}\frac{a_{m,n}}{a_{m+1,n}} + o(z^{-1}).\end{aligned}$$

Similarly, we obtain the asymptotics

$$\begin{aligned}\frac{r_{m,n}(z)}{r_{m,n+1}(z) - r_{m,n}(z)} &= -1 + \frac{r_{m,n+1}(z)}{r_{m,n+1}(z) - r_{m,n}(z)} = -1 - z^{-1}\frac{a_{m,n+1}}{a_{m,n}} + o(z^{-1}),\\ \frac{r_{m,n}(z)}{r_{m-1,n}(z) - r_{m,n}(z)} &= -1 + \frac{r_{m-1,n}(z)}{r_{m-1,n}(z) - r_{m,n}(z)} = -1 - z^{-1}\frac{a_{m-1,n}}{a_{m,n}} + o(z^{-1}),\\ \frac{z - z_{m+n}}{z - z_{m+n+1}} &= 1 + z^{-1}(z_{m+n+1} - z_{m+n}) + o(z^{-1}).\end{aligned}$$

Consequently, (4.13) implies (4.14). ∎

Table 1. Singular block in the Newton–Padé table.

$\tau_{\mu,\nu}$	$\nu=-1$	0	1	2	3	4	5
$\mu=-1$		∞	∞	∞	∞	∞	∞
0	0		NE_0	NE_1	NE_2	NE_3	
1	0	SW_0	C	C	C	NE_4	
2	0	SW_1	C	C	C	NE_5	NE_6
3	0	SW_2	C	C	C	C	NE_7
4	0	SW_3	SW_4	SW_5	C	C	NE_8
5	0			SW_6	SW_7	SW_8	

5. The Nonnormal Newton–Padé Approximation Table

In the previous section we considered the normal Newton–Padé approximation table where "neighbors" are pairwise distinct. In this section we drop this assumption and assume that some neighbors are equal to $r_{m,n} =: C$. Then the set of coordinates (μ, ν) with $r_{\mu,\nu} = r_{m,n} = C$ is called a *singular block*. Claessens described the structure of the singular block with starting point (m, n) as a union of squares forming a symmetric tail along the diagonal through (m, n). This is illustrated in Table 1 with $(m, n) = (1, 1)$.

This block property, proved, e.g., in [5] and [18], can be formulated as follows. A starting point (m, n), $1 \le m, n$, is characterized by $r_{m,n} =: C$ and $r_{m-1,n} \neq C \neq r_{m,n-1}$. Then all interpolants lying in a square with upper left corner $(m + l, n + l)$ and lower right corner $(m + l + k - 1, n + l + k - 1)$ are equal to C, i.e.,

$$C = r_{m+l+\kappa,n+l+\lambda} \qquad (\kappa, \lambda = 0, 1, \dots, k - 1), \tag{5.15}$$

provided that $r_{m+k+l-1,n+l} = C$ or $r_{m+l,n+l+k-1} = C$ for any integers l, k.

In order to choose notation along the antidiagonals $m + n = j - 1$, $j \ge 0$, let k_j describe the number of occurrences of C on this antidiagonal. Then $k_0 = 0$ and there exists a positive integer p (or $p = \infty$ if the block is infinite) with $k_1, \dots, k_{2p-1} > 0$, $k_{2p} = 0$. Note that the upper left corner of the square (5.15) (induced by the equal interpolants on the antidiagonal $m + n + j - 1$) must take the form $(m + l_j, n + l_j)$ with an integer $l_j \ge 0$. Consequently, on the antidiagonal $m + n + j - 1$ we have

$$\begin{aligned} SW_j &:= r_{m+l_j+k_j,n+l_j-1} \neq C \\ &= r_{m+l_j+k_j-1,n+l_j} = r_{m+l_j+k_j-2,n+l_j+1} = \cdots = r_{m+l_j,n+l_j+k_j-1} \\ &= C \neq r_{m+l_j-1,n+l_j+k_j} =: NE_j, \end{aligned} \tag{5.16}$$

compare Table 1. The notation is as in [2] and [3], and we remark that indeed $j = k_j + 2l_j$, so that one of the variables j, k_j, l_j could be neglected.

The characteristic numbers l_j, k_j determining the shape of the singular block (see Figure 1) can be described with the help of interpolating properties of $r_{m,n}$: Let A (resp. U) denote the set of indices $j \in \{0, \dots, 2p - 1\}$ such that $r_{m,n}$ interpolates (does not interpolate) the function f at knot z_{m+n+j}. Then $l_j + k_j$ (resp. l_j) equals the number of elements smaller than j in A (resp. U), which, in fact, coincides with the number of attainable (resp. unattainable) knots of $r_{m+l_j+k_j-1,n+l_j}$ in $(z_{m+n}, \dots, z_{m+n+j-1})$, counting multiplicities. This follows from the well-known representation [5] and [18] for the

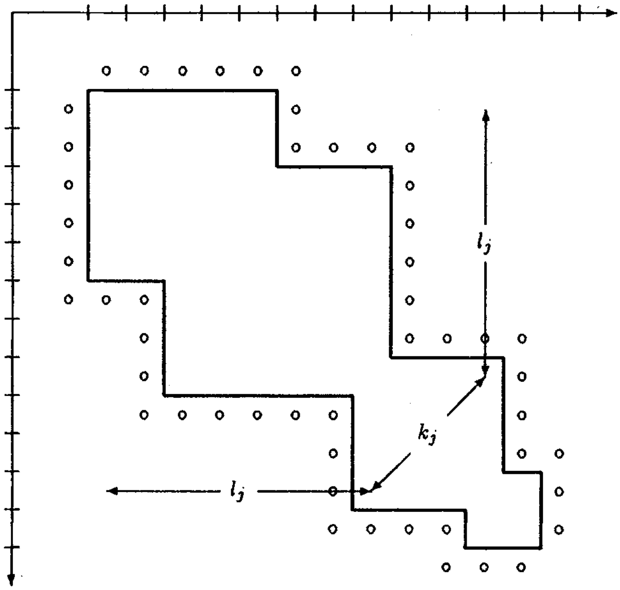

Fig. 1. A singular block with characteristic numbers (the circles characterize neighbors).

corresponding minimal solution of the linearized problem

$$\begin{pmatrix} p^*_{m+l_j+k_j-1-i,n+l_j+i} \\ q^*_{m+l_j+k_j-1-i,n+l_j+i} \end{pmatrix} = \prod_{\nu<j,\nu\in U} (z - z_{m+n+\nu}) \cdot \begin{pmatrix} p^*_{m,n} \\ q^*_{m,n} \end{pmatrix}, \tag{5.17}$$

for $0 < j < 2p$, $0 \le i < k_j$.

