Computing 57, 301–322 (1996)

Computing

A Posteriori Error Estimate for the Symmetric Coupling of Finite Elements and Boundary Elements

C. Carstensen, Kiel

Dedicated to Professor Dr.-Ing. Wolfgang Wendland on the occasion of his 60th birthday

Received December 29, 1995; revised March 27, 1996

Abstract — Zusammenfassung

A Posteriori Error Estimate for the Symmetric Coupling of Finite Elements and Boundary Elements. In this note we study a posteriori error estimates for a model problem in the symmetric coupling of boundary element and finite elements methods. Emphasis is on the use of the Poincaré-Steklov operator and its discretization which are analyzed in general for both a priori and a posteriori error estimates. Combining arguments from [6] and [9, 10] we refine the a posteriori error estimate obtained in [9, 10]. For quasi-uniform meshes on the boundary, we prove some inequality of a reverse type using techniques from [5] and [36]. This indicates efficiency of the new estimate as illustrated in a numerical example.

AMS Subject Classifications: 65N38, 65N15, 65R20, 45L10.

Key words: Coupling of finite elements and boundary elements, a posteriori error estimates, adaptive mesh-refinement, efficiency.

A posteriori Fehlerabschätzung für die symmetrische Kopplung von Finiten Elementen und Randelementen. In dieser Arbeit werden a posteriori Fehlerabschätzungen für ein Modellproblem der symmetrischen Kopplung von Finiten Elementen und Randelementen untersucht. Dabei wird die Rolle des Poincaré-Steklov Operators und seiner Diskretisierung hervorgehoben, die für a priori und a posteriori Fehlerabschätzungen analysiert wird. Die a posteriori Fehlerabschätzungen aus [9, 10] werden verbessert mit Argumenten aus [6] und [9, 10]. Für quasiuniforme Randnetze können mit [5] und [36] Abschätzungen in der umgekehrten Richtung bewiesen werden. Dieses und numerische Beispiele zeigen die Effizienz der Abschätzung.

1. Introduction, Model Problem

In recent decades adaptive mesh refining seemingly became of high practical importance in numerical analysis of partial differential equations and integral equations. We mention only [1, 18, 19, 20, 25, 30, 31, 36, 38, 39]. The framework of adaptive methods, introduced by Eriksson and Johnson [18, 19] for finite elements, is studied in [5, 7–10] for boundary element methods and its coupling with finite element methods. Here we present a new a posteriori error estimate for the coupling of finite element and boundary element methods in the following model problem and analyze its efficiency.

In a bounded two-dimensional Lipschitz domain Ω with boundary $\Gamma = \partial\Omega$ and exterior domain $\Omega_c := \mathbb{R}^2 \setminus \overline{\Omega}$ we are given a possibly nonlinear mapping $A: L^2(\Omega)^2 \to L^2(\Omega)^2$ and a right hand side $f \in L^2(\Omega)$ and look for a function $u \in H^1_{loc}(\Omega \cup \Omega_c)$ and real constants a and b satisfying

$$-\operatorname{div} A(Du) = f \quad \text{in } \Omega \tag{1.1}$$

$$\Delta u = 0 \quad \text{in } \Omega_c \tag{1.2}$$

$$u(x) = a \cdot \log(x) + b + o(1) \quad \text{as} |x| \to \infty \tag{1.3}$$

$$u|_\Omega = u|_{\Omega_c} \quad \text{on } \Gamma \tag{1.4}$$

$$A(Du|_\Omega) \cdot n = Du|_{\Omega_c} \cdot n \quad \text{on } \Gamma. \tag{1.5}$$

Here, D denotes the gradient and Δ denotes the Laplacian; $o(1)$ is the Landau symbol with $\lim_{|x| \to \infty} o(1) = 0$; n is the unit normal of Γ pointing into Ω_c.

Remark 1.1. The model situation could be modified to other operators, e.g., to linear elasticity, or other dimensions (with other radiation conditions (1.3)). Moreover we might add Dirichlet, Neumann or mixed boundary conditions and, furthermore, also could analyze the case that $\Omega_c \subset \mathbb{R}^2 \setminus \overline{\Omega}$ is a (e.g., multiply connected) bounded domain. Finally, we could prescribe the jumps of displacements of tractions in (1.4) or (1.5) or add a right hand side in (1.2) leading to a modified right hand side below. But we restrict ourselves to the above assumptions for notational simplicity.

In the discretization, the exterior problem (1.2), (1.3) is rewritten using integral operators and then treated with a Galerkin scheme leading to the boundary elements on Γ. The interior problem (1.1) is considered in its standard weak form so that a Galerkin scheme on Ω can be performed with finite elements (cf. Section 2 for details). This 'mariage á la mode' was initiated by engineers. Its mathematical justification started in the later seventies with papers by Brezzi, Johnson, Nedelec, Bielak, MacCamy among others. Quasi-optimal a priori error estimates for the coupling of finite and boundary elements were then obtained for Lipschitz boundaries, systems of equations, and nonlinear problems (approximated by finite elements), e.g., in [13, 17, 22, 23, 24, 37] (see also the literature quoted therein); the symmetric coupling, under consideration here, was introduced mathematically by Costabel in [13].

To improve the convergence behavior, the *h*-, *p*- and *hp*-version of the coupling were developed [11, 25] still using a priori information. Recently, self-adaptive mesh refinement strategies and a posteriori error estimates have been established in [9, 10] for the same purpose.

In this paper we treat the model problem [9, 23] and refine the a posteriori error estimate in [9]. For this sharper estimate we prove efficiency in case that the mesh on the interface is quasi-uniform.

The paper is organized as follows. For convenience of the reader we sketch the weak form in Section 2 and recall its discretization in Section 3. In contrast to descriptions in [13, 17, 22–24, 37], we stress the role of the Poincaré-Steklov operator (see below). Since this approach is successful in nonlinear interface problems (see, e.g., [2,3] for variational inequalities and time dependent problems) we study general estimates for this operator and its discretization in some detail in Section 4. Thereby we reveal a simple proof for the a priori estimate found formerly in [9,17,22,23].

In Section 5 we use the refined a posteriori error estimate from [6] to improve the a posteriori estimate in [9]. In some sense, the estimate obtained here gives a more natural adaptive error indicator than that from [9]. In order to prove efficiency of this a posteriori error estimate we prove some reverse estimates in Section 6 combining arguments from [5] and [36]. A numerical example in Section 7 illustrates the results of this paper.

We use the following notations. $H^s(\Omega)$ denotes the usual Sobolev spaces [28] ($s \in \mathbb{R}$) with the trace spaces $H^{s-1/2}(\Gamma)$ ($s > 1/2$) for a bounded Lipschitz domain Ω with boundary Γ. $\|\cdot\|_{H^k(\omega)}$ denotes the norm $H^k(\omega)$ for $\omega \subseteq \Omega$ and an integer k. The duality $\langle \cdot,\cdot \rangle$ between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$ is given by extending the scalar product in $L^2(\Gamma)$; so $H^0(\Gamma) := L^2(\Gamma)$ and $H^s(\Gamma)$ is the dual of $H^{-s}(\Gamma)$ for $s < 0$.

2. Weak Form of the Model Problem

In order to rewrite the exterior problem, we need some boundary integral operators. Given $v \in H^{1/2}(\Gamma)$ and $\phi \in H^{-1/2}(\Gamma)$ we define for $z \in \Gamma$

$$(V\phi)(z) := -\frac{1}{2\pi}\int_\Gamma \phi(\zeta)\log|z-\zeta|\,ds_\zeta$$

$$(Kv)(z) := -\frac{1}{2\pi}\int_\Gamma v(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,ds_\zeta$$

$$(K'\phi)(z) := -\frac{1}{2\pi}\int_\Gamma \phi(\zeta)\frac{\partial}{\partial n_z}\log|z-\zeta|\,ds_\zeta$$

$$(Wv)(z) := \frac{1}{2\pi}\frac{\partial}{\partial n_\zeta}\int_\Gamma v(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,ds_\zeta.$$

This defines linear and bounded boundary integral operators when mapping between the following Sobolev-spaces [14], for $s \in [-1/2, 1/2]$,

$$V: H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma), \quad K: H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma),$$
$$K': H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma), \quad W: H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma).$$

The single layer potential V is symmetric, the double layer potential K has the dual K' and the hyper singular operator W is symmetric. V and W are strongly elliptic in the sense that they satisfy a Gårding inequality (in the above spaces with $s = 0$) [14]. Assuming that the capacity of Γ is smaller than one, the single layer potential V is positive definite, W is positive semi-definite and its kernal is $\mathbb{R}$. We refer, e.g., to [9, 14, 21, 32, 34, 35] for proofs and more details.

Remark 2.1. For a definition of the capacity of Γ, we refer, e.g., to [32] and mention here the sufficient condition that Ω lies in a ball with radius less than 1. Thus, this condition on Γ can always be achieved by scaling [21, 32] and hence may and will be assumed throughout this paper.

Since V is positive definite it is invertible and we may consider the Poincaré-Steklov operator $S: H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, defined as

$$S := W + (K' - 1/2)V^{-1}(K - 1/2),$$

which is linear, symmetric and positive definite [9]. S is a Dirichlet-Neumann map as shown in the following known lemma (see, e.g., [2, 15–17]). For uniqueness of solutions we may prescribe one of the constants a and b in (1.3) and we prescribe $a = 0$ in the sequel.

