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On the h-adaptive coupling of FE and BE for viscoplastic and elasto-plastic interface problems [1]

Carsten Carstensen [a,*], Darius Zarrabi [a], Ernst P. Stephan [b]

[a] *Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, D-24098 Kiel, FRG*

[b] *Institut für Angewandte Mathematik, Universität Hannover, D-30167 Hannover, FRG*

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Abstract

This paper presents a posteriori error estimates for the symmetric finite element and boundary element coupling for a nonlinear interface problem: A bounded body with a viscoplastic or plastic material behaviour is surrounded by an elastic body. The nonlinearity is treated by the finite element method while large parts of the linear elastic body are approximated using the boundary element method. Based on the a posteriori error estimates we derive an algorithm for the adaptive mesh refinement of the boundary elements and the finite elements. Its implementation is documented and numerical examples are included.

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1. Introduction

Within engineering computations, the boundary element method and the finite element method are well-established tools for the numerical approximation of real-life problems where analytical solutions are mostly unknown or available under unrealistic modelling only.

Both methods are somehow complementary: the finite element method seems to be more general and applicable to essential nonlinear problems while the boundary element method is restricted to certain linear problems with constant coefficients. On the other hand, the finite element method

* Corresponding author. E-mail: cc@numerik.uni-kiel.de.

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requires a bounded domain while the boundary element method models an unbounded exterior body as well. Since the work [23] by Zienkiewicz et al. the combination of the two methods is of advantage and, starting in the late seventies with papers by Johnson, Nedelec, Brezzi and others (cf. [21] for a summary and an extensive collection of relevant references) its mathematical justification reached a high level. The symmetric coupling due to Costabel appeared to be a proper tool and its convergence is guaranteed for problems with corners or for problems in quasi-linear elasticity. See, e.g., [7–9] with a material described in [18]. For strong nonlinearities as a plastic material behaviour [13, 19, 20] the convergence of the coupling is proved in [1–3].

A very important tool in Galerkin methods based on meshes or grids like the finite element method or the boundary element method is the adaptive mesh-refinement to reduce the approximation error. Various techniques are applied, some of them based on heuristic arguments, some of them based on a posteriori error estimates. In plasticity, for example, a mesh refinement is forced in plastic zones as pointed out in [15, 14].

This work extends recent work on the adaptive coupling of finite elements and boundary elements for linear or quasi-linear problems in [4, 5] to viscoplastic and plastic material. This makes engineering applications possible to geomechanics (deep tunnel with surrounding viscous material) or to local crack or support problems in huge shells as in Kirsch's problem where we model the real problem as a hole in an unbounded two-dimensional ductile material.

We stress that the coupling might be reasonable also if we have large parts of a body where a linear elastic material behaviour is good enough. Then, the coupling of boundary elements and finite elements is also possible as illustrated in an example given in Section 10.

This paper extends adaptive coupling techniques of FE and BE to the class of nonlinear interface problems which include viscoplastic and perfectly plastic material behaviour.

An outline of the paper is as follows. The interface problem is stated in Section 2 to fix notation and to give a precise example where the coupling of finite elements and boundary elements is advantageous. As a nonlinear material behaviour in a bounded domain, we consider viscoplasticity and plasticity and give an introduction in Section 2.1. For a convenient reading we briefly recall the definitions and a few properties of boundary integral operators of the first kind involved in the symmetric coupling in Section 3. Then, in Section 4, we introduce the complete coupled problem and its discretization is recalled in Section 5. The implementation of the nonlinear problem is reported in Section 6 while, in Section 7, we emphasize the regularization of the discontinuity in the stress–strain relation in viscoplasticity. The main contribution is the derivation of a posteriori error estimates where we can indeed follow [4, 5] in the viscoplastic case but have to take [14, 15] into account in case of perfect plasticity. Based on these a posteriori estimates we derive an adaptive feedback procedure in Section 9. Numerical examples are presented in Section 10 which illustrate the advantage of the coupling and underline the necessity of adaptive grid refinements in the numerical analysis.

2. The interface problem

In this section we consider the interior and the exterior problem and a reformulation of the latter using boundary integral operators. Throughout the paper, let $\Omega \subset \mathbb{R}^d$ where $d = 2$ or $d = 3$, be a bounded Lipschitz domain, the *interior domain*. Let $\Omega_c := \mathbb{R}^d \setminus \overline{\Omega}$ and let $\Gamma = \partial\Omega = \overline{\Omega} \cap \overline{\Omega}_c$ be partitioned into $\Gamma_1 \cup \Gamma_2$. We use the usual Sobolev spaces ([11, 17]) $H^s(\mathbb{R}^d, \mathbb{R}^d)$, $H^s(\Omega, \mathbb{R}^d)$ and

$$\begin{aligned}
&H^s_{\text{loc}}(\Omega_c, \mathbb{R}^d) := \{u : u \in H^s(\omega) \text{ for any } \omega \subset\subset \Omega_c\}\\
&H^s(\Gamma, \mathbb{R}^d) := \{u|_\Gamma : u \in H^{s+1/2}(\mathbb{R}^d, \mathbb{R}^d)\} \quad (s > 0)\\
&H^0(\Gamma, \mathbb{R}^d) := L_2(\Gamma, \mathbb{R}^d) \quad (s = 0)\\
&H^s(\Gamma, \mathbb{R}^d) := (H^{-s}(\Gamma, \mathbb{R}^d))^* \quad (s < 0)\\
&\tilde{H}^s(\Gamma_j, \mathbb{R}^d) := \{u \in H^{-s}(\Gamma_j, \mathbb{R}^d)\}^* \quad (s < 0)
\end{aligned}$$

with * denoting duality extending the L_2 scalar product. For brevity we use the notation

$$\mathscr{H} := H^1(\Omega, \mathbb{R}^d) \quad \text{and} \quad \mathscr{H}_u := \{u \in \mathscr{H} : u|_{\Gamma_1} = 0\}.$$

2.1. *The interior problem: viscoplasticity and perfect plasticity*

The basic step in the modelling of (small strain) elastoplasticity is the additive split of the total strain $\varepsilon(u)$. Given a displacement field $u \in \mathscr{H}_u$ we define

$$\varepsilon(u) := \operatorname{sym}\operatorname{grad} u = \left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)_{i,j=1}^{d} \in \mathbb{L}_2(\Omega) := L_2(\Omega, \mathbb{R}^{d\times d}_{\text{sym}})$$

where $\mathbb{R}^{d\times d}_{\text{sym}}$ denotes the set of real symmetric $d \times d$-matrices. In the announced additive split

$$\varepsilon(u) = \varepsilon^{\text{e}} + \varepsilon^{\text{p}}, \tag{1}$$

the elastic contribution is $\varepsilon^{\text{e}} = E\sigma$ with the inverse E of the linear elasticity operator,

$$E : \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}^{d\times d}_{\text{sym}}, \ \tau \mapsto \frac{1}{2\mu}\tau^{\text{D}} + c_d \cdot \operatorname{tr}\tau \cdot I_d. \tag{2}$$

Here λ and μ are the Lamé constants, $c_2 = 1/2(\lambda + \mu)$ and $c_3 = 1/3(3\lambda + 2\mu)$, I_d is the $d \times d$ unit matrix and

$$\sigma^{\text{D}} := \sigma - \frac{1}{d} \cdot \operatorname{tr}\sigma \cdot I_d \quad (\operatorname{tr}\sigma := \sigma_{11} + \cdots + \sigma_{dd})$$

is the deviatoric part of the stress tensor $\sigma \in \mathbb{R}^{d\times d}_{\text{sym}}$. As in linear elasticity, the stress field $\sigma \in \mathbb{L}_2(\Omega)$ satisfies local equilibrium conditions

$$\operatorname{div}\sigma + f = 0 \quad \text{in } \Omega, \tag{3}$$

$$\sigma \cdot n = g \quad \text{on } \Gamma_2, \tag{4}$$

where Γ_2 is the Neumann part of the boundary and we are given an applied volume force $f \in L^2(\Omega, \mathbb{R}^d)$ and an applied surface load $g \in \tilde{H}^{-1/2}(\Gamma_2, \mathbb{R}^d)$.