As proved in [2] and [3] the neighbors of a singular block are connected by cross-rule type identities. Let α_j be a meromorphic function such that

$$\frac{1}{SW_j - C} - \frac{1}{NE_j - C} = \alpha_j \cdot \frac{\prod_{i\in U,i<j}(z - z_{m+n+i})}{\prod_{i\in A,i<j}(z - z_{m+n+i})} \cdot \frac{{q^*_{m,n}}^2}{a_{m,n} \cdot \omega_{0,m+n}}. \tag{5.18}$$

Note that, by (5.18), any recurrence relation for α_j leads to an identity in the nonnormal Newton–Padé approximation table and vice versa.

Theorem 5.1 (See [2] and [3]). *$\alpha_0, \alpha_1, \alpha_2, \ldots$ are monic polynomials of degree $k_0, k_1, k_2, \ldots$ called Kronecker polynomials. There holds $\alpha_0 = 1$ and for any $j \ge 0$*

$$\begin{aligned} \alpha_{j+1}(z) &= \alpha_j(z) \cdot (z - z_{m+n+j}) - c_j \qquad &\text{if } j \in A, \\ \alpha_j(z) &= \alpha_{j+1}(z) \cdot (z - z_{m+n+j}) - c_{j+1} \qquad &\text{if } j \in U, \end{aligned}$$

where

$$c_j = \frac{b_{m,n}}{b_{m+l_j+k_j,n+l_j-1}} + \frac{b_{m+l_j-1,n+l_j+k_j}}{b_{m,n}} \in \mathbf{C}. \tag{5.19}$$

Remark 3. If we have a trivial block containing only (m, n), i.e., if $r_{m,n}$ is different from $r_{m+1,n}, r_{m,n+1}, r_{m-1,n}$, and $r_{m,n-1}$, then $k_0 = k_2 = l_0 = l_1 = 0, k_1 = l_2 = 1, \alpha_0 = \alpha_2 = 1$, and from Theorem 5.1, after eliminating α_1, we obtain $c_2 = c_0 + z_{m+n} - z_{m+n+1}$ or

$$\frac{a_{m,n}}{b_{m,n-1}} + \frac{b_{m-1,n}}{a_{m,n}} = \frac{a_{m,n}}{b_{m+1,n}} + \frac{b_{m,n+1}}{a_{m,n}} + z_{m+n+1} - z_{m+n}, \tag{5.20}$$

which generalizes Theorem 4.1.

In the sequel, we say that the coordinate (μ, ν) lies at the northern border of a block induced by $r_{m,n}$ if $r_{\mu,\nu} = r_{m,n}$ but $r_{\mu-1,\nu} \neq r_{m,n}$; we will use the short notation $(\mu, \nu) = CN$. Similarly, we will write CW, CS, and CE for coordinates at the other borders and note that cases like $CN = CE$, etc., can occur. In addition, coordinates in the neighborhood of a singular block will be denoted by N, W, S, and E, e.g., $E = (\mu, \nu)$ is an eastern neighbor if $r_{\mu,\nu} \neq r_{m,n}$ but $r_{\mu,\nu-1} = r_{m,n}$. Finally, for two coordinates $C_1 = (\mu_1, \nu_1)$, $C_2 = (\mu_2, \nu_2)$ lying in the common singular block induced by $r_{m,n}$, $\delta_1 := \mu_1 + \nu_1 - m - n \leq \delta_2 := \mu_2 + \nu_2 - m - n$, we will use the short notation

$$\omega^U_{C_1,C_2}(z) = \prod_{\delta_1 < i \leq \delta_2, i \in U} (z - z_{m+n+i}), \qquad \omega^A_{C_1,C_2}(z) = \prod_{\delta_1 \leq i \leq \delta_2, i \in A} (z - z_{m+n+i}),$$

such that, for instance,

$$p^*_{C_2} = \omega^U_{C_1,C_2} \cdot p^*_{C_1}, \qquad q^*_{C_2} = \omega^U_{C_1,C_2} \cdot q^*_{C_1}, \tag{5.21}$$

and if

$$\delta_1 \in A\colon \omega^U_{C_1,C_2} \cdot \omega^A_{C_1,C_2} = \omega_{\mu_1+\nu_1,\mu_2+\nu_2+1}. \tag{5.22}$$

Remark 4. If (m, n) is a starting point of a singular block, i.e., $r_{m-1} \neq r_{m,n} \neq r_{m,n-1}$, then necessarily $\deg p^*_{m,n} = m$ and $\deg q^*_{m,n} = n$ holds (see [3, Theorem 1] or Lemma 7.2 below). From (5.17) we can conclude that the degrees of the numerators $p^*_{\mu,\nu}$ all have maximal degree μ for $(\mu, \nu) = CN, CE, W, S$, and the degrees of the denominators $q^*_{\mu,\nu}$ all have maximal degree ν for $(\mu, \nu) = CW, CS, N$, and E.

6. The Nonnormal b-Table

Of course, the asymptotic coefficient $b_{\mu,\nu}$ introduced in Section 2 is also well defined in the nonnormal case, but now it might take one of the values $0, \infty$ different from the leading coefficient $a_{\mu,\nu}$. A singular block in the table of Newton–Padé approximants $(r_{\mu,\nu})$ with starting point (m, n) canonically leads to a related "singular block" in the b-table with the same coordinates. From $\deg p^*_{m,n} = m$ and $\deg q^*_{m,n} = n$ (see Remark 4) and (5.17) we can conclude that the nonvanishing complex number $b_{m,n} = a_{m,n}$ can be found on the diagonal of the block through (m, n) while all other entries corresponding to the singular block are zero or infinity. The b-table related to Table 1 is given in Table 2.

Note that due to Remark 4 the coefficients $b_{\mu,\nu}$ are different from zero for $(\mu, \nu) = CN, CE, W$, and S and different from infinity for $(\mu, \nu) = CW, CS, N$, and E. Therefore we are able to determine singular blocks uniquely if the b-table is (partly) known. Also, the nonnormal a-table is determined by the b-table: The leading coefficient $a_{\mu,\nu}$ with (μ, ν) lying in a singular block with starting point (m, n) is given by $a_{\mu,\nu} = b_{m,n}$.

Table 2. Singular block in the b-table.

$b_{\mu,\nu}$	$\nu = -1$	0	1	2	3	4	5
$\mu = -1$		∞	∞	∞	∞	∞	∞
0	0	$*$	$\neq\infty$	$\neq\infty$	$\neq\infty$	$\neq\infty$	
1	0	$\neq 0$	$a_{1,1}$	∞	∞	$\neq\infty$	
2	0	$\neq 0$	0	$a_{1,1}$	∞	$\neq\infty$	$\neq\infty$
3	0	$\neq 0$	0	0	$a_{1,1}$	∞	$\neq\infty$
4	0	$\neq 0$	$\neq 0$	$\neq 0$	0	$a_{1,1}$	$\neq\infty$
5	0			$\neq 0$	$\neq 0$	$\neq 0$	

Theorem 6.1. *Using the initializations and recurrence relations of Theorem* 4.1 *and* 5.1, *the nonnormal b-table as well as the corresponding Kronecker polynomials can be computed without any a priori information about singular blocks.*

Proof. By applying (5.20) we are able to detect a singular block in the b-table and to determine its shape and all western, southern, and some of its northern neighbors. The process is similar to the strategy described in [3] for computing the values in the nonnormal Newton–Padé approximation table along antidiagonals via the reliable modification of Claessens' cross-rule. Using Theorem 5.1, we obtain the first Kronecker polynomials α_j for $j \in A$. Again Theorem 5.1 yields the missing polynomials α_j (for $j \in U$) together with the eastern neighbors b_E of the singular block considered. For details we refer to [3]. ■

Instead of giving a complete algorithmic description, let us illustrate the preceding theorem by the following example where the singular block of Tables 1 and 2 is discussed. The entries of the nonnormal b-table and the Kronecker polynomials will be required for the modification of the continued fractions of Theorem 3.1 in Sections 8 and 9.