Remark 2.2. The condition $u(x)$ bounded as $|x| \to \infty$ leads to fix $b = 0$ which is equivalent to $\int_\Gamma Du|_\Gamma \cdot n ds = 0$ [15]. As a consequence we require that $\int_\Omega f\, dx = 0$. Therefore, this case $b = 0$ can be easily adopted from the case $a = 0$ analyzed here by replacing the space $H^{-1/2}(\Gamma)$ with $H_0^{-1/2}(\Gamma) := \{\phi \in H^{-1/2}(\Gamma) : \int_\Gamma \phi\, ds = 0\}$ (and modifying the discrete subspaces correspondingly).

Lemma 2.1. *Let $u \in H^1_{loc}(\Omega_c)$ satisfy* (1.2) *and* (1.3) *with $a = 0$, then*

$$Du|_\Gamma \cdot n = -Su|_\Gamma. \tag{2.1}$$

Conversely, for $w \in H^{1/2}(\Gamma)$ there exists a unique function $u \in H^1_{loc}(\Omega_c)$ satisfying (1.2), (1.3) (*with $a = 0$*) *and*

$$u|_\Gamma = w \quad \text{and} \quad (Du \cdot n)|_\Gamma = -Sw. \tag{2.2}$$

The function u is given by the representation formula, for $x \in \Omega_c$,

$$u(x) = -\frac{1}{2\pi}\int_\Gamma (Sw)(z) \cdot \log|x - z|\, ds_z - \frac{1}{2\pi}\int_\Gamma w(z) \cdot \frac{\partial}{\partial n_z} \log|x - z|\, ds_z. \quad \square \tag{2.3}$$

Using Lemma 2.1 (to replace the traction on the interface) and standard arguments one gains the weak form of the interface problem (1.1)–(1.5): Given $f \in L^2(\Omega)$ find $u \in H^1(\Omega)$ satisfying

$$\int_\Omega A(Du) \cdot D\eta\, dx + \langle Su|_\Gamma, \eta|_\Gamma \rangle = \int_\Omega f \cdot \eta\, dx \quad (\eta \in H^1(\Omega)). \tag{2.4}$$

In addition to the assumptions in Section 1, we consider $A: L^2(\Omega)^2 \to L^2(\Omega)^2$ to be uniformly monotone and Lipschitz continuous, i.e., there exists positive constants c_A and C_A with

$$c_A\|\sigma-\tau\|^2_{L^2(\Omega)^2} \le \int_\Omega (A(\sigma)-A(\tau))(\sigma-\tau)\,dx$$

$$\|A(\sigma)-A(\tau)\|_{L^2(\Omega)^2} \le C_A\|\sigma-\tau\|_{L^2(\Omega)^2} \quad \left(\sigma,\tau\in L^2(\Omega)^2\right). \tag{2.5}$$

The problems (1.1)–(1.5) and (2.4) are equivalent in the following sense.

Theorem 2.1. *If $u\in H^1_{loc}(\Omega\cup\Omega_c)$ is a solution of (1.1)–(1.5), then $u|_\Omega$ solves (2.4). Conversely, if $u\in H^1(\Omega)$ is a solution of (2.4), then u can be extended by using the representation formula (2.3) to a function $u\in H^1_{loc}(\Omega\cup\Omega_c)$ which solves (1.1)–(1.5).*

Proof: The proof is based on standard arguments and the use of Lemma 2.1. We refer to [9] and [2, 17, 23, 24] for details and related results. □

The left hand side in (2.4) defines an operator B as

$$B(u)(\eta) := \int_\Omega A(Du)\cdot D\eta\,dx + \langle Su|_\Gamma, \eta|_\Gamma\rangle$$

which maps $H^1(\Omega)$ into its dual $H^1(\Omega)^*$. Then, Equation (2.4) reads

$$B(u)=f$$

where f is regarded as an element in $H^1(\Omega)^*$. Since S is bounded and positive definite, B inherits monotonicity and Lipschitz continuity from A. Hence, from standard results in the theory of monotone operators, we gain existence and uniqueness of solutions in our model problem.

Theorem 2.2. *The operator B is uniform monotone and Lipschitz continuous. The problems (1.1)–(1.5) and (2.4) have unique solutions.* □

3. Discretization

For simplicity, we assume that Ω is a polygon and consider finite partitions $\mathcal{T}_h$ of Ω,

$$\overline{\Omega} = \cup_{T\in\mathcal{T}_h}\overline{T}$$

such that $T\in\mathcal{T}_h$ is (the interior of) a triangle with angles greater than $c_\theta>0$ and diameter $h_T>0$. (c_θ is a global constant and independent of $\mathcal{T}_h$). We assume that two non-identical triangles or so-called finite elements in $\mathcal{T}_h$ share at most a common edge or a common vertex.

With any partition $\mathcal{T}_h$ we associate the piecewise linears $S^1_h(\Omega)$, i.e., continuous functions which are affine on each triangle T in $\mathcal{T}_h$. The element sides

$$\mathcal{E}_h := \{E : E \text{ open edge of } T\in\mathcal{T}_h\}$$

(an open edge is an edge without the nodes) consist of interior elements sides

$$\mathcal{E}_h(\Omega) := \{E\in\mathcal{E}_h : E\subset\Omega\}$$

or of boundary sides

$$\mathscr{E}_h(\Gamma) := \{E \in \mathscr{E}_h : E \subset \Gamma\}.$$

For each $E \in \mathscr{E}_h = \mathscr{E}_h(\Omega) \cup \mathscr{E}_h(\Gamma)$ let h_E denote the length of E. Let

$$h_{\Gamma,max} := \max\{h_E : E \in \mathscr{E}_h(\Gamma)\},$$
$$h_{\Gamma,min} := \min\{h_E : E \in \mathscr{E}_h(\Gamma)\},$$
$$h_{\Omega,max} := \max\{h_T : T \in \mathscr{T}_h\}.$$

For each $E \in \mathscr{E}_h(\Omega)$ we choose and fix some orientation of a unit normal vector n_E on E while $n_E = n$ for $E \in \mathscr{E}_h(\Gamma)$, i.e., we choose the exterior unit normal on the boundary.

In order to discretize the boundary integral operators, we associate the discrete function spaces of piecewise linears $S_h^1(\Gamma)$ and piecewise constant $S_h^0(\Gamma)$ to the partition $\mathscr{E}_h(\Gamma)$ of the boundary, i.e. $S_h^1(\Gamma)$ (resp. $S_h^0(\Gamma)$) consist of continuous (resp., in general, discontinuous) functions on Γ which are affine (resp. constant) on each so-called boundary element E in $\mathscr{E}_h(\Gamma)$.

Remark 3.1. The boundary element method is related to $S_h^1(\Gamma)$ and $S_h^0(\Gamma)$ which are defined with respect to the same mesh as the finite element trial space $S_h^1(\Omega)$. This is by no means necessary. However, since it might yield a convenient data handling and since it simplifies notations, we restrict the model problem to this special case.

Given $\mathscr{T}_h$, define discrete operators

$$V_h := i_h^* V i_h, \qquad K_h := i_h^* K j_h,$$
$$K_h^* := j_h^* K^* i_h = (K_h)^*, \quad W_h := j_h^* W j_h,$$
$$A_h := k_h^* D^* A D k_h, \qquad B_h := A_h + S_h,$$
$$S_h := W_h + \left(K_h^* - \tfrac{1}{2} j_h^* i_h\right) V_h^{-1} \left(K_h - \tfrac{1}{2} i_h^* j_h\right)$$

which are well defined as mappings between the appropriate discrete spaces indicated by the canonical embeddings

$$i_h := S_h^0(\Gamma) \hookrightarrow H^{-1/2}(\Gamma)$$
$$j_h := S_h^1(\Gamma) \hookrightarrow H^{1/2}(\Gamma)$$
$$k_h := S_h^1(\Omega) \hookrightarrow H^1(\Omega)$$

and their duals i_h^*, j_h^*, k_h^*.

Remark 3.2. We note that $\gamma k_h = j_h$ where $\gamma := \cdot|_\Gamma$ is the trace operator. For example, for $w \in H^{1/2}(\Gamma)$, we have $i_h^* w \in S_h^0(\Gamma)^*$ with $(i_h^* w)(v_h) = \int_\Gamma w \cdot v_h \, ds$.

The discrete problem reads: Find $u_h \in S_h^1(\Omega)$ satisfying

$$B_h(u_h) = k_h^* f \tag{3.1}$$

(i.e., $(B_h(u_h))(v_h) = f(v_h)$ for all $v_h \in S_h^1(\Omega)$).

The discrete problem (3.1) is described in different notations in [17,22,23]. In the present notations it is easily seen that B_h inherits the Lipschitz and monotonicity properties of B. A simple proof of the following a priori error estimate is given by Section 5.

Theorem 3.1 ([17,23]). *There exist positive constants c_0 and h_0 such that for all partitions $\mathscr{T}_h$ with $h < h_0$ the discrete problem (3.1) has a unique solution u_h and there holds*

$$\|u - u_h\|_{H^1(\Omega)} \le c_0 \cdot \big(\operatorname{dist}_{H^1(\Omega)}(u; S_h^1(\Omega)) + \operatorname{dist}_{H^{-1/2}(\Gamma)}(Su|_\Gamma; S_h^0(\Gamma))\big).$$

($\operatorname{dist}_X(w; Y)$ denotes the best approximation error in the norm of X when approximating $w \in X$ with functions in Y.)