To describe the material law for ε^{p} in plasticity we need the concept of admissible stresses. In addition to (3) and (4), the stress σ has to satisfy the von Mises yield condition

$$|\sigma^{\text{D}}| \leqslant \sigma_{\text{Y}}, \tag{5}$$

where the yield stress $\sigma_Y > 0$ is a given material parameter and $|\cdot|$ denotes the Frobenius norm in $\mathbb{R}^{d\times d}_{\text{sym}}$, i.e., $|\sigma|^2 = \sum_{i,j=1}^{d} \sigma_{ij}$.

In this work we include viscoplasticity as a related material behaviour with a vicosity parameter $\varrho > 0$. If P is the set of all $\sigma \in \mathbb{R}^{d\times d}_{\text{sym}}$ which satisfy (5), the dissipation functionals are defined by

$$\varphi_\varrho(\sigma^D) = \frac{1}{2\varrho} \inf_{\tau\in P} |\sigma - \tau|^2 \quad (\varrho > 0), \tag{6}$$

$$\varphi_0(\sigma^D) = \begin{cases} 0 & \text{if } \sigma^D \in P \\ \infty & \text{if } \sigma^D \notin P \end{cases} \quad (\varrho = 0). \tag{7}$$

It can be proved that φ_ϱ tends to φ_0 in the sense of epigraphs as $\varrho \to 0$ [19], which justifies the notation φ_0. Given a dissipation functional φ_ϱ, the Prandtl–Reuß flow rule reads

$$\varepsilon^p \in \partial\varphi_\varrho(\sigma^D) \tag{8}$$

and links the plastic strain ε^p (see (1)) with the stress field. In (8) the subgradient is defined as

$$\partial\varphi_\varrho(\sigma) = \{\nabla\varphi_\varrho(\sigma)\} = \left\{\frac{1}{\varrho}(\sigma^D - \Pi\sigma^D)\right\} \quad (\varrho > 0), \tag{9}$$

$$\partial\varphi_0(\sigma) = \left\{\eta \in \mathbb{L}_2(\Omega) : \int_\Omega \eta : (\sigma - \tau)\,dx \geqslant 0 \ \forall\tau \in P\right\} \quad (\varrho = 0), \tag{10}$$

with the scalar product in $\mathbb{R}^{d\times d}_{\text{sym}}$ defined by

$$\sigma : \tau = \sum_{i,j=1}^{d} \sigma_{ij} \cdot \tau_{ij}.$$

We stress that (1), $\varepsilon^e = E\sigma$ and (8) give the stress–strain relation

$$\varepsilon(u) = E\sigma + \partial_\varrho(\sigma^D). \tag{11}$$

Remark 1. The condition (11) is one step of a time discretization of a time-dependent Prandtl–Reuß flow rule which is of some importance in engineering and material science to describe plastic deformations of ductile solids as metals. We refer to [19] for the setting and a discussion of related references.

The stress–strain relation can be inverted.

Theorem 2. *For $\varrho > 0$, there exists a bijective operator A_ϱ mapping $\mathbb{L}_2(\Omega)$ onto $\mathbb{L}_2(\Omega)$ defined by*

$$\tau = A_\varrho\eta \iff \eta \in E\tau + \partial\varphi_\varrho(\tau^D). \tag{12}$$

A_ϱ is uniformly bounded and uniformly monotone, i.e, there exist ϱ-dependent constants $\alpha, \beta > 0$ such that

$$\alpha\|\xi - \eta\|^2_{\mathbb{L}_2(\Omega)} \leqslant \int_\Omega (A_\varrho\xi - A_\varrho\eta) : (\xi - \eta)\,dx \leqslant \beta\|\xi - \eta\|^2_{\mathbb{L}_2(\Omega)} \quad (\eta, \xi \in \mathbb{L}_2(\Omega)). \tag{13}$$

For (2), (9) *and* $\varrho \in [0,\infty]$ *we have*

$$A_\varrho \eta = \frac{\varrho}{1+\frac{\varrho}{2\mu}}\eta^{\mathrm{D}} + \frac{1}{1+\frac{\varrho}{2\mu}}\Pi\left(2\mu\eta^{\mathrm{D}}\right) + \frac{3\lambda+2\mu}{3}\cdot \operatorname{tr}\eta \cdot I_d. \tag{14}$$

Proof. By changing the norm in the Hilbert space $\mathbb{L}_2(\Omega)$ we may and will assume for this proof that $E = I$. By a standard result from convex analysis $\partial\varphi_\varrho$ is maximal monotone: for any given $\eta \in \mathbb{L}_2(\Omega)$ there exists a unique $\sigma \in \mathbb{L}_2(\Omega)$ satisfying $\eta \in \sigma + \partial\varphi_\varrho(\sigma)$. Thus, A_ϱ is well defined. For $\varrho > 0$, the subdifferential reduces to the gradient and A_ϱ is bijective.

To prove boundedness of A_ϱ we set $\sigma = A_\varrho(\xi)$, $\tau = A_\varrho(\eta)$ and estimate

$$\begin{aligned}\int_\Omega (A_\varrho\xi - A_\varrho\eta) : (\xi-\eta)\,\mathrm{d}x &= \int_\Omega (\sigma-\tau) : (A_\varrho^{-1}\sigma - A_\varrho^{-1}\tau)\,\mathrm{d}x \\ &\leqslant (1+\varrho^{-1})\|\sigma-\tau\|^2_{\mathbb{L}_2(\Omega)}\end{aligned} \tag{15}$$

where we have used Cauchy's inequality and, by Lipschitz continuity

$$\|\xi-\eta\|_{\mathbb{L}_2(\Omega)} \leqslant \left(1+\frac{1}{\varrho}\right)\|\sigma-\tau\|_{\mathbb{L}_2(\Omega)}. \tag{16}$$

According to the monotonicity of $\partial\varphi_\varrho$,

$$\int_\Omega (A_\varrho\xi - A_\varrho\eta) : (\xi-\eta)\,\mathrm{d}x \geqslant \|\sigma-\tau\|^2_{\mathbb{L}_2(\Omega)} \tag{17}$$

and so (16) implies

$$\int_\Omega (A_\varrho\xi - A_\varrho\eta) : (\xi-\eta)\,\mathrm{d}x \geqslant \left(1+\frac{1}{\varrho}\right)^{-2}\|\xi-\eta\|^2_{\mathbb{L}_2(\Omega)}. \tag{18}$$

Combining (17) with (15) and using Cauchy's inequality again, we obtain

$$\|\sigma-\tau\|_{\mathbb{L}_2(\Omega)} \leqslant \left(1+\frac{1}{\varrho}\right)^{-1}\|\xi-\eta\|_{\mathbb{L}_2(\Omega)},$$

which finishes the proof of (13).

The representation of A_ϱ in (14) can be checked by hand utilizing that

$$\Pi_P\sigma = \begin{cases} \sigma & \text{for } \sigma \in P, \\ \sigma_{\mathrm{Y}}\dfrac{\sigma}{|\sigma^{\mathrm{D}}|} & \text{for } \sigma \notin P \end{cases}. \qquad \square$$

The weak formulation of the problem under consideration is obtained as follows: multiplying $\operatorname{div}\sigma + f = 0$ with a test function $v \in \mathscr{H}_u$, integrating $v \cdot \operatorname{div}\sigma$ over Ω and using Green's formula results in the weak form. Given $f \in L_2(\Omega, \mathbb{R}^d)$, $g \in \tilde{H}^{-1/2}(\Gamma_2, \mathbb{R}^d)$ and $\varrho \in (0,\infty]$, find $u \in \mathscr{H}_u$ such that, for all $v \in \mathscr{H}_u$,

$$\int_\Omega (A_\varrho\varepsilon(u)) : \varepsilon(v)\,\mathrm{d}x = \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_2} g\cdot v|_\Gamma\,\mathrm{d}s \tag{19}$$

Theorem 3. *There exists a unique* $u \in \mathscr{H}_u$ *to* (19).