Example 6.1. Taking the coordinates as described in Table 1, we suppose that all western and southern neighbors b_{SW_j}, $j = 0, 1, 2, 4, 5, 7, 8$ and the northern neighbors b_{NE_j} for $j = 0, 1, 2$ have already been computed by Theorem 4.1 or by a singular rule with respect to another singular block. Hence the shape of the singular block (with $A = \{0, 1, 2, 5\}$ and $U = \{3, 4, 6, 7\}$) as well as the quantities c_0, c_1, c_2 of (5.19) are known. By Theorem 5.1 we may initialize $\alpha_0 = 1$ and compute $\alpha_1, \alpha_2, \alpha_3$. Moreover, α_4, α_5 are also completely determined since $3, 4 \in U$. This gives c_4, c_5 and by (5.19) we obtain b_{NE_j} for $j = 4, 5$. Theorem 5.1 enables us again to compute $\alpha_6, \alpha_7, \alpha_8 = 1$ and c_7, c_8 leading to b_{NE_j} for $j = 7, 8$.

The missing quantities b_{NE_j} for $j = 3, 6$ can be obtained by applying Theorem 4.1 or (for $j = 6$) by recognizing that $b_{NE_5} = 0$ implies that b_{NE_4} and b_{NE_5} induce a new singular block and hence $b_{NE_6} = b_{NE_4}$.

7. Tools

The following two simple lemmas are useful tools. $U, V, W \in \mathbf{N}_0^2$ denote coordinates in the Newton–Padé table.

Lemma 7.1 (See [9]). *We have*

$$\det[V, W] \cdot \begin{pmatrix} p_U^* \\ q_U^* \end{pmatrix} = \det[U, W] \cdot \begin{pmatrix} p_V^* \\ q_V^* \end{pmatrix} - \det[U, V] \cdot \begin{pmatrix} p_W^* \\ q_W^* \end{pmatrix} \tag{7.23}$$

with

$$\det[U, V] := \det \begin{pmatrix} p_U^* & p_V^* \\ q_U^* & q_V^* \end{pmatrix}.$$

Proof. Expand

$$0 = \det \begin{pmatrix} p_U^* & p_V^* & p_W^* \\ q_U^* & q_V^* & q_W^* \\ p_U^* & p_V^* & p_W^* \end{pmatrix} = \det \begin{pmatrix} p_U^* & p_V^* & p_W^* \\ q_U^* & q_V^* & q_W^* \\ q_U^* & q_V^* & q_W^* \end{pmatrix}$$

with respect to the last row. ∎

Lemma 7.2 (See [9] and [13]). *Let $m, n, \mu, \nu \in \mathbf{N}_0$ with $m + n \leq \mu + \nu$. Then there exists a unique polynomial α of degree less than or equal to* $\max\{\mu - m - 1, \nu - n - 1\}$ *with*

$$\det[(m, n), (\mu, \nu)] = \det \begin{pmatrix} p_{m,n}^* & p_{\mu,\nu}^* \\ q_{m,n}^* & q_{\mu,\nu}^* \end{pmatrix} = \alpha \cdot \omega_{0,m+n+1}.$$

Moreover, if $\deg p_{m,n}^* + \deg q_{\mu,\nu}^* - \deg p_{\mu,\nu}^* - \deg q_{m,n}^* > 0$ *(resp. < 0), then the leading coefficient of α is equal to $a_{m,n}$ (resp. equal to $a_{\mu,\nu}$).*

Proof. The polynomial

$$p_{m,n}^* q_{\mu,\nu}^* - p_{\mu,\nu}^* q_{m,n}^* = q_{\mu,\nu}^* (p_{m,n}^* - f q_{m,n}^*) - p_{m,n}^* (p_{\mu,\nu}^* - f q_{\mu,\nu}^*)$$

has at least the zeros $z_0, \ldots, z_{m+n}$, counting multiplicities (due to the interpolation conditions of the linearized problem), and is of degree $\leq \max\{m + \nu, n + \mu\}$. This yields the lemma. ∎

8. Reliable Modification of the Ascending Continued Fraction

In this section we consider the ascending continued fraction T_k^{asc} of Theorem 3.1(b) [8], [16] which determines $(r_{k,0}, r_{k-1,0}, r_{k-1,1}, r_{k-2,1}, \ldots, r_{0,k})$ for any integer $k > 0$.

Let (5.16) hold for some integers $m, n, k_j, l_j \geq 0$ and suppose that both SW_j and NE_j lie on the ascending staircase T_k^{asc} and that the successor of SW_j in T_k^{asc} is equal to C. Note that this situation covers a singular block (where we want to skip from C to NE_j) as well as a normal case ($k_j = 0, l_j = 0, 1$) in the antidiagonal $(m + n + j - 1)$ $(j := k_j + 2l_j)$.

Table 3. Notation in Theorem 8.1.

		N		CE	E
	$\cdot$	CN		$\cdot$	$\cdot$
$\cdot$	$\cdot$		CS	$\cdot$	
W	CW		S		
	(a)			(b)	

We have to distinguish between the two cases $(SW_j, NE_j) = (r_W, r_N)$ and $(SW_j, NE_j) = (r_S, r_E)$ as illustrated in Table 3. In case (a), the singular block becomes wider while going from antidiagonal $(m+n+j-1)$ to $(m+n+j)$, and hence $j \in A$, whereas in (b) it becomes narrower while going from antidiagonal $(m+n+j-2)$ to $(m+n+j-1)$ $(j-1 \in U)$.

Theorem 8.1. *Suppose that* (5.16) *holds.*

(a) *Let* $r_{m+l_j+k_j,n+l_j} = C$ *and define* (*see Table* 3) $CW = (m+l_j+k_j, n+l_j)$, $CN = (m+l_j, n+l_j+k_j)$, $W = (m+l_j+k_j, n+l_j-1)$, $N = (m+l_j-1, n+l_j+k_j)$. *Then,*

$$\begin{pmatrix} p_N^* \\ q_N^* \end{pmatrix} = \alpha_{N,W} \cdot \begin{pmatrix} p_{CW}^* \\ q_{CW}^* \end{pmatrix} - \frac{a_{CW}}{a_W} \cdot \begin{pmatrix} p_W^* \\ q_W^* \end{pmatrix}$$

with $\alpha_{N,W}$ *being equal to the Kronecker polynomial* α_j *of* (5.18).