4. A Posteriori Error Estimate

While we considered a priori estimates so far we present an a posteriori error estimate in this section and sketch an adaptive feedback procedure for automatic mesh refinements. To describe the computable upper error bound we need further notation.

Given a partition $\mathscr{T}_h$ of Ω as in Section 3, we define n_E to be the exterior unit normal vector for $E \in \mathscr{E}_h(\Gamma)$ and chose some fixed unit normal n_E for $E \in \mathscr{E}_h(\Omega)$. Let $u_h \in S_h^1(\Omega)$ be a (discrete) solution of (3.1). Then we define

$$\sigma_h := A(Du_h) \quad \text{and} \quad \phi_h := V_h^{-1}(K_h - \tfrac{1}{2} i_h^* j_h) u_h|_\Gamma$$

as the discrete analogs of

$$\sigma := A(Du) \quad \text{and} \quad \phi := V^{-1}(K - 1/2) u|_\Gamma .$$

Note that Du_h and ϕ_h are piecewise constant. We assume that $\sigma_h|_T \in C^0(\overline{T})$ and $\operatorname{div}\sigma_h|_T \in L^2(T)$ for any $T \in \mathscr{T}_h$. On $E \in \mathscr{E}_h(\Gamma)$, $[\sigma_h n_E]$ is defined as the jump of the discrete tractions $\sigma_h n_E$ across the common element edge E. Let $\frac{\partial}{\partial s}$ denote the derivative with respect to the arc-length (at least in the distributional sense).

Then, for each triangle T we may define a non-negative real number $\eta_h(T)$ as

$$\begin{aligned}
\eta_h(T)^2 := {} & h_T^2 \cdot \|f + \operatorname{div}\sigma_h\|_{L^2(T)}^2 \\
& + \frac{1}{2} \sum_{E \subset \partial T,\, E \in \mathscr{E}_h(\Omega)} h_E \cdot \|[\sigma_h n_E]\|_{L^2(E)}^2 \\
& + \sum_{E \subset \partial T,\, E \in \mathscr{E}_h(\Gamma)} h_E \cdot \|\sigma_h n_E + W u_h|_\Gamma + (K' - 1/2)\,\phi_h\|_{L^2(E)}^2 \\
& + \sum_{E \subset \partial T,\, E \in \mathscr{E}_h(\Gamma)} h_E \cdot \Big\|\frac{\partial}{\partial s}(V\phi_h - (K - 1/2) u_h|_\Gamma)\Big\|_{L^2(E)}^2 .
\end{aligned}$$

Remark 4.1. The first two terms in the definition of $\eta_h(T)$ are well established in residual based a posteriori error estimates for the finite element method. The third term is the residual on the interface Γ caused by tractions coming from the finite elements and by tractions resulting from the related boundary integral operators. Its sum, related to $\sigma_h n + S_h u_h|_\Gamma$, is a residual of $\sigma n = -Su|_\Gamma$. The last contribution in $\eta_h(T)$ is caused by the approximation V_h^{-1} to V^{-1} giving the residual $V\phi_h - (K-1/2)u_h|_\Gamma$ (cf. $V\phi - (K-1/2)u|_\Gamma = 0$).

The following a posteriori error estimate is proved in Section 5.

Theorem 4.1. *There exists a constant $c > 0$, which depends only on Ω and c_θ, c_A and C_A, such that*

$$\|u - u_h\|_{H^1(\Omega)} + \|\phi - \phi_h\|_{H^{-1/2}(\Gamma)} \le c \cdot \left(\sum_{T \in \mathcal{T}_h} \eta_h(T)^2 \right)^{1/2}. \tag{4.1}$$

Remark 4.2. We emphasize that $\eta_h(T)$ can be computed—at least approximated numerically—once the discrete solution u_h is known; see [9, 10] for details in the implementation.

Remark 4.3. The estimates in [9, 10] differ from (4.1) only in the last contribution in $\eta_h(T)$ related to the $V\phi_h - (K-1/2)u_h$. The right hand side in Theorem 4.1 includes the corresponding summand

$$\sum_{E \in \mathcal{E}_h(\Gamma)} h_E \cdot r_E^2$$

where $r_E := \|\frac{\partial}{\partial s}(V\phi_h - (K-1/2)u_h|_\Gamma)\|_{L^2(E)}$. Using its upper bound

$$\left(\sum_{E \in \mathcal{E}_h(\Gamma)} h_E^{1/2} \cdot r_E \right)^2$$

in Theorem 4.1 we gain the estimate in [9] (and similarly [10]). In this sense, Theorem 4.1 refines [9, 10].

Based on Theorem 4.1 we may define an adaptive algorithm for automatic mesh-refinements following the literature. The heuristic idea is first to regard $\eta_h(T)$ as (an approximation to the unknown) local error related to the element T and secondly to refine T if $\eta_h(T)$ is comparably large. Here, the refinement is steered by a parameter θ with $0 \le \theta \le 1$.

Algorithm ($\mathbf{A}_\theta$)

(a) Start with a coarse initial mesh $\mathcal{T}_{h_0}$. Put $k = 0$.
(b) Solve the discrete problem u_{h_k} with respect to the actual mesh $\mathcal{T}_{h_k}$.
(c) Compute $\eta_{h_k}(T)$ for each T in $\mathcal{T}_{h_k}$.

(d) Compute the upper bound (4.1) and decide to stop (then terminate computation) or to refine (then go to (e)).
(e) Refine (e.g., halve the largest edge of) $T \in \mathscr{T}_{h_k}$ provided

$$\eta_{h_k}(T) \geq \theta \cdot \max_{T' \in \mathscr{T}_{h_k}} \eta_{h_k}(T'). \tag{4.2}$$

(f) Refine further triangles to avoid hanging nodes. This defines a new mesh $\mathscr{T}_{h_{k+1}}$. Replace k by $k+1$ and go to (b).

Remark 4.4. The parameter θ affects the mesh-refinement such that for small $\theta \geq 0$ we expect a global almost uniform refinement ($\theta = 0$ means quasi-uniform refinement) and for $\theta \geq 1$ near 1 we expect a local refinement of only some triangles. The value $\theta = 1/2$ lead to a good performance of the algorithm.

Remark 4.5. Adaptive algorithms similar to Algorithm (A_θ) are considered in [9,10]. We remark that the refinement step (4.2) (for the quantities $\eta(T)$ given here) reads more natural and appears to be more consistent than the related step in [9,10].

Remark 4.6. Though the constant in (4.1) is known in principle, it might be expansive or even difficult to compute an accurate reliable upper bound. Hence, the constant may be unknown in step (d) where we have to decide whether our approximant is accurate enough or not. However, in this case one can estimate the relative improvement comparing the computed upper bound in the first step with the present upper bound (neglecting the constants). This gives a hint to some relative improvement during the computation and suggests a termination criterion.

5. Proofs

The aim of this section is a proof of Theorem 3.1 and 4.1 where emphasis is on the role of the Poincaré-Steklov operator S and its discrete counterpart S_h. We define another related operator $\hat{S}_h : S_h^1(\Gamma) \to H^{-1/2}(\Gamma)$ by

$$\hat{S}_h := Wj_h + (K' - 1/2) i_h V_h^{-1}\left(K_h - \tfrac{1}{2} i_h^* j_h\right).$$

We put stress on S and S_h by first proving quite general estimates of auxiliary character and secondly showing how these estimates can be used to control certain typical terms which arise naturally in an a priori and a posteriori error analysis.

Proposition 5.1. *There exist positive constants C_1, C_2, C_3, C_4 (depending only on Γ) such that*

$$C_1\left(\|\psi - \psi_h\|^2_{H^{-1/2}(\Gamma)} + \|v - v_h\|^2_{H^{1/2}(\Gamma)}\right)$$
$$\leq \langle Sv - \hat{S}_h v_h, v - v_h \rangle + \langle V(\psi_h - \psi_h^*), \psi_h - \psi \rangle \tag{5.1}$$

for arbitrary $v \in H^{1/2}(\Gamma)$ *and* $v_h \in S_h^1(\Gamma)$ *letting* $\psi := V^{-1}(K-1/2)v$, $\psi_h^* := V^{-1}(K-1/2)v_h$ *in* $H^{-1/2}(\Gamma)$ *and* $\psi_h := V_h^{-1} i_h^* = V_h^{-1} i_h^*(K-1/2)v_h$ *in* $S_h^0(\Gamma)$.