Proof. According to Theorem 2, $A_\varrho : \mathbb{L}_2(\Omega) \to \mathbb{L}_2(\Omega)$ is uniformly monotone and (according to (14)) Lipschitz continuous. By Korn's inequality this holds for the operator $\mathscr{H}_u \to \mathscr{H}_u^*$ given by the left-hand side of (19) as well. Then standard results in the theory of monotone operators (as, e.g., in [22]) conclude the proof. □

2.2. The exterior problem: linear elasticity

In $\Omega_c = \mathbb{R}^d \backslash \overline{\Omega}$, the *exterior domain*, we have linear elastic material modelled by the homogeneous Navier–Lamé equations of linearized elasticity

$$-\Delta^* u_c := \mu_c \Delta u_c + (\lambda_c + \mu_c)\,\mathrm{grad}\,\mathrm{div}\,u_c = 0 \quad \text{in } \Omega_c. \tag{20}$$

For the displacement field u_c we assume regularity at infinity (see, e.g., [16]), i.e., u_c satisfies a radiation condition of the form

$$u_c = \mathrm{O}(|x|^{-1}) \quad \text{and} \quad \nabla u_c = \mathrm{O}(|x|^{-2}) \quad \text{as } |x| \to \infty. \tag{21}$$

Let $T_c(u_c)$ denote the *conormal derivative* related to the operator $-\Delta^*$ which is, in its strong form, defined by

$$T_c(u_c) = 2\mu_c \partial_n u_c + \lambda_c n\,\mathrm{div}\,u_c + \mu_c n \times \mathrm{curl}\,u_c$$

where ∂_n denotes the normal derivative and n is the unit normal on Γ pointing into Ω_c. The weak form of T_c is defined by Green's formula (cf., e.g., [7]).

2.3. The interface problem

The transmission problem under consideration in this paper combines the interior and exterior problems. With the operator A_ϱ introduced in Theorem 2 the equilibrium (3) reads

$$\mathrm{div}(A_\varrho \varepsilon(u)) + f = 0 \quad \text{in } \Omega. \tag{22}$$

The two displacements are coupled on the interface $\Gamma = \overline{\Omega} \cap \overline{\Omega}_c$ where we have, in the simplest case, continuity of the displacements and equilibrium of the tractions, i.e.,

$$u|_\Gamma = u_c|_\Gamma \quad \text{and} \quad (A_\varrho \varepsilon(u)) \cdot n = T_c(u_c) \quad \text{on } \Gamma. \tag{23}$$

Definition 4. Given $f \in L_2(\Omega, \mathbb{R}^d)$ the transmission problem $(\mathrm{TP})_\varrho$ consists in finding $(u, u_c) \in \mathscr{H} \times H^1_{\mathrm{loc}}(\Omega_c, \mathbb{R}^d)$ satisfying (11), (3), (20)–(23) and $A_\varrho \varepsilon(u) \cdot n = g$ on Γ_2.

3. Boundary integral operators

For convenient reading we briefly summarize well-known properties of some integral operators related to Δ^*. For proofs and details we refer, e.g., to [12, 16, 21].

The fundamental solution for the Lamé operator $-\Delta^*$ has a kernel $G(x,y)$, the *Kelvin matrix*, for $d=2,3$

$$G(x,y)=\frac{\lambda_c+3\mu_c}{4\pi\mu_c(\lambda_c+2\mu_c)}\left\{\log\frac{1}{|x-y|}I_2+\frac{\lambda_c+\mu_c}{\lambda_c+3\mu_c}\frac{(x-y)(x-y)^T}{|x-y|^2}\right\},$$

$$G(x,y)=\frac{\lambda_c+3\mu_c}{8\pi\mu_c(\lambda_c+2\mu_c)}\left\{\frac{1}{|x-y|}I_3+\frac{\lambda_c+\mu_c}{\lambda_c+3\mu_c}\frac{(x-y)(x-y)^T}{|x-y|^3}\right\}.$$

Since G is analytic in $\mathbb{R}^d\times\mathbb{R}^d$ without the diagonal, the traction may be defined by

$$T(x,y):=T_{c,y}(G(x,y))^T \quad \text{for } x\neq y.$$

Proposition 5 (Hsiao and Wendland [12]). *Each solution* $u_c\in H^1_{loc}(\Omega_c,\mathbb{R}^d)$ *of* (20), (21) *satisfies the Betti representation formula*

$$u_c(x)=\int_\Gamma T(x,y)\cdot v(y)\,ds_y-\int_\Gamma G(x,y)\cdot\phi(y)\,ds_y \quad (x\in\Omega_c) \tag{24}$$

with $v=u|_\Gamma$ *and* $\phi=T_c(u_c)$. □

For any $x\in\Omega_c$, (24) can be differentiated, giving a representation for the stresses $T_c(u_c)$. If the classical jump relations for $x\to\Gamma$ are taken into account, the following identities hold for a piecewise smooth boundary (see, e.g., [7]).

$$\begin{pmatrix} u|_\Gamma \\ T_c(u_c)\end{pmatrix}=\begin{bmatrix}\frac{1}{2}+K & -V\\ -W & \frac{1}{2}-K'\end{bmatrix}\cdot\begin{pmatrix} u|_\Gamma \\ T_c(u_c)\end{pmatrix}, \tag{25}$$

where

$$(V\phi)(x)=\int_\Gamma G(x,y)\cdot\phi\,ds_y, \qquad (Kv)(x)=\int_\Gamma T(x,y)\cdot v\,ds_y,$$

$$(K'\phi)(x)=T_{c,x}\int_\Gamma G(x,y)\cdot\phi\,ds_y, \qquad (Wv)(x)=-T_{c,x}\int_\Gamma T(x,y)\cdot v\,ds_y.$$

The next result recalls some mapping properties of the above boundary integral operators where $\mathscr{L}(X,Y)$ denotes the real Banach space of bounded linear operators mapping X into Y. Let $H^{1/2}=H^{1/2}(\Gamma,\mathbb{R}^d)$ and $H^{-1/2}=H^{-1/2}(\Gamma,\mathbb{R}^d)$.

Lemma 6 (Costabel [6] and Costabel and Stephan [7]). *We have*

$$V\in\mathscr{L}\left(H^{-1/2};H^{1/2}\right), \qquad K\in\mathscr{L}\left(H^{1/2};H^{1/2}\right),$$

$$K'\in\mathscr{L}\left(H^{-1/2};H^{-1/2}\right), \qquad W\in\mathscr{L}\left(H^{1/2};H^{-1/2}\right).$$

Moreover, W and V are symmetric, the hypersingular operator W is positive semi-definite and its kernel consists of the rigid body motions. The single layer potential V is positive definite for $d=3$. Finally, the double layer potential K has the dual operator K'. □

4. Equivalent form of the interface problem

The transmission problem may be reformulated using boundary integral operators. Let the mapping $B : (\mathscr{H} \times H^{-1/2})^2 \to \mathbb{R}$ and the linear form $L : \mathscr{H} \times H^{-1/2} \to \mathbb{R}$ be defined by

$$B_\varrho\left(\begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \psi \end{pmatrix}\right) := \int_\Omega (A_\varrho \varepsilon(u)) : \varepsilon(v)\,\mathrm{d}x + \int_\Gamma Wu|_\Gamma \cdot v|_\Gamma\,\mathrm{d}s$$
$$+ \int_\Gamma (K' - 1/2)\phi \cdot v|_\Gamma\,\mathrm{d}s + \int_\Gamma V\phi \cdot \psi\,\mathrm{d}s + \int_\Gamma (1/2 - K)u|_\Gamma \cdot \psi\,\mathrm{d}s$$

$$L\left(\begin{pmatrix} v \\ \psi \end{pmatrix}\right) := \int_\Omega f \cdot v\,\mathrm{d}x.$$

Definition 7. Given $f \in L_2(\Omega, \mathbb{R}^d)$ problem $(P)_\varrho$ consists in finding $(u, \phi) \in \mathscr{H} \times H^{-1/2}$ satisfying

$$B_\varrho((u, \phi), (v, \psi)) = L((v, \psi)) \tag{26}$$

for all $(v, \psi) \in \mathscr{H} \times H^{-1/2}$, respectively.