(b) *Let* $r_{m+l_j+k_j-1,n+l_j-1} = C$ *and define* (*see Table* 3) $CS = (m+l_j+k_j-1, n+l_j-1)$, $CE = (m+l_j-1, n+l_j+k_j-1)$, $S = (m+l_j+k_j, n+l_j-1)$, $E = (m+l_j-1, n+l_j+k_j)$. *Then,*

$$\begin{pmatrix} p_E^* \\ q_E^* \end{pmatrix} = \alpha_{S,E} \cdot (z - z_{m+n+j-1}) \cdot \begin{pmatrix} p_{CS}^* \\ q_{CS}^* \end{pmatrix} - \frac{a_{CS}}{a_S} \cdot \begin{pmatrix} p_S^* \\ q_S^* \end{pmatrix}.$$

with $\alpha_{S,E}$ *being equal to the Kronecker polynomial* α_j *of* (5.18).

Proof. We will only show the assertion of case (a) since a proof of (b) can be given using the same methods. From Remark 4 and Lemma 7.2 we know that

$$\det[W, CW] = a_W \cdot \omega_{0,m+n+j} \qquad \text{and} \qquad \det[CN, N] = a_{CW} \cdot \omega_{0,m+n+j}$$

and that we may define a (monic) polynomial $\alpha_{W,N}$ by

$$\det[W, N] = a_W \cdot \alpha_{W,N} \cdot \omega_{0,m+n+j}.$$

Moreover, by assumption we have $p_{CW}^* = p_{CN}^*$ and $q_{CW}^* = q_{CN}^*$ since the coordinates CW and CN lie on the same antidiagonal in the singular block. Hence the given identities for numerators and denominators follow from Lemma 7.1. It remains to prove $\alpha_{W,N} = \alpha_j$ where α_j satisfies (5.18). We get

$$\begin{aligned} \frac{1}{SW_j - C} - \frac{1}{NE_j - C} &= \frac{r_N - r_W}{(r_N - r_{CN}) \cdot (r_W - r_{CW})} \\ &= q_{CN}^* \cdot q_{CW}^* \cdot \frac{\det[W, N]}{\det[CN, N] \cdot \det[W, CW]} \\ &= \alpha_{W,N} \cdot \frac{q_{CN}^* \cdot q_{CW}^*}{a_{CW} \cdot \omega_{0,m+n+j}}. \end{aligned}$$

Consequently, comparing $q_{CN}^* = q_{CW}^*$ and $q_{m,n}^*$ with the help of (5.17) completes the proof of (a). ■

9. Reliable Modification of Thiele's Fraction

In this section we modify the Thiele-type fraction of Theorem 3.1(a) by deriving some relations for the minimal polynomials on the descending staircase. Similarly, as in the last section, we have to distinguish between two cases: We can enter a singular block at the western or the northern border. Note that if the last interpolant before the singular block has the local coordinates $SW_i = r_W$ and $NE_i = r_N$, then we always leave the singular block at interpolants $SW_j = r_S$ and $NE_j = r_E$, respectively, with $k_i = k_j$ and $l_i < l_j$ (and $i \in A$, $j - 1 \in U$), since they must lie on the same diagonal.

Theorem 9.1. *Let $m, n, i, j \geq 0$ be integers and let* (5.16) *hold both for j and for j replaced by i with $k_j = k_i$, $l_i < l_j$ ($i = 2l_i + k_i < j = 2l_j + k_j$).*

(a) *Define $W = (m + l_i + k_i, n + l_i - 1)$, $CW = (m + l_i + k_i, n + l_i)$, $CS = (m + l_j + k_j - 1, n + l_j - 1)$, $S = (m + l_j + k_j, n + l_j - 1)$ belonging all to the same descending staircase, as shown in Table* 4(a). *Assume $r_W \neq r_{CW} = C = r_{CS} \neq r_S$ so that CW and CS belong to the same singular block. Then there exists a unique polynomial $\beta_{W,S}$ of degree less than $l_j - l_i$, called the Werner polynomial, with*

$$\frac{1}{a_S}\begin{pmatrix} p_S^* \\ q_S^* \end{pmatrix} = \frac{\omega_{CW,CS}^A}{a_W}\begin{pmatrix} p_W^* \\ q_W^* \end{pmatrix} + \beta_{W,S} \cdot \begin{pmatrix} p_{CW}^* \\ q_{CW}^* \end{pmatrix}, \tag{9.24}$$

$$\frac{1}{r_S - C} - \frac{1}{r_W - C} = \beta_{W,S} \cdot \frac{q_{CW}^* \cdot q_{CS}^*}{\omega_{0,m+n+j-1}}. \tag{9.25}$$

(b) *Define $N = (m + l_i - 1, n + l_i + k_i)$, $CN = (m + l_i, n + l_i + k_i)$, $CE = (m + l_j - 1, n + l_j + k_j - 1)$, and $E = (m + l_j - 1, n + l_j + k_j)$ belonging all to the same descending staircase, as shown in Table* 4(b). *Assume $r_N \neq r_{CN} = C = r_{CE} \neq r_E$ so that CW and CS belong to the same singular block. Then there exists a unique polynomial $\beta_{N,E}$ of degree less than $l_j - l_i$, called the Werner polynomial, with*

$$\begin{pmatrix} p_E^* \\ q_E^* \end{pmatrix} = \omega_{CN,CE}^A \cdot \begin{pmatrix} p_N^* \\ q_N^* \end{pmatrix} + \frac{\beta_{N,E}}{a_{CN}} \cdot \begin{pmatrix} p_{CN}^* \\ q_{CN}^* \end{pmatrix}, \tag{9.26}$$

$$\frac{1}{r_N - C} - \frac{1}{r_E - C} = \beta_{N,E} \cdot \frac{q_{CN}^* \cdot q_{CE}^*}{a_{CN}^2 \cdot \omega_{0,m+n+j-1}}. \tag{9.27}$$

Table 4. Notation in Theorem 9.1.

<table>
<tr><td>W CW N
       $\cdot$ $\cdot$
            CS
            S
(a)</td><td>CN $\cdot$
       $\cdot$ $\cdot$
        CE E
(b)</td></tr>
</table>

Proof. As in the proof of Theorem 8.1 we get from Remark 4 and Lemma 7.2, together with (5.21), (5.22),

$$\begin{aligned}
\det[W, CW] &= a_W \cdot \omega_{0,m+n+i},\\
\det[S, CW] &= \det[S, CS] \cdot (\omega^U_{CW,CS})^{-1} = a_S \cdot \omega_{0,m+n+j-1} \cdot (\omega^U_{CW,CS})^{-1}\\
&= a_S \cdot \omega_{0,m+n+i} \cdot \omega^A_{CW,CS}
\end{aligned}$$

(note that $i \in A$), and we may define a polynomial $\beta_{W,N}$ of degree less than $l_j - l_i$ by

$$\det[W, S] = a_W \cdot a_S \cdot \beta_{W,S} \cdot \omega_{0,m+n+i}.$$

Then the identities for numerators and denominators follow from Lemma 7.1. In addition,

$$\frac{1}{r_S - C} - \frac{1}{r_W - C} = q^*_{CW} \cdot q^*_{CW} \cdot \frac{\det[W, S]}{\det[S, CW] \cdot \det[W, CW]},$$

which gives the rest of assertion (a). The proof of (b) is similar; we omit the details. ■

10. Recurrence Relations for Werner Polynomials

The following theorem gives recurrence relations for the polynomials β required in Theorem 9.1.