Furthermore, for any $\eta_h \in S_h^0(\Gamma)$, *there holds*

$$\langle V(\psi_h - \psi_h^*), \psi_h - \psi \rangle = \langle V(\psi_h - \psi_h^*), \eta_h - \psi \rangle \tag{5.2}$$

$$\begin{aligned} \|\psi_h - \psi_h^*\|_{H^{-1/2}(\Gamma)} &\le C_2 \cdot \operatorname{dist}_{H^{-1/2}(\Gamma)}\big(\psi_h^*; S_h^0(\Gamma)\big) \\ &\le C_3 \cdot \operatorname{dist}_{H^{-1/2}(\Gamma)}\big(\psi; S_h^0(\Gamma)\big) + C_4 \cdot \|v - v_h\|_{H^{1/2}(\Gamma)}. \end{aligned} \tag{5.3}$$

Proof: Using the notation of the proposition we calculate

$$\begin{aligned} \langle S(v - v_h), v - v_h \rangle &= \langle W(v - v_h), v - v_h \rangle \\ &\quad + \langle VV^{-1}(K-1/2)(v - v_h), V^{-1}(K-1/2)(v - v_h) \rangle \\ &= \langle W(v - v_h), v - v_h \rangle + \langle V(\psi - \psi_h^*), \psi - \psi_h^* \rangle \end{aligned}$$

and

$$\begin{aligned} \langle (S - \hat{S}_h) v_h, v - v_h \rangle &= \langle V^{-1}(K-1/2)v_h - V_h^{-1}\big(K_h - \tfrac{1}{2} i_h^* j_h\big) v_h, \\ &\qquad (K-1/2)(v - v_h) \rangle \\ &= \langle V(\psi_h^* - \psi_h), V^{-1}(K-1/2)(v - v_h) \rangle \\ &= \langle V(\psi_h^* - \psi_h), \psi - \psi_h^* \rangle. \end{aligned}$$

Combining the two identities, one verifies

$$\begin{aligned} &\langle Sv - \hat{S}_h v_h, v - v_h \rangle + \langle V(\psi_h - \psi_h^*), \psi_h - \psi \rangle \\ &\quad = \langle W(v - v_h), v - v_h \rangle + \langle V(\psi - \psi_h), \psi - \psi_h \rangle. \end{aligned} \tag{5.4}$$

Note that V is positive definite on $H^{-1/2}(\Gamma)$, and W is positive definite on $H^{1/2}(\Gamma)/\mathbb{R}$. We may identify $H^{-1/2}(\Gamma)/\mathbb{R}$ with $H_0^{-1/2}(\Gamma)$ where

$$H_0^s(\Gamma) := \{v \in H^s(\Gamma) : Mv = 0\}, \quad Mv := |\Gamma|^{-1} \int_\Gamma v\, ds,$$

$|\Gamma|$ is the length of Γ and Mv is the integral mean of v. Define $P := I - M$, I being the identity. Then, P and M are projectors and establish a decomposition of

$$H^s(\Gamma) \equiv H_0^s(\Gamma) \otimes \mathbb{R}. \tag{5.5}$$

In particular, the norm in $H^s(\Gamma)$ is equivalent to $(|M\cdot|^2 + \|P\cdot\|^2_{H_0^s(\Gamma)})^{1/2}$.

From (5.4) and the definiteness of W and V, we infer existence of $c_1 > 0$ such that

$$\begin{aligned} &c_1\Big(\|\psi - \psi_h\|^2_{H^{-1/2}(\Gamma)} + \|Pv - Pv_h\|^2_{H_0^{1/2}(\Gamma)}\Big) \\ &\qquad \le \langle Sv - \hat{S}_h v_h, v - v_h \rangle + \langle V(\psi_h - \psi_h^*), \psi_h - \psi \rangle. \end{aligned} \tag{5.6}$$

Since $1 \in S_h^0(\Gamma)$ we obtain $\langle 1, V(\psi_h^* - \psi_h) \rangle = 0$ and hence, using $(K-1/2)1 = -1$,

$$\begin{aligned} \langle 1, V(\psi - \psi_h) \rangle &= \langle 1, V(\psi - \psi_h^*) \rangle = \langle (K' - 1/2)1, v - v_h \rangle \\ &= -|\Gamma|(Mv - Mv_h) + \langle (K' - 1/2)1, Pv - Pv_h \rangle. \end{aligned}$$

Therefore and since the boundary integral operators are bounded we verify

$$|Mv - Mv_h|^2 \le c_2|\langle (K' - 1/2)1, Pv - Pv_h\rangle|^2 + c_3|\langle V1, \psi - \psi_h\rangle|^2$$
$$\le c_4\|Pv - Pv_h\|^2_{H_0^{1/2}(\Gamma)} + c_5\|\psi - \psi_h\|^2_{H^{-1/2}(\Gamma)}$$

for positive constants $c_1, \ldots, c_5$. Note that this is bounded by the lower bound in (5.6) as well. Thus, we have proved that $\|\psi - \psi_h\|^2_{H^{-1/2}(\Gamma)}$, $|Mv - Mv_h|^2$ and $\|Pv - Pv_h\|^2_{H0^{1/2}(\Gamma)}$ are, up to a constant factor, upper bounded by the right hand side in (5.1). According to the decomposition (5.5) and equivalence of norms, this concludes the proof of (5.1).

To verify (5.2) we notice that, for all $\eta_h \in S_h^0(\Gamma)$, $V_h\psi_h = i_h^* V\psi_h^*$, i.e., for all $\eta_h \in S_h^0(\Gamma)$, $\langle V\psi h, \eta_h\rangle = \langle V\psi_h^*, \eta_h\rangle$. Hence (5.3) follows from Cea's lemma and the triangle inequality. □

We now illustrate that Proposition 5.1 is a useful tool in proofs of a priori error estimates.

Proof of Theorem 3.1: Let $v_h \in S_h^1(\Omega)$ (resp. $\eta_h \in S_h^0(\Gamma)$) be defined as the best approximant of u (resp. ϕ). If we subtract (2.4) and (3.1) we obtain, for any $w_h \in S_h^1(\Omega)$,

$$\int_\Omega (A(Du) - A(Du_h))Dw_h\, dx + \langle Su|_\Gamma - \hat{S}_h u_h|_\Gamma, w_h|_\Gamma\rangle = 0.$$

According to the uniform monotonicity of A and Proposition 5.1 this shows, with $\phi_h^* := V^{-1}(K - 1/2)v_h$,

$$\begin{aligned} c_1\|u - u_h\|^2_{H^1(\Omega)} + c_2\|\phi - \phi_h\|_{H^{-1/2}(\Gamma)} &\le \int_\Omega (A(Du) - A(Du_h))(Du - Du_h)\,dx \\ &\quad + \langle Su|_\Gamma - \hat{S}_h u_h|_\Gamma, u|_\Gamma - u_h|_\Gamma\rangle \\ &\quad + \langle V(\phi_h - \phi_h^*), \phi_h - \phi\rangle \\ &= \int_\Omega (A(Du) - A(Du_h))(Du - Dv_h)\,dx \\ &\quad + \langle Su|_\Gamma - \hat{S}_h u_h|_\Gamma, u|_\Gamma - v_h|_\Gamma\rangle \\ &\quad + \langle V(\phi_h - \phi_h^*), \phi_h - \phi\rangle. \end{aligned} \tag{5.7}$$

According to Proposition 5.1 and Lipschitz continuity of A, this and the identity

$$\begin{aligned} &\langle Su|_\Gamma - \hat{S}_h u_h|_\Gamma, u|_\Gamma - v_h|_\Gamma\rangle \\ &\quad = \langle W(u - u_h)|_\Gamma, (u - v_h)|_\Gamma\rangle + \langle \phi - \phi_h, (K - 1/2)(u - v_h)|_\Gamma\rangle \end{aligned}$$

prove, with positive constants $c_1, \ldots, c_5$,

$$\begin{aligned} &c_1\|u - u_h\|^2_{H^1(\Omega)} + c_2\|\phi - \phi_h\|^2_{H^{-1/2}(\Gamma)} \\ &\quad \le c_3\|u - u_h\|_{H^1(\Omega)}\|u - v_h\|_{H^1(\Omega)} + c_4\|\phi - \phi_h\|_{H^{-1/2}(\Gamma)}\|u - v_h\|_{H^1(\Omega)} \\ &\qquad + c_5\|\phi - \phi_h\|_{H^{-1/2}(\Gamma)}\|\phi - \eta_h\|_{H^{-1/2}(\Gamma)}. \end{aligned}$$

Then, a standard application of Young's inequality concludes the proof. □

A posteriori error estimates for the boundary integral equations can be obtained from the following result [6].

Theorem 5.1 ([6]). *Assume $f \in H^1(\Gamma)$ has at least one zero in each of the elements in $\mathscr{E}_h(\Gamma)$. Then,*

$$\|f\|_{H^{1/2}(\Gamma)} \leq c(\log(1+\kappa))^{1/2} \left(\sum_{E \in \mathscr{E}_h(\Gamma)} h_E \cdot \| \frac{\partial}{\partial s} f \|^2_{L^2(E)} \right)^{1/2}$$

where the constant $c > 0$ is independent of $\mathscr{T}_h$ and κ is the largest quotient of neighbouring element sides on Γ. □

To be self-contained and to give a simpler proof under the present assumptions, we state and prove the following weaker but more explicit modification of Theorem 5.1. Note that κ is h-independently bounded because of the angle condition (c_θ is a global bound).