For $\varrho > 0$, the problems $(\mathrm{TP})_\varrho$ and $(P)_\varrho$ are equivalent, as proved, e.g. in [1, 2, 4, 5, 7] for similar problems.

Theorem 8 (Carstensen and Stephan [5]). *Assuming $\varrho > 0$, the problems $(\mathrm{TP})_\varrho$ and $(P)_\varrho$ are equivalent in the following sense: if $(u, u_c) \in H^1(\Omega, \mathbb{R}^d) \times H^1_{\mathrm{loc}}(\Omega_c, \mathbb{R}^d)$ is a solution of $(\mathrm{TP})_\varrho$ then $(u, \phi) \in H^1(\Omega, \mathbb{R}^d) \times H^{-1/2}(\Gamma, \mathbb{R}^d)$ solves $(P)_\varrho$ with $\phi = T_c(u_c)$. If, conversely, (u, ϕ) is a solution of problem $(P)_\varrho$ then (u, u_c) solves $(\mathrm{TP})_\varrho$ with $u_c \in H^1_{\mathrm{loc}}(\Omega_c, \mathbb{R}^d)$ defined by the representation formula* (24).

For $\varrho > 0$, problem $(P)_\varrho$ has a unique solution. The proof in [7] is based on the fact that (26) are the Euler–Lagrange equations of a $\mathscr{C}^1$-functional with a unique saddle point (for a different proof, see [5]).

5. Discretization

Let $\mathscr{T}_h = \{\Delta\}_{\Delta \in \mathscr{T}_h}$ be a regular finite element subdivision of the polygonal domain Ω into nonoverlapping tetrahedrons Δ of diameter $h_\Delta \leqslant h$ and let $\mathbb{P}_k(D, \mathbb{R}^m)$ denote the set of polynomials on a domain D of degree at most k with values in $\mathbb{R}^m$. Define

$$\begin{aligned} \mathscr{H}_h &:= \{u \in \mathscr{C}(\Omega, \mathbb{R}^d) : u|_\Delta \in \mathbb{P}_1(\Delta, \mathbb{R}^d) \quad \forall \Delta \in \mathscr{T}_h\}, \\ H_h^{-1/2} &:= \left\{\phi \in L_\infty(\Gamma, \mathbb{R}^d) : \phi|_{\bar{\Delta} \cap \Gamma} \in \mathbb{P}_0(\bar{\Delta} \cap \Gamma, \mathbb{R}^d) \quad \forall \Delta \in \mathscr{T}_h\right\}, \\ \mathbb{L}_h^2 &:= \left\{\sigma \in L_\infty(\Omega, \mathbb{R}^{d \times d}_{\mathrm{sym}}) : \sigma|_\Delta \in \mathbb{P}_0(\Delta, \mathbb{R}^{d \times d}_{\mathrm{sym}}) \quad \forall \Delta \in \mathscr{T}_h\right\}. \end{aligned} \tag{27}$$

Then, the symmetric coupling of finite elements and boundary elements is the following Galerkin procedure.

Definition 9. Problem $(P)_{\varrho,h}$ consists in finding $(u_h,\phi_h)\in\mathscr{H}_h\times H_h^{-1/2}$ such that, for all $(v_h,\psi_h)\in\mathscr{H}_h\times H_h^{-1/2}$,

$$B_\varrho((u_h,\phi_h),(v_h,\psi_h)) = L((v_h,\psi_h)). \tag{28}$$

For the discrete subspaces introduced above there holds the approximation property (i.e., the best approximation error, e.g., in the right-hand side of (29) below tends to zero as $h\to 0$) . Therefore the next a priori estimate shows quasi-optimality of the Galerkin scheme (28). In particular, we get convergence of the algorithm.

Theorem 10 (Carstensen and Stephan [5]). *Suppose $\varrho>0$. Then there exist constants $c_0>0$ and $h_0>0$ such that for any $h<h_0$ the problem $(P)_{\varrho,h}$ has a unique solution (u_h,ϕ_h) and, if (u,ϕ) denotes the solution of $(P)_\varrho$, then*

$$\|(u-u_h,\phi-\phi_h)\| \leqslant c_0\cdot \inf_{\binom{v_h}{\psi_h}\in\mathscr{H}_h\times H_h^{-1/2}} \|(u-v_h,\phi-\psi_h)\|, \tag{29}$$

where the error is measured in the norm of the space $\mathscr{H}\times H^{-1/2}$.

6. Implementation

In this section we describe some details of the implementation of method (28). Let $\chi_1,\ldots,\chi_N$ be a (nodal) basis of $\mathscr{H}_h$ and write $u_h := \sum_{i=1}^N x_i\chi_i$, with $\boldsymbol{x}=(x_1,\ldots,x_N)^{\mathrm{T}}\in\mathbb{R}^N$. Similarly, let $\phi_h := \sum_{i=1}^M y_i\eta_i$ for a basis $\eta_1,\ldots,\eta_M$ of $H_h^{-1/2}$, $\boldsymbol{y}=(y_1,\ldots,y_M)^{\mathrm{T}}\in\mathbb{R}^M$.

We define functions $\vartheta,\theta_\varrho:[0,\infty)\to\mathbb{R}$ and the functional $F_\varrho:\mathscr{H}\to\mathbb{R}$ by

$$\vartheta(x) = \begin{cases} \mu x & \text{if } x\leqslant(\sigma_{\mathrm{Y}}/2\mu)^2, \\ \sigma_{\mathrm{Y}}\sqrt{x}-\dfrac{\sigma_{\mathrm{Y}}^2}{4\mu} & \text{if } x>(\sigma_{\mathrm{Y}}/2\mu)^2, \end{cases} \tag{30}$$

$$\theta_\varrho(x) = \frac{1}{1+\frac{\varrho}{2\mu}}\vartheta(x) + \frac{\varrho\cdot x}{1+\frac{\varrho}{2\mu}}, \tag{31}$$

$$F_\varrho(u) = \int_\Omega \theta_\varrho(|\varepsilon^{\mathrm{D}}(u)|^2) + \frac{3\lambda+2\mu}{6}\cdot\operatorname{tr}^2\varepsilon(u)\,\mathrm{d}x. \tag{32}$$