Theorem 10.1.

(a) *In addition to the definitions in case* (a) *of Theorem* 9.1 *let* $\overline{W} = (m + l_i + k_i + 1, n + l_i - 1)$, $\overline{CW} = (m + l_i + k_i + 1, n + l_i)$, $\overline{CS} = (m + l_j + k_j - 1, n + l_j - 2)$, *and* $\overline{S} = (m + l_j + k_j, n + l_j - 2)$ *such that, if* W, CW, CS, *and* S *belong to the staircase* T^{desc}_k, *then* $\overline{W}$, $\overline{CW}$, $\overline{CS}$, *and* $\overline{S}$ *belong to the descending staircase* T^{desc}_{k+1}; *see Table* 5(a).

Assume $r_{\overline{W}} \neq r_{\overline{CW}} = r_{CW} = r_{CS} = r_{\overline{CS}} \neq r_{\overline{S}}$ *so that* CW, $\overline{CW}$, $\overline{CS}$, *and* CS *belong to the same singular block. Then we have the Froebenius identities*

$$\begin{smallmatrix}\bullet & \bullet\\ \circ & \end{smallmatrix} \quad \begin{pmatrix} p^*_{\overline{W}} \\ q^*_{\overline{W}} \end{pmatrix} = \frac{a_{\overline{W}}}{a_W} \cdot (z - z_{m+n+i}) \cdot \begin{pmatrix} p^*_W \\ q^*_W \end{pmatrix} - \frac{a_{\overline{W}}}{b_W} \cdot \begin{pmatrix} p^*_{CW} \\ q^*_{CW} \end{pmatrix}, \tag{10.28}$$

$$\begin{smallmatrix} & \bullet\\ \circ & \bullet \end{smallmatrix} \quad \begin{pmatrix} p^*_{\overline{S}} \\ q^*_{\overline{S}} \end{pmatrix} = \frac{a_{\overline{S}}}{a_S} \cdot \begin{pmatrix} p^*_S \\ q^*_S \end{pmatrix} - \frac{a_{\overline{S}}}{b_S} \cdot \begin{pmatrix} p^*_{CS} \\ q^*_{CS} \end{pmatrix}, \tag{10.29}$$

and the recurrence relation

$$\beta_{W,S} = \beta_{\overline{W},\overline{S}} - \frac{1}{b_W}\omega^A_{\overline{CW},CS} + \frac{1}{b_S}\omega^U_{\overline{CW},CS}. \tag{10.30}$$

Table 5. Notation in Theorem 10.1.

W	CW			N	$\overline{N}$		
$\overline{W}$	$\overline{CW}$			CN	$\overline{CN}$		
		$\overline{CS}$	CS			$\overline{CE}$	$\overline{E}$
		$\overline{S}$	S			CE	E
(a)				(b)			

(b) *In addition to the definitions in case* (b) *of Theorem* 9.1 *let* $\overline{N} = (m + l_i - 1, n + l_i + k_i + 1)$, $\overline{CN} = (m + l_i, n + l_i + k_i + 1)$, $\overline{CE} = (m + l_j - 2, n + l_j + k_j - 1)$, *and* $\overline{E} = (m + l_j - 2, n + l_j + k_j)$ *such that, if* N, CN, CE, *and* E *belong to the staircase* T_k^{desc}, *then* $\overline{N}$, $\overline{CN}$, $\overline{CE}$, *and* $\overline{E}$ *belong to the descending staircase* T_{k-1}^{desc}; *see Table* 5(b).

Assume $r_{\overline{N}} \neq r_{\overline{CN}} = r_{CN} = r_{CE} = r_{\overline{CE}} \neq r_{\overline{E}}$ *so that* $\overline{CN}$, CN, CE, *and* $\overline{CE}$ *belong to the same singular block. Then we have the Froebenius identities*

$$\begin{array}{c}\bullet\ \circ\\ \bullet\end{array}\quad \begin{pmatrix} p_{\overline{N}}^* \\ q_{\overline{N}}^* \end{pmatrix} = (z - z_{m+n+i}) \cdot \begin{pmatrix} p_N^* \\ q_N^* \end{pmatrix} - \frac{b_N}{a_{CN}} \cdot \begin{pmatrix} p_{CN}^* \\ q_{CN}^* \end{pmatrix}, \tag{10.31}$$

$$\begin{array}{c}\ \ \circ\\ \bullet\ \bullet\end{array}\quad \begin{pmatrix} p_{\overline{E}}^* \\ q_{\overline{E}}^* \end{pmatrix} = \begin{pmatrix} p_E^* \\ q_E^* \end{pmatrix} - \frac{b_E}{a_{CE}} \cdot \begin{pmatrix} p_{CE}^* \\ q_{CE}^* \end{pmatrix}, \tag{10.32}$$

and the recurrence relation

$$\beta_{N,E} = \beta_{\overline{N},\overline{E}} - b_N \omega^A_{\overline{CN},CE} + b_E \omega^U_{\overline{CN},CE}. \tag{10.33}$$

Proof. As before, we only show (a) since (b) follows analogously or by a duality argument, see Remark 2. If $r_W = r_{\overline{W}}$ and consequently $b_W = \infty$, then identity (10.28) holds trivially. But from $r_W \neq r_{\overline{W}}$ we can conclude that all rational interpolants involved are distinct, and hence (10.28) follows from (2.9). Identity (10.29) can be proved using the same arguments. With the new notation $\overline{W}$, $\overline{S}$, $\overline{CW}$, and $\overline{CS}$, (9.24) becomes

$$\frac{p_{\overline{S}}^*}{a_{\overline{S}}} = \omega^A_{\overline{CW},\overline{CS}} \cdot \frac{p_{\overline{W}}^*}{a_{\overline{W}}} + \beta_{\overline{W},\overline{S}} \cdot p_{\overline{CW}}^*.$$

Replacing $p_{\overline{S}}^*$ and $p_{\overline{W}}^*$ with the help of (10.28) and (10.29) leads to

$$\begin{aligned} \frac{1}{a_S} \cdot p_S^* - \frac{1}{b_S} \cdot p_{CS}^* &= \omega^A_{\overline{CW},\overline{CS}} \cdot \left(\frac{1}{a_W} \cdot (z - z_{m+n+i}) \cdot p_W^* - \frac{1}{b_W} \cdot p_{CW}^* \right) \\ &\quad + \beta_{\overline{W},\overline{S}} \cdot p_{\overline{CW}}^*, \end{aligned}$$

which after rearranging and using (5.21) gives

$$\begin{aligned} \frac{p_S^*}{a_S} &= \omega^A_{\overline{CW},\overline{CS}} \cdot (z - z_{m+n+i}) \cdot \frac{p_W^*}{a_W} \\ &\quad + \left(\frac{1}{b_S} \cdot \omega^U_{CW,CS} - \omega^A_{\overline{CW},\overline{CS}} \cdot \frac{1}{b_W} + \beta_{\overline{W},\overline{S}} \right) \cdot p_{CW}^*. \end{aligned}$$