Theorem 5.2. *Let the norm in $H^{1/2}(\Gamma)$ be (equivalently) defined by complex interpolation and recall $h_{\Gamma,max} := max\{h_E : E \in \mathscr{E}_h(\Gamma)\}$. Assume $f \in H^1(\Gamma)$ has at least one zero in each of the elements in $\mathscr{E}_h(\Gamma)$. Then,*

$$\|f\|_{H^{1/2}(\Gamma)} \leq (1+\kappa)^{1/2} \cdot (1 + 4h^2_{\Gamma,max})^{1/4} \cdot \left(\sum_{j=1}^{N} h_j \cdot \left\| \frac{\partial}{\partial s} f \right\|^2_{L^2(\Gamma_j)} \right)^{1/2}.$$

Proof: Let y_j denote the zero of f in Γ_j where $\Gamma_1, \ldots, \Gamma_N$ is the partition $\mathscr{E}_h(\Gamma)$ of the boundary such that Γ_j is a neighbour of Γ_{j+1} for $j = 1, \ldots, N$, $\Gamma_{N+1} = \Gamma_1$. For $j = 1, \ldots, N$ define f_j on Γ by $f_j(x) = f(x)$ if $x \in \Gamma$ belongs to the subarc with start point y_j and end point y_{j+1}, $y_{N+1} := y_1$ and let $f_j = 0$ otherwise. By construction, f_j is continuous at y_{j-1} and y_j and equals piecewise an absolute continuous function. Furthermore, the derivative f' belongs to $L^2(\Gamma)$, so that f_j belongs to $H^1(\Gamma)$. But, as used, e.g., in [20, 29, 33] functions with disjoint support satisfy

$$\left\| \sum_{j=1}^{n} f_j \right\|^2_{H^{1/2}(\Gamma)} \leq C \cdot \sum_{j=1}^{n} \|f_j\|^2_{H^{1/2}(\Gamma)} \tag{5.8}$$

where $C = 1$ if we use complex interpolation to define $H^{1/2}(\Gamma)$. Moreover, by complex interpolation,

$$\|f_j\|^2_{H^{1/2}(\Gamma)} \leq \|f_j\|_{L^2(\Gamma)} \|f_j\|_{H^1(\Gamma)}. \tag{5.9}$$

According to the fundamental theorem on calculus and Hölder's inequality, one verifies

$$\|f_j\|_{L^2(\Gamma)} \leq (h_j + h_{j+1}) \|f'_j\|_{L^2(\Gamma)} \tag{5.10}$$

using essentially the fact that f_j has a zero on Γ_j and Γ_{j+1}; $h_j := |\Gamma_j|$. By definition of the $H^1(\Gamma)$-norm, (5.9) and (5.10) yield

$$\|f_j\|^2_{H^{1/2}(\Gamma)} \le (h_j + h_{j+1})\Big(1 + (h_j + h_{j+1})^2\Big)^{1/2} \|f_j'\|^2_{L^2(\Gamma)}. \tag{5.11}$$

By definition of κ we have $h_j + h_{j+1} \le (1+\kappa)\min\{h_j, h_{j+1}\}$ and, with $h_{\Gamma,max} := \max\{h_1, \ldots, h_N\}$, (5.11) leads to

$$\|f_j\|^2_{H^{1/2}(\Gamma)} \le \left(1 + 4\cdot h^2_{\Gamma,max}\right)^{1/2} \min\{h_j, h_{j+1}\}(1+\kappa)\|f_j'\|^2_{L^2(\Gamma)}.$$

Using this in (5.8) we conclude the proof. □

We will prove Theorem 4.1 by combining Proposition 5.1 and Theorem 5.2 with arguments from the finite element literature on a posteriori error estimates.

Proof of Theorem 4.1: Let us define $e_h := u - u_h$. We start as in the proof of Theorem 3.1 and obtain as in (5.7), for all $v_h \in S_h^1(\Omega)$ and $\eta_h \in S_h^0(\Gamma)$,

$$c_1\|e_h\|^2_{H^1(\Omega)} + c_2\|\phi - \phi_h\|^2_{H^{-1/2}(\Gamma)} \le \int_\Omega (\sigma - \sigma_h)(De_h - Dv_h)\, dx$$
$$+ \langle Su|_\Gamma - \hat{S}_h u_h|_\Gamma, e_h|_\Gamma - v_h|_\Gamma\rangle + \langle V(\phi_h - \phi_h^*), \eta_h - \phi\rangle.$$

Using $B(u) = f$ we obtain that the upper bound is equal to

$$\int_\Omega f(e_h - v_h)\, dx - \int_\Omega \sigma_h(De_h - Dv_h)\, dx$$
$$- \langle \hat{S}_h u_h|_\Gamma, e_h|_\Gamma - v_h|_\Gamma\rangle + \langle V(\phi_h - \phi_h^*), \eta_h - \phi\rangle.$$

The first three terms are treated as in finite element a posteriori analysis: By using integration by parts on each element

$$\int_\Omega f(e_h - v_h)\, dx - \int_\Omega \sigma_h(De_h - Dv_h)\, dx - \langle \hat{S}_h u_h|_\Gamma, e_h|_\Gamma - v_h|_\Gamma\rangle$$
$$= \sum_{T\in\mathcal{T}_h} \int_T (f + \operatorname{div}\sigma_h)(e_h - v_h)\, dx + \sum_{E\in\mathcal{E}_h(\Omega)} \int_E [\sigma_h \cdot n_E](e_h - v_h)\, ds$$
$$- \langle \sigma_h n + \hat{S}_h u_h|_\Gamma, e_h|_\Gamma - v_h|_\Gamma\rangle.$$

As it is well-established, one can construct an approximant $v_h \in S_h^1(\Omega)$ to e_h with the following properties:

$$\|e_h - v_h\|_{L^2(T)} \le C_1 h_T \|e_h\|_{H^1(\mathcal{N}_h(T))} \tag{5.12}$$

$$\|e_h - v_h\|_{L^2(E)} \le C_2 h_E^{1/2} \|e_h\|_{H^1(\mathcal{N}_h(E))} \tag{5.13}$$

for all $T \in \mathcal{T}_h$, $E \in \mathcal{E}_h(\Omega)$. Here, $\mathcal{N}_h(T)$ (resp. $\mathcal{N}_h(E)$) is the union of T and at most $\lfloor 2\pi/c_\theta \rfloor$ other elements which share a common node with T (resp. the union of at most two elements which share E as a common side). The constants $C_1, C_2 > 0$ in (5.12), (5.13) depend only on c_θ but neither on h nor on $u - u_h$. For a proof of (5.12) and (5.13) we refer to [12]. Using (5.12) and Cauchy's inequality we gain

$$\sum_{T\in\mathscr{T}_h}\int_T (f+\operatorname{div}\sigma_h)(e_h-v_h)\,dx \le C_3\|h\cdot(f+\operatorname{div}\sigma_h)\|_{L^2(\Omega)}\|e_h\|_{H^1(\Omega)} \quad (5.14)$$

where we used that the number of elements in $\mathscr{N}_h(T)$ is smaller than $\lfloor 6\pi/c_\theta\rfloor$. In (5.14), h is regarded as the piecewise constant function with $h|_T=h_T$ for $T\in\mathscr{T}_h$. Similarly,

$$\sum_{E\in\mathscr{E}_h(\Omega)}\int_E [\sigma_h\cdot n_E](e_h-v_h)\,ds \le C_4\Bigg(\sum_{E\in\mathscr{E}_h(\Omega)} h_E\|[\sigma_h\cdot n_E]\|^2_{L^2(\Omega)}\Bigg)^{1/2}\|e_h\|_{H^1(\Omega)}. \quad (5.15)$$

In the same way we get (h is the above defined piecewise constant function on Ω and on Γ as well)

$$-\langle \sigma_h n+\hat{S}_h u_h|_\Gamma, e_h|_\Gamma - v_h|_\Gamma\rangle \le C_5\|h\big(\sigma_h n+\hat{S}_h u_h|_\Gamma\big)\|_{L^2(\Gamma)}\|e_h\|_{H^1(\Omega)}. \quad (5.16)$$

Combining (5.14)–(5.16) we obtain, with Young's inequality,

$$\begin{aligned}
c_3\|e_h\|^2_{H^1(\Omega)} &+ c_4\|\phi-\phi_h\|^2_{H^{-1/2}(\Gamma)}\\
&\le \|h\cdot(f+\operatorname{div}\sigma_h)_{L^2(\Omega)}\|e_h\|^2_{H^1(\Omega)} + \sum_{E\in\mathscr{E}_h(\Omega)} h_E\|[\sigma_h\cdot n_E]\|^2_{L^2(\Omega)} \quad (5.17)\\
&\quad + \|h\big(\sigma_h n+\hat{S}_h u_h|_\Gamma\big)\|^2_{L^2(\Gamma)} + \langle V(\phi_h-\phi_h^*),\eta_h-\phi\rangle.
\end{aligned}$$

It remains to apply an a posteriori error estimate to $\langle V(\phi_h-\phi_h^*),\eta_h-\phi\rangle$ from Theorem 5.2. Since η_h is arbitrary, $R:=V(\phi_h-\phi_h^*)$ is $L^2(\Gamma)$-orthogonal to $S_h^0(\Gamma)$. Thus, the integral means of the continuous function R vanish for each element and so R has at least one zero on each boundary element. Hence, by Theorem 5.2,

$$\begin{aligned}
\langle V(\phi_h-\phi_h^*),\eta_h-\phi\rangle = \langle V(\phi_h-\phi_h^*),\phi_h-\phi\rangle &\le \|R\|_{H^{1/2}(\Gamma)}\|\phi_h-\phi\|_{H^{-1/2}(\Gamma)}\\
&\le C_5\|h^{1/2}\frac{\partial R}{\partial s}\|_{L^2(\Gamma)}\|\phi_h-\phi\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

Using (5.18) in (5.17) one concludes the proof with Young's inequality again. □

6. Efficiency

To study the sharpness of the a posteriori error estimate, we start proving a local estimate of a reverse type. We follow [36] and include ideas from [5]. To employ the analysis of [36, §2] we assume that σ_h is piecewise constant for simplicity.

Remark 6.1. The assumption $\sigma_h\in S_h^0(\Omega)$ is satisfied, e.g., if $A:L^2(\Omega)^2\to L^2(\Omega)^2$ is local in the sense that there is an appropriate function $a:\Omega\times\mathbb{R}^2\to\mathbb{R}^2$ with

$$(A\tau)(x)=a(x,\tau(x)) \quad \text{for a.a. } x \text{ in } \Omega\ \big(\tau\in L^2(\Omega)^2\big)$$

and $a(\cdot,\tau)$ is piecewise constant for a fixed $\tau\in\mathbb{R}^2$ (notice $Du_h\in S_h^0(\Omega)$ for a proof).