Since θ_ϱ is continuously differentiable, F_ϱ admits a Gâteaux derivative. Therefore, we may write (28) as

$$\overrightarrow{DF_\varrho}\,(u_h) + \mathscr{A}\cdot(u_h,\phi_h) - \boldsymbol{b} = 0 \tag{33}$$

with

$$\mathscr{A} = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{K} - 1/2 \\ (\boldsymbol{K} - 1/2)^{\mathrm{T}} & -\boldsymbol{V} \end{bmatrix},$$

$$\overrightarrow{DF_\varrho}\,(u_h) = \Big(DF_\varrho(u_h,\chi_1),\ldots,DF_\varrho(u_h,\chi_N),\underbrace{0,\ldots,0}_{M\ \text{times}}\Big)^{\mathrm{T}},$$

$$\boldsymbol{b} = \Big(\int_\Omega f\cdot\chi_1\,\mathrm{d}x,\ldots,\int_\Omega f\cdot\chi_N\,\mathrm{d}x,\underbrace{0,\ldots,0}_{M\ \text{times}}\Big)^{\mathrm{T}},$$

$$\boldsymbol{V} = \Big(\int_\Gamma \eta_j(\zeta)\int_\Gamma G(\zeta,\xi)\cdot\eta_i(\xi)\,\mathrm{d}s_\xi\,\mathrm{d}s_\zeta\Big)_{i,j=1}^{M},$$

$$\boldsymbol{K} - 1/2 = \Big(\int_\Gamma \eta_j(\zeta)\int_\Gamma T(\zeta,\xi)\chi_i(\xi)|_\Gamma\,\mathrm{d}s_\xi\,\mathrm{d}s_\zeta - \frac{1}{2}\int_\Gamma \eta_j(\xi)\chi_i(\xi)|_\Gamma\,\mathrm{d}s_\xi\Big)_{i,j=1}^{N,M}, \tag{34}$$

$$\boldsymbol{W} = \Big(-\int_\Gamma \chi_j(\zeta)\cdot T_{\mathrm{c},x}\int_\Gamma T(\zeta,\xi)\cdot\chi_i(\xi)\,\mathrm{d}s_\xi\,\mathrm{d}s_\zeta\Big)_{i,j=1}^{N},$$

$$T(x,y) = \frac{\mu}{2\pi(\lambda+2\mu)}\left\{\left[I + \frac{2(\lambda+\mu)}{\mu}\frac{(x-y)(x-y)^{\mathrm{T}}}{|x-y|^2}\right]\frac{\partial}{\partial n_y} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial s_y}\right\}\log|x-y| \quad (d=2),$$

$$DF_\varrho(u,\chi_j) = \int_\Omega 2\theta_\varrho'(|\varepsilon^{\mathrm{D}}(u)|^2)\cdot\varepsilon^{\mathrm{D}}(u) : \varepsilon^{\mathrm{D}}(\chi_j) + \frac{3\lambda+2\mu}{3}\cdot\operatorname{tr}\varepsilon(u)\cdot\operatorname{tr}\varepsilon(\chi_j)\,\mathrm{d}x$$

using the Kelvin matrix G as given in Section 3. The entries of the stiffness matrix of the hypersingular operator can be reduced to the terms appearing in the integration of the matrix $\boldsymbol{V}$ (see, e.g., Appendix of [10]). It is emphasized that those trial functions which are not related to nodes on Γ vanish on the boundary and the corresponding matrix entries are 0.

Remark 11. The stiffness matrices $\boldsymbol{V}$, $\boldsymbol{K}$ and $\boldsymbol{W}$ have to be computed only once while the nonlinear part requires an iterative process.

7. Regularization

In [14] a Picard-type iteration is proposed where a sequence (u_h^ν) is defined by

$$(G_\varrho(u_h^{\nu-1},u_h^\nu,\chi_1),\ldots,G_\varrho(u_h^{\nu-1},u_h^\nu,\chi_N),0,\ldots,0) + \mathscr{A}\cdot(u_h^\nu,\phi_h^\nu) - \boldsymbol{b} = 0, \tag{35}$$

$$G_\varrho(u_h^{\nu-1},u_h^\nu,\chi_j) = \int_\Omega 2\theta_\varrho'(|\varepsilon^{\mathrm{D}}(u_h^{\nu-1})|^2)\cdot\varepsilon^{\mathrm{D}}(u_h^\nu) : \varepsilon^{\mathrm{D}}(\chi_j) + \frac{3\lambda+2\mu}{3}\operatorname{tr}\varepsilon(u_h^\nu)\cdot\operatorname{tr}\varepsilon(\chi_j)\,\mathrm{d}x,$$

cf. (32) and (33). Although the function θ_ϱ from (31) is not twice differentiable, a formal application of the second Gâteaux differential results in

$$D^2F_\varrho(u,v,w) = \int_\Omega 2\theta'_\varrho(|\varepsilon^{\mathrm{D}}(u)|^2)\cdot\varepsilon^{\mathrm{D}}(w):\varepsilon^{\mathrm{D}}(v) + 4\theta''_\varrho(|\varepsilon^{\mathrm{D}}(u)|^2)\left((\varepsilon^{\mathrm{D}}(u)\otimes\varepsilon^{\mathrm{D}}(u)):\varepsilon^{\mathrm{D}}(w)\right):\varepsilon^{\mathrm{D}}(v) + \frac{3\lambda+2\mu}{3}\mathrm{tr}\,\varepsilon(w)\mathrm{tr}\,\varepsilon(v)\,\mathrm{d}x, \tag{36}$$

where $\otimes$ denotes the tensorial product $\tau\otimes\sigma=(\tau_{ij}\sigma_{kl})_{ijkl}$.

In order to apply Newton–Raphson's method, we regularize the function ϑ as depicted in Fig. 1 (details are given in the appendix). This regularization technique allows usage of the Hessian

$$\boldsymbol{H}^\delta_\varrho(v_h) = \begin{bmatrix} \boldsymbol{L}+\boldsymbol{W} & \boldsymbol{K}-1/2 \\ (\boldsymbol{K}-1/2)^{\mathrm{T}} & -\boldsymbol{V} \end{bmatrix} \tag{37}$$

with the tangential stiffness matrix $\boldsymbol{L}=\left(D^2F^\delta_\varrho(v_h,\chi_i,\chi_j)\right)_{i,j=1}^N$.

Then, Newton–Raphson's method is as follows:

(i) *Initialize* $\nu=0,\ \delta=1$.

(ii) *Solve the linear elastic interface problem: compute solution of* $\overrightarrow{DF^0_\infty}(u^0_h)+\mathscr{A}\cdot(u^0_h,\phi^0_h)=\boldsymbol{b}$.

(iii) *If* $\delta>10^{-12}$ *then replace* δ *by* $\delta/50$.

(iv) *Increment* ν.

(v) *Solve the linear system of equations*

$$\boldsymbol{H}^\delta_\varrho(u^{\nu-1}_h)\cdot(u,\phi)=\overrightarrow{DF^0_\varrho}(u^{\nu-1}_h)+\mathscr{A}\cdot(u^{\nu-1}_h,\phi^{\nu-1}_h)-\boldsymbol{b}.$$

(vi) *Set* $(u^\nu_h,\phi^\nu_h)=(u^{\nu-1}_h,\phi^{\nu-1}_h)-(u,\phi)$.

(vii) *If* $\|\overrightarrow{DF^\delta_\varrho}(u^\nu_h)+\mathscr{A}\cdot(u^\nu_h,\phi^\nu_h)-b\|_{\mathbb{R}^{N+M}}<10^{-13}$ *then STOP and accept* (u^ν_h,ϕ^ν_h) *as discrete solution, else goto* (iii).

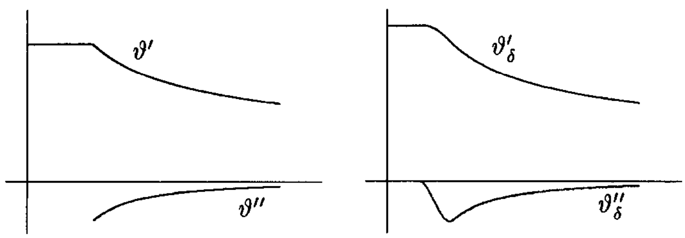

Fig. 1. The functions ϑ', ϑ'' and their regularizations.

8. A posteriori error estimates

Let $\mathcal{T}_h$ be a partition of the two dimensional polygon Ω and let $\mathcal{S}_h$ denote the set of sides split into boundary and interior sides,

$$\mathcal{S}_h = \{E : E \text{ is a side of } \Delta \in \mathcal{T}_h\},$$

$$\mathcal{G}_h = \{E \in \mathcal{S}_h : E \subset \Gamma\} \quad \text{and} \quad \mathcal{S}_h^0 = \mathcal{S}_h \backslash \mathcal{G}_h.$$

The angles of any $\Delta \in \mathcal{T}_h$ are bounded below by some $\Theta > 0$. With h we identify the piecewise constant function on $\mathcal{T}_h$ such that $h|_\Delta = \operatorname{diam}(\Delta)$ for $\Delta \in \mathcal{T}_h$ and $h|_E = \operatorname{diam}(E)$ for $E \in \mathcal{S}_h$.