Comparing this equation with (5.15) leads to (5.18) since with (5.22) we have $\omega^A_{\overline{CW},\overline{CS}} \cdot (z - z_{m+n+i}) = \omega^A_{CW,CS}$ due to $i \in A$, $j - 1 \in U$, and $\omega^A_{\overline{CW},\overline{CS}} = \omega^A_{\overline{CW},CS}$ due to $j - 1 \in U$, and $\omega^U_{CW,CS} = \omega^U_{\overline{CW},CS}$ due to $i + 1 \in A$. ∎

Remark 5. Note that the Froebenius identities (10.28) and (10.29) of Theorem 10.1(a) hold as well if we have the trivial case $i = j - 2$ and with $\overline{S} = \overline{W}$, $CS = CW$, and

$\overline{CS}$, $\overline{CS}$ being undefined. Hence comparing (10.28) and (10.29) with (9.24) gives the initialization for constant β (since $l_j - l_i - 1 = 0$)

$$\beta_{WS} = \frac{1}{b_S} - \frac{1}{b_W} \quad \text{if} \quad CW = CS. \tag{10.34}$$

Similarly, from (10.31) and (10.32) we obtain

$$\beta_{NE} = b_E - b_N \quad \text{if} \quad CN = CE. \tag{10.35}$$

Hence Theorem 8.1 (for $k_j = 0$) and Theorem 9.1 (for $i = j - 2$) together imply Theorem 2.1 as a particular case.

Remark 6. Due to the block structure, the polynomials $\omega^A_{\overline{CW},CS} = \omega^A_{\overline{CN},CE}$ and $\omega^U_{\overline{CW},CS} = \omega^U_{\overline{CN},CE}$ occurring in (10.30) and (10.33) all have degree $l_j - l_i - 1$, whereas $\beta_{W,S}$, $\beta_{N,E}$ and $\beta_{\overline{W},\overline{S}}$, $\beta_{\overline{N},\overline{E}}$ have degrees bounded by $l_j - l_i - 1$ and $l_j - l_i - 2$, respectively. Hence the coefficients of $z^{l_j - l_i - 1}$ in $\beta_{W,S}$ and $\beta_{N,E}$ are $1/b_S - 1/b_W$ and $b_E - b_N$, respectively (as already shown for the trivial case $i = j - 2$ in Remark 5).

Remark 7. In the notation of Theorem 10.1, the following recurrence relations of Theorem 5.1 hold:

$$\begin{aligned} \alpha_{\overline{W},\overline{N}} &= \alpha_{W,N} \cdot (z - z_{m+n+i}) - \frac{a_{CW}}{b_W} - \frac{b_N}{a_{CW}}, \\ \alpha_{\overline{S},\overline{E}} &= \alpha_{S,W} \cdot (z - z_{m+n+j-1}) - \frac{a_{CW}}{b_S} - \frac{b_E}{a_{CW}} \end{aligned}$$

(also valid if $\overline{W} = \overline{S}$ and $\overline{N} = \overline{E}$). This can be shown following the ideas of the proof of Theorem 10.1.

Using the initializations of Remark 5 and the recurrence relations of Theorem 10.1 the required polynomials β in Theorem 9.1 can be calculated from the b-table. The computation may be organized along diagonals with increasing degrees as illustrated in the following example.

Example 10.1. Taking the coordinates as in Table 1, we can initialize the polynomials β_{SW_2,SW_4}, β_{SW_5,SW_7}, β_{NE_2,NE_4}, and β_{NE_5,NE_7} as in Remark 5. Afterwards, we may compute β_{SW_1,SW_5} by (10.30) and β_{NE_1,NE_5} by (10.33). Knowing β_{SW_1,SW_7} enables us to determine β_{SW_0,SW_8} by (10.30). We then require a formula connecting β_{SW_1,SW_7} with β_{SW_1,SW_5} and β_{SW_5,SW_7} (and similarly for the north-eastern coordinates) which will be derived in the next theorem. Note that such situations do not occur for the special case of Padé approximation since these singular blocks always have the shape of a square.

Theorem 10.2. *Let the assumptions of Theorem* 9.1 *be true for two pairs* (i_0, j_0) *and* (i_1, j_1) *with* $j_0 = i_1$.

(a) *As in Theorem* 9.1(*a*), *we obtain two tuples of coordinates* W_0, CW_0, CS_0, S_0 *and* W_1, CW_1, CS_1, S_1 *with* $S_0 = W_1$ *all belonging to the same descending staircase, see Table* 6. *Then there holds the recurrence relation*

$$\beta_{W_0,S_1} = \beta_{W_0,S_0} \cdot \omega^A_{CW_1,CS_1} + \beta_{W_1,S_1} \cdot \omega^U_{CW_0,CW_1}. \tag{10.36}$$

Table 6. Notation in Theorem 10.2.

$W_0 \quad CW_0$ $\cdot \quad \cdot$ $\cdot \quad CS_0$ $S_0 \quad CW_1$ $\cdot \quad \cdot$ $\cdot \quad CS_1$ S_1 Case (a), here $W_1 = S_0$	N_0 $CN_0 \quad \cdot$ $\cdot \quad \cdot$ $CE_0 \quad E_0$ $CN_1 \quad \cdot$ $\cdot \quad \cdot$ $CE_1 \quad E_1$ Case (b), here $N_1 = E_0$

(b) *As in Theorem* 9.1(*b*), *we obtain two tuples of coordinates* N_0, CN_0, CE_0, E_0 *and* N_1, CN_1, CE_1, E_1 *with* $E_0 = N_1$ *all belonging to the same descending staircase, see Table* 6. *Then there holds the recurrence relation*

$$\beta_{N_0,E_1} = \beta_{N_0,E_0} \cdot \omega^A_{CN_1,CE_1} + \beta_{N_1,E_1} \cdot \omega^U_{CN_0,CN_1}. \tag{10.37}$$

Proof. Applying Theorem 9.1 three times for (i, j) equal to (i_0, j_0), (i_1, j_1), and (i_0, j_1) gives the identities

$$\frac{p^*_{S_0}}{a_{S_0}} = \omega^A_{CW_0,CS_0} \cdot \frac{p^*_{W_0}}{a_{W_0}} + \beta_{W_0,S_0} \cdot p^*_{CW_0},$$
$$\frac{p^*_{S_1}}{a_{S_1}} = \omega^A_{CW_1,CS_1} \cdot \frac{p^*_{W_1}}{a_{W_1}} + \beta_{W_1,S_1} \cdot p^*_{CW_1},$$

and

$$\frac{p^*_{S_1}}{a_{S_1}} = \omega^A_{CW_0,CS_1} \cdot \frac{p^*_{W_0}}{a_{W_0}} + \beta_{W_0,S_1} \cdot p^*_{CW_0}.$$