Adopting notation from Section 4 we define, $f_T\in S_h^0(\Omega)$ and $t_T\in S_h^0(\Gamma)$ for $T\in\mathcal{T}_h$, $E\in\mathcal{E}_h, t:=-Su|_\Gamma(=\phi)$,

$$f_T:=|T|^{-1}\cdot\int_T f\,dx \quad\text{and}\quad t_E:=|E|^{-1}\cdot\int_E t\,ds,$$

where $|T|$ (resp. $|E|$) is the two dimensional (resp. one dimensional) Lebesgue measure of T (resp. E).

The first estimate shows that the error indicator η_h is locally bounded by a constant times local error terms and local approximation error of the data f and the boundary stresses t.

Proposition 6.1. *Assuming $\sigma_h\in S_h^0(\Omega)$ there is a constant $C>0$ which depends on c_θ only such that for each $T\in\mathcal{T}_h$*

$$\begin{aligned}\frac{1}{C}\cdot\eta_h(T)^2\le{}&\|\sigma-\sigma_h\|^2_{L^2(\mathcal{N}_h(T))}\\&+\|h_T(f-f_T)\|^2_{L^2(\mathcal{N}_h(T))}+\|h_E^{1/2}(t-t_E)\|^2_{L^2(\Gamma\cap\partial T)}\\&+\|h_E^{1/2}W(u|_\Gamma-u_h|_\Gamma)\|^2_{L^2(\Gamma\cap\partial T)}\\&+\|h_E^{1/2}(K'-1/2)(\phi-\phi_h)\|^2_{L^2(\Gamma\cap\partial T)}\\&+\left\|h_E^{1/2}\frac{\partial}{\partial s}V(\phi-\phi_h)\right\|^2_{L^2(\Gamma\cap\partial T)}\\&+\left\|h_E^{1/2}\frac{\partial}{\partial s}(K-1/2)(u|_\Gamma-u_h|_\Gamma)\right\|^2_{L^2(\Gamma\cap\partial T)}.\end{aligned}$$

Proof: Arguing as in [36, Eq. (2.15)] we estimate $\|h\cdot f_T\|_{L^2(T)}$ and obtain

$$3/\sqrt{20}\cdot\|h\cdot f_T\|_{L^2(T)}\le\|h_T(f-f_T)\|_{L^2(T)}+c_5\cdot\|\sigma-\sigma_h\|_{L^2(T)}. \tag{6.1}$$

The constant $c_5>0$ depends only on c_θ (see [36, Lemma 1.3]).

For $E\in E_h(\Omega)$ let $\mathcal{N}_h(E)=:T_1\cup T_2$ be the union of two triangles $T_1,T_2\in\mathcal{T}_h$ sharing the common side E. Define $b_E:=4\lambda_{T_i,1}\cdot\lambda_{T_i,2}$ on T_i, $i=1,2$ and $b_E=0$ on $\Omega\backslash\mathcal{N}_h(E)$. Here, $\lambda_{T,1},\lambda_{T,2},\lambda_{T,3}$ are the barycentric coordinates of $T\in\mathcal{T}_h$ and $\lambda_{T_i,1}$ and $\lambda_{T_i,2}$ are such that b_E is nonzero along E. As in [36, Eq. (2.19)] one proves

$$\begin{aligned}&2/9\cdot\|h_E^{1/2}[n_E\cdot\sigma_h]\|^2_{L^2(E)}\\&\quad\le c_4\|h_T\cdot(f-f_T)\|^2_{L^2(\mathcal{N}_h(E))}+c_4c_6^2\cdot\|\sigma-\sigma_h\|_{L^2(\mathcal{N}_h(E))}.\end{aligned} \tag{6.2}$$

The constants $c_4,c_6>0$ depend only on c_θ (see [36, Lemma 1.3]).

To estimate the third term in the definition of $\eta_h(T)$, let $E \in \mathscr{E}_h(\Gamma)$ and consider, with $t = -Su|_\Gamma$ and the triangle inequality,

$$h_E \int_E |\sigma_h \cdot n_E + \hat{S}_h u_h|_\Gamma|^2 \, ds$$

$$\leq 2h_E \int_E |\sigma_h \cdot n_E - t|_\Gamma|^2 \, ds + 2h_E \int_E |\hat{S}_h u_h|_\Gamma - Su|_\Gamma|^2 \, ds. \tag{6.3}$$

The first term on the right hand side in (6.3) might be estimated exactly as in [36, Eq. (2.23)] which results here in

$$2h_E \int_E |\sigma_h \cdot n_E - t|_\Gamma|^2 \, ds \leq 13 h_E \int_E |t - t_E|^2 \, ds$$

$$+ 27 c_4 h_E^2 \|f - f_T\|_{L^2(T)}^2 + 27 c_4 c_6^2 \|\sigma - \sigma_h\|_{L^2(T)}^2. \tag{6.4}$$

The second term on the right hand side in (6.3) may be rewritten by using $\phi = V^{-1}(K - 1/2)u|_\Gamma$, $\phi_h^* = V^{-1}(K - 1/2)u_h|_\Gamma$, and $\phi_h = V_h^{-1} i_h^* V \phi_h^*$ in the definition of S and $\hat{S}_h$,

$$2h_E \int_E |\hat{S}_h u_h|_\Gamma - Su|_\Gamma|^2 \, ds = 2h_E \int_E |W(u|_\Gamma - u_h|_\Gamma) - (K' - 1/2)(\phi - \phi_h)|^2 \, ds$$

$$\leq 4h_E \int_E |W(u|_\Gamma - u_h|_\Gamma)|^2 \, ds$$

$$+ 4h_E \int_E |(K' - 1/2)(\phi - \phi_h)|^2 \, ds. \tag{6.5}$$

The fourth term in the definition of $\eta_h(T)$ is estimated, for $E \in \mathscr{E}_h(\Gamma)$, with the triangle inequality and the identity $V\phi = (K - 1/2)u|_\Gamma$,

$$h_E \cdot \left\| \frac{\partial}{\partial s}(V\phi_h - (K - 1/2)u_h|_\Gamma) \right\|_{L^2(E)}^2 \leq 2h_E \cdot \left\| \frac{\partial}{\partial s}(V(\phi_h - \phi)) \right\|_{L^2(E)}^2$$

$$+ 2h_E \cdot \left\| \frac{\partial}{\partial s}((K - 1/2)(u|_\Gamma - u_h|_\Gamma)) \right\|_{L^2(E)}^2. \tag{6.6}$$

Gathering the estimates (6.1)–(6.6) we obtain the assertion (note $\operatorname{div}\sigma_h|_T = 0$). □

From Proposition 6.1 we infer a reverse inequality to (4.1). We recall $h_{\Gamma,max} := \max\{h_E : E \in \mathscr{E}_h(\Gamma)\}$ and $h_{\Gamma,min} := \min\{h_E : E \in \mathscr{E}_h(\Gamma)\}$.

Theorem 6.1. *Assuming $\sigma_h \in S_h^0(\Omega)$ there is a constant $C > 0$ which depends on c_θ*

only such that for each $T \in \mathcal{T}_h$

$$\frac{1}{C} \cdot \sum_{T \in \mathcal{T}_h} \eta_h(T)^2 \leq \|h_T(f - f_T)\|^2_{L^2(\Omega)} + \|\sigma - \sigma_h\|^2_{L^2(\Omega)} + \frac{h_{\Gamma,max}}{h_{\Gamma,min}} \cdot \left(\|(u - u_h)|_\Gamma\|^2_{H^{1/2}(\Gamma)} + \|\phi - \phi_h\|^2_{H^{-1/2}(\Gamma)}\right) + \frac{h^2_{\Gamma,max}}{h_{\Gamma,min}} \cdot \left(\mathrm{dist}_{H^1(\Gamma)}(u; S^1_h(\Gamma))^2 + \mathrm{dist}_{L^2(\Gamma)}(Su|_\Gamma; S^0_h(\Gamma))^2\right). \tag{6.7}$$

Proof: Let $C > 0$ be a generic h-independent constant in this proof. We consider the sum over all elements T in Proposition 6.1 and obtain

$$\begin{aligned}\frac{1}{C} \cdot \sum_{T \in \mathcal{T}_h} \eta_h(T)^2 \leq{}& \|\sigma - \sigma_h\|^2_{L^2(\Omega)} \\ &+ \|h_T(f - f_T)\|^2_{L^2(\Omega)} + h_{\Gamma,max}\|t - t_E\|^2_{L^2(\Gamma)} \\ &+ h_{\Gamma,max}\|W(u|_\Gamma - u_h|_\Gamma)\|^2_{L^2(\Gamma)} \\ &+ h_{\Gamma,max}\|(K' - 1/2)(\phi - \phi_h)\|^2_{L^2(\Gamma)} \\ &+ h_{\Gamma,max}\|\frac{\partial}{\partial s}V(\phi - \phi_h)\|^2_{L^2(\Gamma)} \\ &+ h_{\Gamma,max}\|\frac{\partial}{\partial s}(K - 1/2)(u|_\Gamma - u_h|_\Gamma)\|^2_{L^2(\Gamma)}.\end{aligned}$$