Given an interior side $E \in \mathcal{S}_h^0$ such that Δ and Δ' are the neighbour triangles, let n_E denote that unit normal to E which points from Δ into Δ' and for a given tensor τ let $[\tau \cdot n_S]$ be the jump across the side E.

The following result is taken from [5] and applies to viscous materials where $\varrho > 0$. Given a solution (u_h, ϕ_h) of (28) let $\sigma_h := A_\varrho(\varepsilon(u_h))$ and

$$a_\Delta^2 = \|h_\Delta \cdot (f + \operatorname{div} \sigma_h)\|_{L_2(\Delta)}^2 + \sum_{E \in \mathcal{S}_h^0, E \subset \partial\Delta} \|\sqrt{h_E} \cdot |[\sigma_h \cdot n]|\|_{L_2(E)}^2$$
$$+ \|h_{\partial\Delta} \cdot (\sigma_h \cdot n + W u_h + (K' - 1/2)\phi_h)\|_{L_2(\Gamma \cap \partial\Delta)}^2$$
$$b_E = \|\sqrt{h_E} \cdot \frac{\partial}{\partial s}\{(K - 1/2)u_h - V\phi_h\}\|_{L_2(E)},$$

where $\partial/\partial s$ denotes derivative with respect to the arc-length along E.

Theorem 12 (Carstensen and Stephan [5]). *Let (u, ϕ) and (u_h, ϕ_h) solve problem $(P)_\varrho$ and $(P)_{\varrho,h}$ with $\varrho > 0$, respectively. Then there exist constants c_ϱ, $h_0 > 0$ such that for any $h < h_0$ there holds*

$$\|(u - u_h, \phi - \phi_h)\|_{\mathcal{H} \times H^{-1/2}} \leqslant c_\varrho \cdot \left(\sqrt{\sum_{\Delta \in \mathcal{T}_h} a_\Delta^2} + \sum_{E \in \mathcal{G}_h} b_E \right). \tag{38}$$

Remark 13. The constant c_ϱ tends to infinity as $\varrho \to 0$. Theorem 12 gives an a posteriori error estimate, although c_ϱ is difficult to compute. Nevertheless, (38) justifies an adaptive mesh refinement algorithm as introduced in Section 9.

In order to obtain an a posteriori error estimate for perfectly plastic materials we recall that the elasticity operator is positive definite on $\mathbb{R}^{2\times 2}_{\text{sym}}$ and introduce a norm on this space by

$$\|\tau\|_E^2 = \int_\Omega E\tau : \tau \,\mathrm{d}x = \int_\Omega \frac{1}{2\mu}\tau^{\mathrm{D}} : \tau^{\mathrm{D}} + \frac{1}{2(\lambda + \mu)} \operatorname{tr}^2 \tau \,\mathrm{d}x$$

which is equivalent to the initial norm. Given σ_h, which may be obtained via some post-processing, the triangulation $\mathcal{T}_h$ of Ω splits into an elastic part Ω^{e} and a plastic part Ω^{p} defined by

$$\Omega^{\mathrm{e}} = \Omega_\varrho^{\mathrm{e}} \cap \Omega_{\varrho,h}^{\mathrm{e}} \quad \text{and} \quad \Omega^{\mathrm{p}} = \Omega \backslash \Omega^{\mathrm{e}}, \tag{39a}$$

where

$$\Omega^{\mathrm{e}}_{\varrho} := \left\{ \Delta \in \mathscr{T}_h : |\sigma^{\mathrm{D}}_{\varrho}(x)| \leqslant \sigma_{\mathrm{Y}} \ \forall x \in \Delta \right\}, \tag{39b}$$

$$\Omega^{\mathrm{e}}_{\varrho,h} := \left\{ \Delta \in \mathscr{T}_h : |\sigma^{\mathrm{D}}_{\varrho,h}(x)| \leqslant \sigma_{\mathrm{Y}} \ \forall x \in \Delta \right\}. \tag{39c}$$

Let $C^s = \|\varepsilon(u)\|_{L_1(\Omega^{\mathrm{p}})} + \|\varepsilon(u_h)\|_{L_1(\Omega^{\mathrm{p}})}$,

$$E_1 = \operatorname{div} \sigma_h + f$$

$$E_2 = \begin{cases} [\sigma_h \cdot n] & \text{on } E \in \mathscr{S}^0_h, \\ g - \sigma_h \cdot n & \text{on } E \in \mathscr{G}_h \end{cases}$$

and consider residuals

$$a^2_{\Delta} = \begin{cases} \|h \cdot E_1\|^2_{L_2(\Delta)} + \sum\limits_{E \in \mathscr{S}^0_h, E \subset \partial\Delta} \|\sqrt{h} \cdot E_2\|^2_{L_2(E)}, & \Delta \in \Omega^{\mathrm{e}} \\ C^s \cdot \|h \cdot E_1\|_{L_\infty(\Delta)} + C^s \cdot \sum\limits_{E \in \mathscr{S}^0_h, E \subset \partial\Delta} \|\sqrt{h} \cdot E_2\|_{L_\infty(E)}, & \Delta \in \Omega^{\mathrm{p}} \end{cases}$$
$$+ \|\sqrt{h} \cdot (\sigma_h \cdot n + W u_h + (K' - 1/2)\phi_h)\|^2_{L_2(\Gamma \cap \partial\Delta)}.$$

Theorem 14. *Let $(\sigma, u, \phi) \in \mathbb{L}_2(\Omega) \times \mathscr{H} \times H^{-1/2}$ solve problem $(P)_0$ and let $(\sigma_h, u_h, \phi_h) \in \mathbb{L}_2(\Omega) \times \mathscr{H} \times H^{-1/2}$ solve problem $(P)_{0,h}$. Then there exists a constant $c > 0$ such that for $h < h_0$*

$$\|\sigma - \sigma_h\|_{\mathbb{L}_2(\Omega)} + \|(u - u_h)|_\Gamma\|_{H^{1/2}} + \|\phi - \phi_h\|_{H^{-1/2}} \leqslant c \cdot \left(\sqrt{\sum_{\Delta \in \mathscr{T}_h} a^2_{\Delta}} + \sum_{E \in \mathscr{G}_h} b_E \right). \tag{40}$$

Proof. We combine the assertion of Theorem 12 and a result from [14]. Assuming $\Omega^{\mathrm{p}} \subset\subset \Omega$ we deduce as in [14]

$$\|\sigma - \sigma_h\|_{E,\Omega^{\mathrm{p}}} \leqslant C^s \cdot \sum_{\Delta \in \Omega^{\mathrm{p}}} \|h \cdot E_1\|_{L_\infty(\Delta)} + \|\sqrt{h} \cdot E_2\|_{L_\infty(\partial\Delta)}.$$

The assertion follows by applying the notion of Theorem 12 to Ω^{e} and using that

$$\|\sigma - \sigma_h\|_{\mathbb{L}_2(\Omega)} + \|(u - u_h)|_\Gamma\|_{H^{1/2}} + \|\phi - \phi_h\|_{H^{-1/2}}$$
$$\leqslant c \cdot \|(u - u_h, \phi - \phi_h)\|_{\mathscr{H} \times H^{-1/2}}$$

by equivalence of norms and positive definiteness of E. □

Remark 15. Note that (40) is not a full a posteriori error estimate since Ω^{e} and C^s depend on u. In addition, we emphasize that under a safe load assumption it may be proved that C^s is bounded but tends to infinity as the *limit load* is approached (*cf*, e.g., [20]).