Similar identities can be derived for the denominators q^*. Since $S_0 = W_1$ we obtain—after elimination of S_1 and S_0 with (5.21)—

$$\begin{aligned}\omega^A_{CW_0,CS_1} \cdot \frac{p^*_{W_0}}{a_{W_0}} &+ \beta_{W_0,S_1} \cdot p^*_{CW_0}\\ &= \omega^A_{CW_1,CS_1} \cdot \omega^A_{CW_0,CS_0} \cdot \frac{p^*_{W_0}}{a_{W_0}}\\ &\quad + \omega^A_{CW_1,CS_1} \cdot \beta_{W_0,S_0} \cdot p^*_{CW_0} + \beta_{W_1,S_1} \cdot \omega^U_{CW_0,CW_1} \cdot p^*_{CW_0}\end{aligned}$$

and again a similar relation for the denominators. Since $r_{CW_0} \neq r_{W_0}$, we may compare coefficients; this leads to (10.36). Similarly, one can show (10.37). ∎

11. Connections Between Kronecker and Werner Polynomials

In this section we fix (i, j) as in Theorem 9.1 and use the notation from Theorem 8.1(b) and from Theorem 8.1(a) with i replacing j (see Table 3). Then the neighbors N, W, E, and S form a particular *Cordellier rectangle*, which is illustrated in Figure 2. Relations between the polynomials α and β are given in the next theorem.

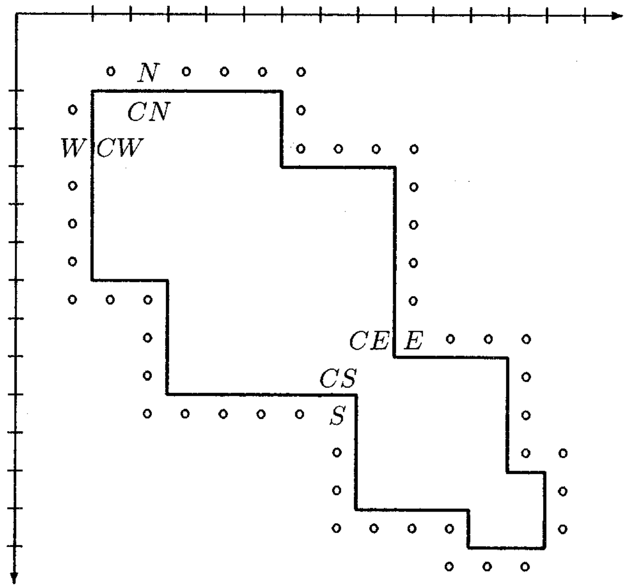

Fig. 2. The Cordellier rectangle.

Theorem 11.1. *With the above notation*

$$\alpha_{S,E} \cdot \omega^{U}_{CW,CS} \cdot (z - z_{m+n+j-1}) - \alpha_{W,N} \cdot \omega^{A}_{CW,CS} = \beta_{W,S} \cdot a_{CW} + \frac{\beta_{N,E}}{a_{CW}}.$$

Proof. From Theorem 8.1(a), (b) and the equation (5.18) we have

$$\begin{aligned} \frac{1}{r_W - C} - \frac{1}{r_N - C} &= \alpha_{W,N} \cdot \frac{q^*_{CN} \cdot q^*_{CW}}{a_{CW} \cdot \omega_{0,m+n+i}}, \\ \frac{1}{r_S - C} - \frac{1}{r_E - C} &= \alpha_{S,E} \cdot \frac{q^*_{CS} \cdot q^*_{CE} \cdot (z - z_{m+n+j-1})}{a_{CW} \cdot \omega_{0,m+n+j-1}}. \end{aligned}$$

Subtracting the first equation from the second gives with (9.25) and (9.27)

$$\begin{aligned} &\frac{1}{r_S - C} - \frac{1}{r_W - C} + \frac{1}{r_N - C} - \frac{1}{r_E - C} \\ &\quad = -\alpha_{W,N} \cdot \frac{q^*_{CN} \cdot q^*_{CW}}{a_{CW} \cdot \omega_{0,m+n+i}} + \alpha_{S,E} \cdot \frac{q^*_{CS} \cdot q^*_{CE} \cdot (z - z_{m+n+j-1})}{a_{CW} \cdot \omega_{0,m+n+j-1}} \\ &\quad = \beta_{W,S} \cdot \frac{q^*_{CW} \cdot q^*_{CS}}{\omega_{0,m+n+j-1}} + \beta_{N,E} \cdot \frac{q^*_{CN} \cdot q^*_{CE}}{a^2_{CN} \cdot \omega_{0,m+n+j-1}}. \end{aligned}$$

Since $\omega_{m+n+i,m+n+j-1} = \omega^{A}_{CW,CS} \cdot \omega^{U}_{CW,CS}$ and $q^*_{CN} = q^*_{CW}$, etc., the assertion of the theorem follows after some simplifications. ∎

Remark 8. Theorem 11.1 also covers the case $k_j = k_i = 0$, i.e., N, W and S, E are neighbors of the north-western and the south-eastern corner of the singular block, respectively. From Theorem 5.1 we know that in this case $\alpha_{W,N} = \alpha_{S,E} = 1$ and therefore Theorem 11.1 gives

$$\beta_{W,S} \cdot a_{CW} + \frac{\beta_{N,E}}{a_{CW}} = \prod_{\nu \in U}(z - z_{m+n+\nu}) - \prod_{\nu \in A}(z - z_{m+n+\nu}).$$

12. The Padé Case

Let us have a closer look at the Padé approximation problem which is included as the particular case of confluent knots ($z_0 = z_1 = \cdots = 0$).

It is well known that singular blocks in the Padé table always have the shape of a square, hence with the notation of the former section, $j \in \{p+1, \ldots, 2p\}$ and necessarily $i = 2p - j$, which implies $N = (m-1, n+2p-j)$, $W = (m+2p-j, n-1)$, $S = (m+p, n+j-p-1)$, and $E = (m+j-p-1, n+p)$. Cordellier [10] showed that

$$\frac{1}{r_S - C} - \frac{1}{r_W - C} + \frac{1}{r_N - C} - \frac{1}{r_E - C} = 0. \tag{12.38}$$