Since t_E is the $L^2(\Gamma)$-projection of $t = \phi - Su|_\Gamma$ onto $S^0_h(\Gamma)$, $\|t - t_E\|_{L^2(\Gamma)} = \mathrm{dist}_{L^2(\Gamma)}(Su|_\Gamma; S^0_h(\Gamma))$ and it remains to consider the other boundary terms: For example, let us estimate $h^{1/2}_{\Gamma,max}\|W(u|_\Gamma - u_h|_\Gamma)\|^2_{L^2(\Gamma)}$. According to mapping properties [14], we have

$$h_{\Gamma,max}\|W(u|_\Gamma - u_h|_\Gamma)\|^2_{L^2(\Gamma)} \leq h_{\Gamma,max}C\|u|_\Gamma - u_h|_\Gamma\|^2_{H^1(\Gamma)} \tag{6.8}$$

Let $u_E \in S^1_h(\Gamma)$ be some approximant to u, e.g., let u_E be the nodal interpolant of u with respect to the mesh on Γ. Then, by the triangle inequality,

$$\|u|_\Gamma - u_h|_\Gamma\|^2_{H^1(\Gamma)} \leq 2\|u|_\Gamma - u_E\|^2_{H^1(\Gamma)} + 2\|u_E - u_h|_\Gamma\|^2_{H^1(\Gamma)}. \tag{6.9}$$

Since, $u_E - u_h|_\Gamma \in S^1_h(\Gamma)$, we obtain from the well-known inverse inequality (cf., e.g., [5, 37]),

$$\begin{aligned}\|u_h|_\Gamma - u_E\|_{H^1(\Gamma)} &\leq Ch^{-1/2}_{\Gamma,min}\|u_h|_\Gamma - u_E\|_{H^{1/2}(\Gamma)} \\ &\leq Ch^{1/2}_{\Gamma,min}\|u_h|_\Gamma - u|_\Gamma\|_{H^{1/2}(\Gamma)} + Ch^{1/2}_{\Gamma,min}\|u|_\Gamma - u_E\|_{H^{1/2}(\Gamma)}.\end{aligned} \tag{6.10}$$

By interpolation in $H^{1/2}(\Gamma)$ and approximation property of u_E, we achieve

$$\begin{aligned}\|u|_\Gamma - u_E\|^2_{H^{1/2}(\Gamma)} &\leq C\|u|_\Gamma - u_E\|_{L^2(\Gamma)}\|u|_\Gamma - u_E\|_{H^1(\Gamma)} \\ &\leq Ch_{\Gamma,max}\|u|_\Gamma - u_E\|^2_{H^1(\Gamma)}.\end{aligned} \tag{6.11}$$

Putting (6.8)–(6.11) together, we conclude

$$h_{\Gamma,max}\|W(u|_\Gamma - u_h|_\Gamma)\|^2_{L^2(\Gamma)} \le C\frac{h_{\Gamma,max}}{h_{\Gamma,min}}\Big(\|u|_\Gamma - u_h|_\Gamma\|^2_{H^{1/2}(\Gamma)}$$

$$+h_{\Gamma,max}\|u|_\Gamma - u_E\|^2_{H^1(\Gamma)}\Big). \qquad (6.12)$$

The arguments used in (6.8)–(6.12) apply to the remaining boundary terms as well and conclude the proof of (6.7). □

To illustrate Theorem 6.1, we discuss (6.7) in a model situation where f is assumed to be (piecewise) smooth such that

$$\|h_T(f-f_T)\|_{L^2(\Omega)} \le Ch^2_{\Omega,max}, \qquad (6.13)$$

$h_{\Omega,} max := \max\{h_T : T \in \mathcal{T}_h\}$.

In the exterior domain the solution is arbitrarily smooth, so we may choose Ω so large that u is smooth near the interface (and then scale Ω into a unit circle to obtain positive definiteness of the single layer potential). Then, it is a natural choice to have a quasi-uniform mesh on the boundary, i.e., to assume

$$1 \le \frac{h_{\Gamma,max}}{h_{\Gamma,min}} \le C \qquad (6.14)$$

with an h-independent constant $C>0$. Still assuming that $u|_\Gamma$ and ϕ are smooth we obtain

$$h^{1/2}_{\Gamma,max}\mathrm{dist}_{H^1(\Gamma)}(u; S^1_h(\Gamma)) + h^{1/2}_{\Gamma,max}\mathrm{dist}_{L^2(\Gamma)}(Su|_\Gamma; S^0_h(\Gamma)) \le Ch^{3/2}_{\Omega,max}.$$

(As seen below, this regularity and approximation assumption could be weakened.)

For a quite large class of meshes, we might generically expect

$$ch_{\Omega,max} \le \|\sigma - \sigma_h\|_{L^2(\Omega)} + \|(u-u_h)|_\Gamma\|_{H^{1/2}(\Gamma)} + \|\phi - \phi_h\|_{H^{-1/2}(\Gamma)}.$$

Altogether, in the present situation, Theorem 4.1 and 6.1 verify

$$C^{-1}\|u-u_h\|_{H^1(\Omega)} \le \left(\sum_{T\in\mathcal{T}_h}\eta_h(T)^2\right)^{1/2} \le C\|u-u_h\|^2_{H^1(\Omega)}, \qquad (6.15)$$

i.e., the a posteriori error estimate is sharp (for meshes being quasi-uniform on Γ).

Remark 6.2. Following the technique in [5], one can replace the assumption that $u|_\Gamma$ and ϕ are arbitrarily smooth by a typical behaviour of corner singularities (cf. [16] for relevant regularity results). Moreover, in this special case, one could even neglect the terms $\mathrm{dist}_{H^1(\Gamma)}(u; S^1_h(\Gamma))$ and $\mathrm{dist}_{L^2(\Gamma)}(Su|_\Gamma; S^0_h(\Gamma))$ in (6.7).

7. Numerical Example

To illustrate the theoretical results we consider a numerical example on the L-shaped domain $\Omega \subset \mathbb{R}^2$ where $\Gamma = \partial\Omega$ is the polygon that connects $(0,0)$, $(1,0), (1,1), (-1,1), (-1,-1), (0,-1)$ and $(0,0)$. The interior problem is linear, $A(Du) = Du$, and the right hand side f is zero; while, on Γ, we prescribe jumps of u and u_c (instead of continuity) and jumps of the stresses (instead of equilibrium) such that u and u_c are known to equal

$$u(x,y) = \Im\{(x+iy)^{2/3}\} \qquad \text{in } \Omega$$
$$u_c(x,y) = \Re\{\log(x + \tfrac{1}{2} + i(y - \tfrac{1}{2}))\} \quad \text{in } \Omega_c.$$

We refer to [9, 10] for further computational details where this example (treated with different estimators) is also under consideration. Note that this example fits in the above assumptions and the minor modifications needed to include the jumps $u|_\Gamma - u_c|_\Gamma$ and $\frac{\partial u}{\partial n}|_\Gamma - \frac{\partial u_c}{\partial n}|_\Gamma$ are straight forward.

For uniform meshes we obtained numerical results as shown in Table 1. We computed the upper bound b_N of the a posteriori error estimate and, since the solution is known, the error in the energy norm e_N by

$$b_N = \left(\sum_{T \in \mathcal{T}_h} \eta(T)^2 \right)^{1/2}$$

$$e_N = \|u - u_N\|_{H^1(\Omega)} + \|\partial_n u - \phi_N\|_{H^{-1/2}(\Gamma)}$$

Table 1. Numerical results for uniform meshes

N	e_N	b_N	$\frac{e_N}{b_N}$
16	0.63636	2.4960	0.2550
19	0.60685	2.0679	0.2935
37	0.42442	1.6991	0.2498
49	0.38701	1.3376	0.2893
97	0.27670	1.1036	0.2507
145	0.24931	0.8637	0.2894
289	0.17789	0.70868	0.2510
481	0.15919	0.55277	0.2880
961	0.12030	0.45182	0.2663
1729	0.10114	0.35165	0.2876

where N is the degree of freedom for the discrete problem at hand. From Table 1 we infer that the error is decreasing as the upper error bound. Of certain interest is the quotient $\frac{e_N}{b_N}$ which is proved to be bounded from above in Theorem 4.1. From Table 1 we observe that this is the case and, moreover, that $\frac{e_N}{b_N}$ seems to be bounded below a well which verifies that the estimate is sharp in harmony with the discussion in Section 6.

Finally, we applied the Algorithm $(A_{1/2})$ introduced in Section 4. The meshes show a certain refinement towards the singularity of the solution at the origin as might be expected; see Fig. 1 for the final mesh and two magnifications of the mesh near the origin. The errors and bounds are shown in Table 2. Again, the errors and upper bounds are decreasing as N increases. The quotients $\frac{e_N}{b_N}$ are bounded from above and sufficiently bounded below indicating that the upper a posteriori error bound is reasonably sharp.