9. Adaptive feedback procedure

Assume that there is given a triangulation $\mathcal{T}_h = \{\Delta_1, \dots, \Delta_N\}$ of Ω with a related partition $\mathcal{G}_h = \{\Gamma_1, \dots, \Gamma_M\}$ of the plane polygon Γ, i.e., $\mathcal{G}_h$ is induced by $\mathcal{T}_h$. Considering one element $\Delta \in \mathcal{T}_h$ we can compute its contributions a_Δ, b_E to the right-hand side of the a posteriori error estimate in Theorem 12. Provided $\mathcal{G}_h$ is induced by the subdivision $\mathcal{T}_h$ of Ω, let

$$c_\Delta = a_\Delta + \sum_{E \in \mathcal{G}_h \cap \partial\Delta} b_E.$$

Note that the sum above is either zero or consists of one or two summands. The meshes in the numerical examples are steered by the following algorithm.

Algorithm (A): Given some mesh and a global parameter $0 \leqslant \theta \leqslant 1$ refine the mesh successively by halving some elements due to the following rule. Divide some element Δ by halving its largest side if

$$c_\Delta \geqslant \theta \cdot N^{-1/2} \cdot \left(\sum_{\Delta \in \mathcal{T}_h} c_\Delta^2 \right)^{1/2}. \tag{41}$$

In a subsequent step all hanging nodes are avoided by a further refinement to get a regular mesh.

Remark 16. Note that $\theta = 0$ in Algorithm (A) implies a uniform refinement of the mesh. The parameter θ steers the refinement: θ small gives a more overall refinement whereas $\theta \approx 1$ refines a small region only.

10. Numerical examples

We report on two two-dimensional model problems of the form studied above. In the first example we consider an unbounded plate with a rectangular hole under uniform pressure $p = 1.5$ as seen in Fig. 2 with material parameters $\sigma_Y = 1$, $\lambda = 576.92$ and $\mu = 384.61$ corresponding to Young's modulus 10^3 and Poisson's ratio 0.3.

By equilibrium, the stress will decrease with the distance from the hole such that the nonlinearity only needs to be considered in some bounded domain surrounding the hole. In this way, we are lead to the coupling of FEM and BEM as shown in Fig. 2. Note we plot the finite elements only but the system continues outside of Γ with a linear elastic material.

As explained in Section 9, we started with a coarse mesh shown in Fig. 2 and run Algorithm (A) with $\theta = 0.8$. If the material was linear, the singular behaviour of the solution at the corners of the rectangular hole would lead the adaptive algorithm to refine the mesh strongly towards the corner points. The final meshes were obtained after 8 refinement steps and shown in Fig. 2 for viscoplastic material with $\rho = 4$ and for plastic material with $\rho = 0$. The expected singularities are not observed in Fig. 2, instead we have a strong refinement towards the rectangular hole but a nearly uniform refinement along its edges for $\rho > 0$. The plastic zones are plotted as follows: a dot is drawn whenever the discrete stress in some element–midpoint satisfies $|\sigma_h^D| \geqslant \sigma_Y$.

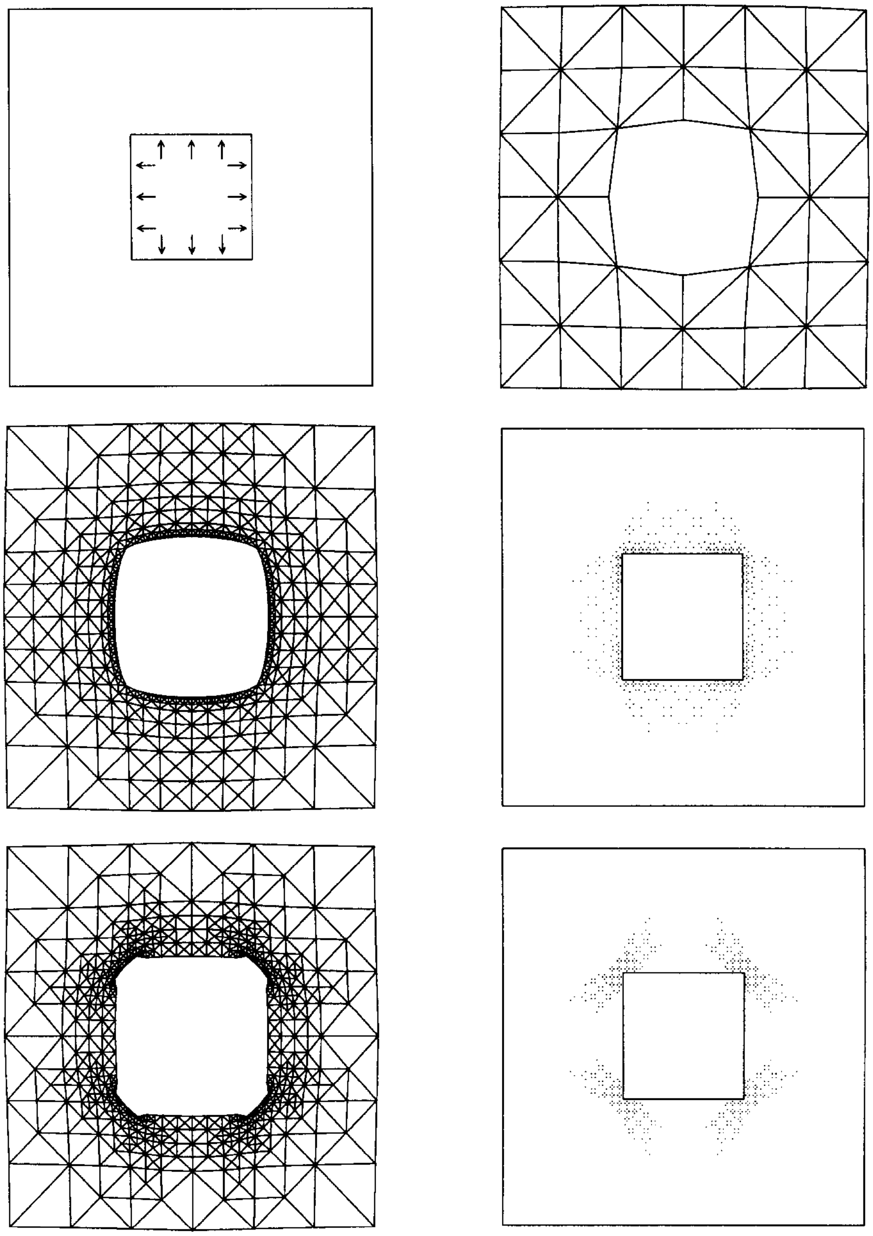

Fig. 2. Deformed meshes and plastic zones for Example 1. First row: system and loading and initial mesh. Second and last row: final meshes and plastic zones for $\varrho > 0$ resp. $\varrho = 0$.

In case $\rho = 0$ we observe some nonsmooth displacements which, regarding the plastic zones ($|\sigma_h^{\mathrm{D}}| = \sigma_{\mathrm{Y}}$), might be caused by a shear band like singularity.

Although no direct comparisons with experiments were possible from the literature, the authors were adviced by engineers that the different phenomena for $\varrho > 0$ and $\varrho = 0$ are reasonable.