From (5.18) we may conclude that $\alpha_{W,N} = \alpha_{S,E}$ (confer [2] and [3]) and by Theorem 11.1 also $\beta_{W,S} \cdot a_{CW} = -\beta_{N,E}/a_{CW}$. In view of Remark 6, this yields the simple Cordellier-type formula for the nonnormal Padé case

$$\frac{a_{CW}}{b_W} + \frac{b_N}{a_{CW}} = \frac{a_{CW}}{b_S} + \frac{b_E}{a_{CW}} \tag{12.39}$$

generalizing (5.20). Obviously, by identity (12.39)—the "limiting analog" of (12.38)—we obtain a direct QD-type algorithm for computing the nonnormal b-table without necessarily computing any auxiliary polynomials or applying Theorem 5.1. Moreover, from Theorems 5.1 and 9.1 the following explicit formulas for the Kronecker and Werner polynomials are immediate:

$$\alpha_{W,N}(t) = \alpha_{S,E}(t) = t^{2p-j} + \sum_{i=0}^{2p-j-1} t^i \cdot \left(\frac{a_{m,n}}{b_{m+i,n-1}} + \frac{b_{m-1,n+i}}{a_{m,n}} \right),$$

$$\beta_{N,E}(t) = -\beta_{W,S}(t) \cdot (a_{m,n})^2 = \sum_{i=p+1}^{j} t^{i-p-1} \cdot (b_{m+i-p-1,n+p} - b_{m-1,n+2p-i}).$$

Consequently, due to the simplicity of the resulting QD-type algorithm, our scaling seems to be preferable to that used in the computational schemes proposed by [9] and [14].

13. Summary and Conclusion

The recurrence relations of Theorems 8.1 and 9.1 enable us now to write down explicitly the modified interpolating continued fraction T_k^{asc} and T_k^{desc} for the nonnormal Newton–Padé approximation problem.

Let $V_0, V_1, V_2, \ldots$ denote the coordinates belonging to T_k^{desc} (resp. to T_k^{asc}), enumerated in the natural order. We define the two subsequences $(X_j)_{j \geq 0}$ and $(Y_j)_{j \geq 1}$ of $(V_\ell)_{\ell \geq 0}$ by $X_0 = V_0$, $\ell(0) = 0$, and for $j \geq 1$

$$X_j = V_{\ell(j)}, \qquad Y_j = V_{\ell(j)-1}, \qquad \text{where} \quad r_{V_{\ell(j-1)}} = r_{V_{\ell(j-1)+1}} = \cdots = r_{V_{\ell(j)-1}} \neq r_{V_{\ell(j)}}.$$

Note that by construction all X_{2j} belong to the diagonal k and all X_{2j+1} to diagonal $k+1$ (antidiagonal k and $k-1$). For the sake of simplicity, we assume that there is no singular block in the first column of the Newton–Padé table so that $\ell(1) = 1$, $\ell(2) = 2$, i.e.,

$$X_0 = Y_1 = (k, 0), \qquad X_1 = Y_2 = (k+1, 0) \qquad \text{and} \qquad X_2 = (k+1, 1)$$

(resp., $X_0 = Y_1 = (k, 0)$, $X_1 = Y_2 = (k-1, 0)$, and $X_2 = (k-1, 1)$). Then the modified descending continued fraction T_k^{desc} reads

$$f(z) = \sum_{j=0}^{k} a_{j,0} \cdot \omega_{0,j}(z) + \frac{\omega_{0,k+1}(z)\,|}{|\, 1/a_{k+1,0}} + \sum_{j=2}^{\infty} \frac{\omega^A_{X_{j-1},Y_j}\,|}{|\, \beta_{Y_{j-2},X_j}} \tag{13.40}$$

and the modified ascending continued fraction T_k^{asc} reads

$$\begin{aligned} f(z) &= \sum_{j=0}^{k} a_{j,0} \cdot \omega_{0,j}(z) - \frac{a_{k,0} \cdot \omega_{0,k}(z)\,|}{|\, 1} \\ &\quad + \sum_{j=1}^{\cdots} \left(- \frac{a_{X_{2j-1}}/a_{X_{2j-2}}\,|}{|\,(z - z_k) \cdot \alpha_{Y_{2j-2},X_{2j}}} - \frac{a_{X_{2j}}/a_{X_{2j-1}}\,|}{|\, \alpha_{Y_{2j-1},X_{2j+1}}} \right). \end{aligned} \tag{13.41}$$

We conclude the paper with some computational remarks.

Remark 9. It is a well-known fact that a knot which once has been an unattainable point cannot later become attainable (i.e., $z_{m+n+i} = z_{m+n+j}$, $i \in U$, $i < j$ implies $j \notin A$) [5, Theorem 4]. Hence with the notation of Theorem 10.2, the polynomials $\omega^A_{CW_1,CS_1}$ and $\omega^U_{CW_0,CS_0}$ cannot have a common zero. Consequently, given β_{N_0,E_1}, we are able to compute both β_{N_0,E_0} and β_{N_1,E_1} by (10.37) since $\deg \beta_{N_0,E_0} \leq \deg \omega^U_{CN_0,CN_1} - 1 = l_{j_0} - l_{i_0} - 1$ and $\deg \beta_{N_1,E_1} \leq \deg \omega^A_{CN_1,CE_1} - 1 = l_{j_1} - l_{i_1} - 1$.

In addition, by (10.33) (see Remark 6) we can determine the quantities $\beta_{\overline{N},\overline{E}}$ and b_E supposing that $\beta_{N,E}$ and b_N are given.

Finally, if N and E are neighbors of a north-eastern corner of the singular block and b_N, $\beta_{N,E}$ are given, then we obtain b_E by the initialization (10.35).

Taking Remarks 8 and 9 into account we see that there are essentially two strategies to compute the polynomials β required for the modified descending interpolating continued fraction.

As proposed in the former sections, we could first compute the nonnormal b-table of asymptotic coefficients together with the polynomials $\alpha_{W,N}$ and $\alpha_{S,E}$, then compute the polynomials $\beta_{W,S}$ with the help of (10.34) and (10.30), and finally compute $\beta_{N,E}$ along diagonals with decreasing degrees by Theorem 11.1.

Alternatively, we can compute directly first the b_W, b_S, some of the b_N and the shape of the block by (5.20) (or a singular rule with respect to another block), afterward $\beta_{W,S}$ with the help of (10.34) and (10.30) along the diagonals with increasing degrees, and, finally, simultaneously the $\beta_{N,E}$, b_E, and the other b_N along diagonals with decreasing degrees using the equation of Remark 8 and the ideas given in Remark 9.

Let us finally raise the important question of numerical stability of the algorithms mentioned above. It is well known that the QD-algorithm and its generalizations are numerically unstable. Our method also has this disadvantage and hence should preferably be applied if exact arithmetic is available such that tests for zero and the detection of singular blocks are unambiguous. There are some approaches to overcome these numerical difficulties. For a class of rhombus-type algorithms in the sense of [11] including Claessens' ε-algorithm and therefore the cross-rule (4.13), Cordellier gave a reliable and stable implementation [12, Annexe 6]. It seems, however, that his methods do not extend to QD-type algorithms. Another approach was proposed by Gutknecht [19] and [20] for Padé approximation where, roughly speaking, one skips not only singular blocks but also blocks corresponding to "not well conditioned" interpolation problems. Here, the connection between two neighboring descending staircases is specified as a matrix recurrence relation. A stable and efficient QD-type algorithm for Newton–Padé approximation will be a subject of further research.

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