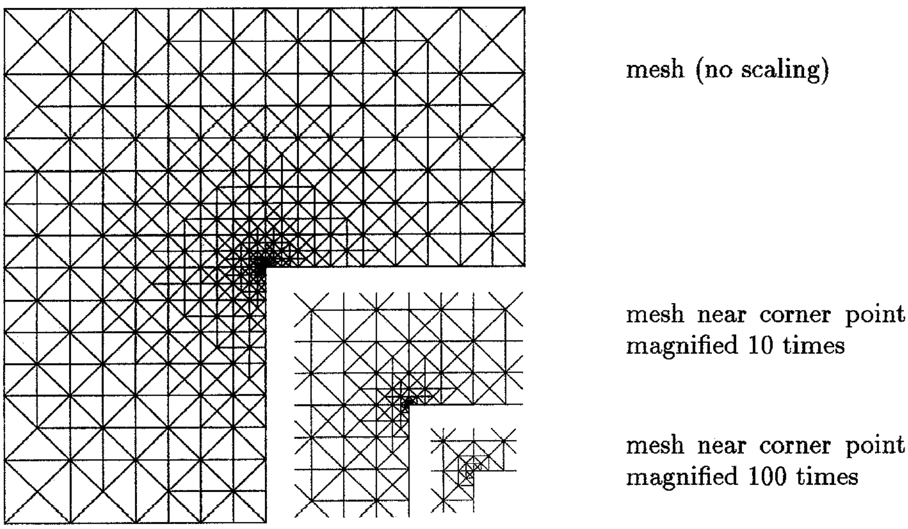

Figure 1. Mesh generated by algorithm $(A_{1/2})$

Table 2. Numerical results for adapted meshes generated by Algorithm $(A_{1/2})$

N	e_N	b_N	$\frac{e_N}{b_N}$
16	0.63636	2.4960	0.2549
19	0.60685	2.0679	0.2935
23	0.47838	1.8632	0.2568
27	0.43429	1.5545	0.2794
32	0.34192	1.3723	0.2492
36	0.32110	1.2134	0.2646
45	0.26198	1.0724	0.2443
57	0.23503	0.94526	0.2486
66	0.19794	0.85006	0.2329
79	0.17396	0.70325	0.2474
92	0.14849	0.63112	0.2353
100	0.14401	0.59589	0.2417
139	0.11496	0.50475	0.2278
156	0.10811	0.46686	0.2316
199	0.08965	0.40986	0.2187
226	0.08250	0.36940	0.2233
287	0.06996	0.32042	0.2183
328	0.06449	0.29169	0.2211
491	0.05064	0.23577	0.2148

Efficiency of the adaptive scheme is supported, e.g., by comparing the computer effort used to obtain $b_N \leq 0.35$, that is, by comparing the effort to handle $N \geq 1729$ degrees of freedom in case of a uniform mesh with the effort to handle $N \leq 287$ degrees of freedom in case of a mesh generated automatically by Algorithm $(A_{1/2})$.

Acknowledgement

The author thanks D. Zarrabi for performing the numerical example.

References

[1] Babuška, I., Miller, A.: A feedback finite element method with a posteriori error estimation. Comp. Meth. Appl. Mech. Eng. *61*, 1–40 (1987).

[2] Carstensen, C.: Interface problem in holonomic elastoplasticity. Math. Meth. Appl. Sci. *16*, 819–835 (1993).

[3] Carstensen, C.: Coupling of FEM and BEM for interface problems in viscoplasticity and plasticity with hardening. SIAM J. Numer. Anal. *33*, 171–206 (1996).

[4] Carstensen, C.: Adaptive boundary element methods and adaptive finite element and boundary element coupling. In: Boundary value problems and integral equations on non-smooth domains (Costabel, M., Dauge, M., Nicaise, S., eds.), pp. 47–58. New York: Marcel Dekker.

[5] Carstensen, C.: Efficiency of a posteriori BEM-error estimates for first kind integral equations on quasi-uniform meshes Math. Comp. *65*, 69–84 (1996).

[6] Carstensen, C.: An a posteriori error estimate for a first kind integral equation; Math. Comp. accepted for publication (1996).

[7] Carstensen, C., Stephan, E. P.: A posteriori error estimates for boundary element methods. Math. Comp. *64*, 483–500 (1994).

[8] Carstensen, C., Stephan, E. P.: Adaptive boundary element methods for some first kind integral equations. SIAM J. Numer. Anal. accepted for publication (1995).

[9] Carstensen, C., Stephan, E. P.: Adaptive coupling of boundary elements and finite elements. Math. Modelling Numer. Anal. *29*, 1–39 (1995).

[10] Carstensen, C., Funken, S., Stephan, E. P.: On the adaptive coupling of FEM and BEM in 2-d-elasticity. Numer. Math. accepted for publication (1996).

[11] Carstensen, C., Stephan, E. P.: Coupling of FEM and BEM for a nonlinear interface problem: the h-p Version. Numer. Meth. Partial Diff. Eq. *111*, 539–554 (1995).

[12] Clement, P.: Approximation by finite element functions using local regularization. RAIRO Sér. Rouge Anal. Numér. *R-2*, 77–84 (1975).

[13] Costabel, M.: Symmetric methods for the coupling of finite elements and boundary elements. In: Boundary elements IX, Vol. 1, (Bretia, C. A., ed.), pp. 411–420 Heidelberg New York Tokyo: Springer 1987.

[14] Costabel, M.: Boundary integral operators on Lipschitz domains: Elementary results. SIAM J. Math. Anal. *19*, 613–626 (1988).

[15] Costabel, M., Stephan, E. P.: Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation. Banach Center Publ. *15*, 175–251 (1985).

[16] Costabel, M., Stephan, E. P.: A direct boundary integral equation method for transmission problems. J. Math. Anal. Appl. *106*, 367–413 (1985).

[17] Costabel, M., Stephan, E.P.: Coupling of finite and boundary element methods for an elastoplastic interface problem. SIAM J. Numer. Anal. *27*, 1212–1226 (1990).

[18] Eriksson, K., Johnson, C.: An adaptive finite element method for linear elliptic problems. Math. Comp. *50*, 361–3883 (1988).

[19] Eriksson, K., Johsnon, C.: Adaptive finite element methods for parabolic problems I. A linear model problem SIAM J. Numer. Anal. *28*, 43–77 (1991).

[20] Faermann, B.: Lokale a-posteriori-Fehlerschätzer bei der Diskretisierung von Randintegralgleichungen. PhD-thesis, University of Kiel, FRG (1993).

[21] Gaier, D.: Integralgleichungen erster Art und konforme Abbildung, Math. Z. *147*, 113–129 (1976).

[22] Gatica, G. N., Hsiao, G. C.: On a class of variational formulations for some nonlinear interface problems. Rendiconti di Mathematica Ser. VII *10*, 681–715 (1990).
[23] Gatica, G. N., Hsiao, G. C.: On the coupled BEM and FEM for a nonlinear exterior Dirichlet problem in R^2. Numer. Math. *61*, 171–214 (1992).
[24] Han, H.: A new class of variational formulations for the coupling of finite and boundary element methods. J. Comput. Math. *8*, 223–232 (1990).
[25] Heuer, N.: hp-Versionen der Randelementemethode. PhD Dissertation, University of Hannover, (1992).
[26] Hsiao, G. C., Wendland, W. L.: The Aubin-Nitsche lemma for integral equations. J Integr. Eq. *3*, 299–315 (1981).
[27] Johnson, C., Hansbo, P.: Adaptive finite element methods in computational mechanics. Comput. Meth. Appl. Mech. Engin. *101*, 143–181 (1992).
[28] Lions, J. L., Magenes, E.: Non-homogeneous boundary value problems and applications, Vol. I. Berlin Heidelberg New York: Springer 1972.
[29] von Petersdorff, T.: Randwertprobleme der Elastizitätstheorie für Polyeder – Singularitäten und Approximation mit Randelementmethoden. PhD-thesis, TH Darmstadt, FRG (1989).
[30] Rank, E.: Adaptive boundary element methods. In: Boundary elements *9* (Brebbia, C. A., Wendland, W. L., Kuhn, G., eds.), Vol. 1, 259–273. Berlin Heidelberg New York Tokyo: Springer 1987.
[31] Saranen, J., Wendland, W. L.: Local residual-type error estimates for adaptive boundary element methods on closed curves. Appl. Anal. *48*, 37–50 (1993).
[32] Sloan, I. H., Spence, A.: The Galerkin method for integral equations of the first kind with logarithmic kernel: theory. IMA J. Numer. Anal. *8*, 105–122 (1988).
[33] Stephan, E. P., Suri, M.: The hp-version of the boundary element method on polygonal domains with quasiuniform meshes. Math. Modell. Numer. Anal. *25*, 783–807 (1991).
[34] Stephan, E. P., Wendland W. L.: Remarks on Galerkin and least squares methods with finite elements for general elliptic problems. Manusc. Geod. *1*, 93–123 (1976).
[35] Stephan, E. P., Wendland, W. L., Hsiao, G. C.: On the integral equation method for the plane mixed boundary value problem of the Laplacian. Math. Meth. Appl. Sci. *1*, 265–321 (1979).
[36] Verfürth, R.: A review of a posteriori error estimation and adaptive mesh-refinement techniques. Teubner Skripten zur Numerik. Stuttgart: B.G. Teubner 1996.
[37] Wendland, W.L.: On asymptotic error estimates for combined BEM and FEM. In: Finite and boundary element techniques from mathematical and engineering point of view (Stein, E., Wendland, W. L., eds.), pp. 273–331. New York: Springer 1988.
[38] Wendland, W. L., Yu, D.: Adaptive boundary element methods for strongly elliptic integral equations. Numer. Math. *53*, 539–558 (1988).
[39] Wendland, W. L., Yu, D.: A posteriori local error estimates of boundary element methods with some pseudo-differential equations on closed curves. J. Comput. Math. *10*, 273–289 (1992).

C. Carstensen
Mathematisches Seminar II
Christian-Albrechts-Universität zu Kiel
Ludwig-Meyn-Str. 4
D-24098 Kiel, Federal Republic of Germany
e-mail: cc@numerik.uni-kiel.de