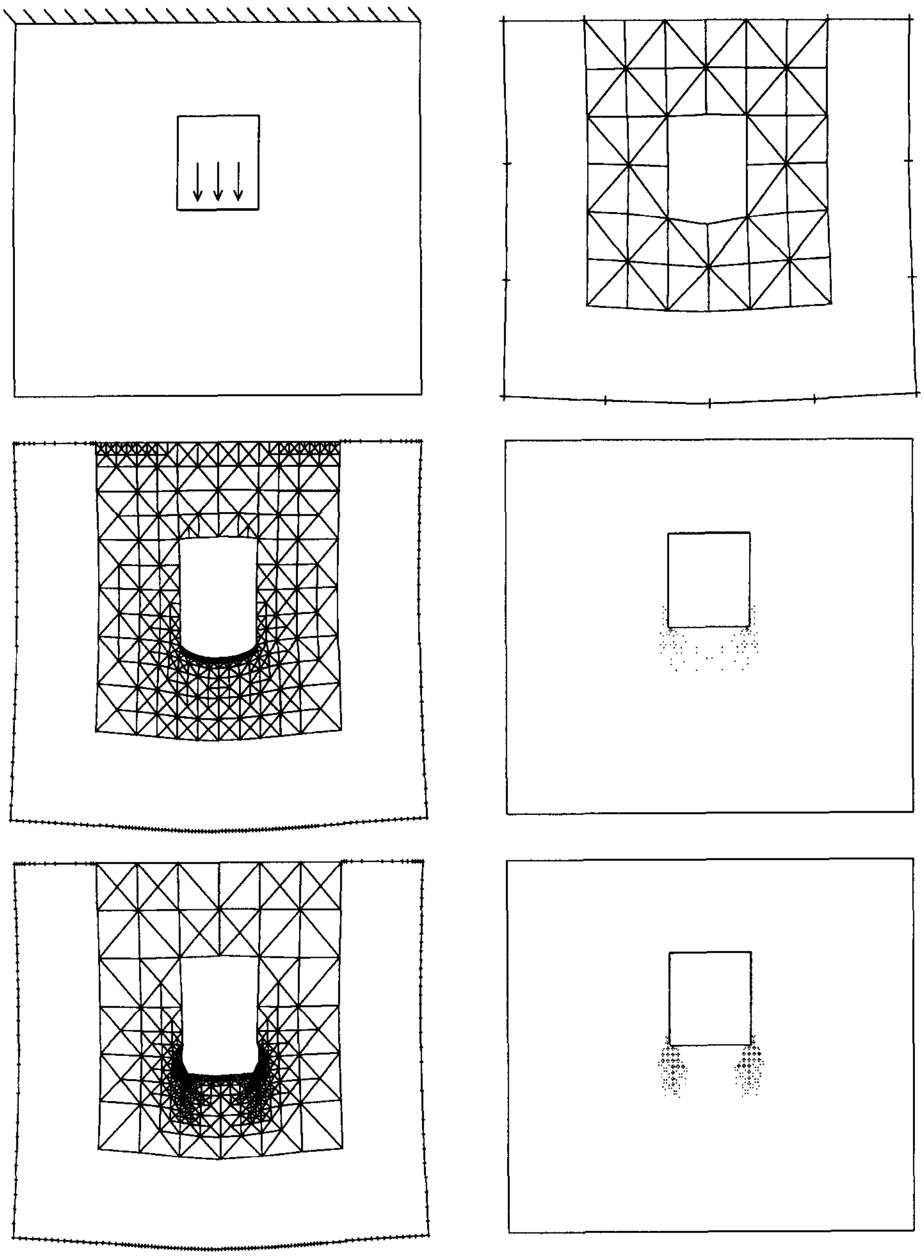

Fig. 3. Deformed meshes and plastic zones for Example 2. First row: system and loading and initial mesh. Second and last row: final meshes and plastic zones for $\varrho > 0$ resp. $\varrho = 0$.

Similar to the first example we apply the finite element method in the second example to a bounded part of Ω near the hole where we might expect plastic zones and hence have to consider the nonlinear material behaviour. Our choice, shown in Fig. 3, was motivated by a rough previous finite element computation. The remaining part of Ω is expected to behave linearly and thus is modelled using the boundary element method. The material constants are the same as above but the main difference is that the body is bounded here and included in the exterior frame which represents a Dirichlet boundary at the top and a homogeneous Neumann boundary at the three remaining sides. We stress that the mesh for the boundary element method is along the exterior frame and along the outer boundary of the finite element mesh. The final meshes were produced automatically by Algorithm (A) as shown in Fig. 3 where, caused by a different loading, the plasic zones are restricted to the lower corners of the hole.

We present two alternative iteration schemes for solving the (nonlinear) discrete problem (28). The Picard-type iteration (35) did converge too slowly as shown in Table 1. There, G denotes the left hand side of (35) with $u_h^{\nu-1}$ replaced by u_h^{ν}. Therefore, we proceeded as indicated in Section 7 and used Newton–Raphson's method instead. In each step of the iteration, the parameter δ, which steers the regularization, has been divided by 50 in order to get as near to the exact discrete solution as possible, cf. Table 2. This shows the superior efficiency of the presented regularization technique.

Table 1
Picard iteration for Example 1

ν	$\|G\|$	ν	$\|G\|$
1	0.64054909858001	6	0.50759329469558
2	0.56715925862251	7	0.50482885662070
3	0.53690978284427	8	0.50347864319758
4	0.52094250303812	9	0.50300490011140
5	0.51247594432732	10	0.50274840558972

Table 2
Newton–Raphson's method for Example 1

ν	δ	$\|\overrightarrow{DF_0^{\delta}}(u_h^{\nu}) + \mathscr{A}\cdot(u_h^{\nu},\phi_h^{\nu}) - b\|$
1	2.00 D-02	5.02748 D-01
2	4.00 D-04	7.28313 D-01
3	8.00 D-06	1.13009 D-01
4	1.60 D-07	1.03294 D-02
5	3.20 D-09	2.37536 D-04
6	6.40 D-11	1.63663 D-07
7	1.28 D-12	8.64827 D-14

Appendix

The regularization of ϑ (as indicated in (30)) and depicted in Fig. 1 is analytically defined in this appendix. For $\delta > 0$ let $x_0 = (\sigma_Y/2\mu)^2$, $x^* = x_0 - \delta$ and define the following coefficients:

$$a = \frac{\sigma_Y(7x_0 - 3x^*)}{160x_0^{5/2}\delta^3}, \qquad b = \frac{\sigma_Y(2x^{*^2} - 3x_0x^* - 3x_0^2)}{32x_0^{5/2}\delta^3},$$

$$c = \frac{\sigma_Y x^*(6x_0^2 - x^* x_0 - x^{*^2})}{16 x_0^{5/2} \delta^3}, \qquad d = \frac{\sigma_Y x^{*^2}(5x^* - 9x_0)}{16 x_0^{3/2} \delta^3}.$$

With $p''(x) = 20ax^3 + 12bx^2 + 6cx + 2d$ the polynomial p' is defined by $p'(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$ where e is choosen such that $p'(x_0) = \vartheta'(x_0)$. With this polynomial the function

$$\vartheta'_\delta(x) = \begin{cases} p'(x^*), & x \leqslant x^*, \\ p'(x), & x^* < x < x_0, \\ \dfrac{\sigma_Y}{2\sqrt{x}}, & x \geqslant x_0 \end{cases}$$

may be regarded as a regularization of ϑ'. ϑ'_δ is twice continously differentiable and converges locally uniformly to ϑ' as $\delta \to 0$. Finally we obtain ϑ_δ simply by integration, $\vartheta_\delta(x) = \int_0^x \vartheta'_\delta(\xi)\,\mathrm{d}\xi$, and after setting

$$\theta_\varrho^\delta(x) = \frac{1}{1 + \frac{\varrho}{2\mu}} \vartheta_\delta(x) + \frac{\varrho \cdot x}{1 + \frac{\varrho}{2\mu}},$$

we introduce the functional $F_\varrho^\delta : \mathscr{H} \to \mathbb{R}$ through

$$F_\varrho^\delta(u) = \int_\Omega \theta_\varrho^\delta(|\varepsilon^{\mathrm{D}}(u)|^2) + \frac{3\lambda + 2\mu}{6} \operatorname{tr}^2 \varepsilon(u)\,\mathrm{d}x,$$

which is now used in Section 6 instead of (32).

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