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An experimental survey of *a posteriori* Courant finite element error control for the Poisson equation

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This comparison of some *a posteriori* error estimators aims at empirical evidence for a ranking of their performance for a Poisson model problem with conforming lowest order finite element discretizations. Modified residual-based error estimates compete with averaging techniques and two estimators based on local problem solving. Multiplicative constants are involved to achieve *guaranteed* upper and lower energy error bounds up to higher order terms. The optimal strategy combines various estimators.

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1. Introduction

Suppose we are given a finite element approximation u_h to the unknown exact solution u of the Poisson problem:

$$\begin{aligned} &\text{Given } f \in L^2(\Omega) \text{ seek } u \in H^1(\Omega) \text{ with} \\ &\Delta u + f = 0 \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here, $\Omega \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain and $H^1(\Omega)$ is the standard Sobolev space. Although the error $e := u - u_h \in H_0^1(\Omega) := \{v \in H^1(\Omega)\colon v|_{\partial\Omega} = 0\}$ is unknown, an *a posteriori* error estimator η_h aims to estimate, e.g., its energy norm $\|\nabla e\|_{L^2(\Omega)}$. The point is that η_h is computable, i.e., a function of u_h, f, Ω, and the mesh $\mathcal{T}$. We call an error estimator reliable and efficient if

$$\|\nabla e\|_{L^2(\Omega)} \leqslant \eta_h \quad \text{and} \quad \eta_h \leqslant \|\nabla e\|_{L^2(\Omega)}, \tag{1.2}$$

respectively. It is clear from (1.2) that an error estimator η_h should *not* be expected to be reliable and efficient at the same time; but more frequently

$$c_1\eta_h + \text{h.o.t.} \leqslant \|\nabla e\|_{L^2(\Omega)} \leqslant c_2\eta_h \tag{1.3}$$

holds for constants $c_1, c_2 > 0$ (and the same estimator η_h on both sides) and higher-order terms h.o.t. For complete error control, c_1 and c_2 (and h.o.t.) have to be evaluated as well. Sometimes one constant c_j is hidden in η_h (replace η_h by $c_j\eta_h$), whence one constant c_1 or c_2 equals one.

A proper introduction into the topic are the books [4,8,40] where equivalence of a series of estimators is shown; here, we focus on modified estimators and sharp constants in this paper. A look in [4,40] reveals that a residual-based *a posteriori* error estimate is essentially equivalent to all the other estimators. But absolute constants are generic in [4,40] and so it remains unclear what estimator is the best in the sense of quality per costs. This paper addresses the question for modified, presumably sharper estimators.

The survey discusses various modified estimators based on different techniques and involves constants for rigorous termination of a finite element simulation. We assume that $\mathcal{T}$ is a regular triangulation of Ω in the sense of [15,28] with nodes $\mathcal{N}$, free nodes $\mathcal{K} := \mathcal{N} \setminus \partial\Omega$, and edges $\mathcal{E}$. Given the discrete solution u_h, let $[\partial u_h/\partial n_{\mathcal{E}}]$ denote the edgewise jump of the $\mathcal{T}$-piecewise constant gradient ∇u_h in the component n_E normal to the edge E; $\bigcup \mathcal{E}_\Omega$ denotes the union of all inner edges in $\mathcal{T}$; $\mathcal{E}_\Omega := \{E \in \mathcal{E}\colon\ E \not\subseteq \partial\Omega\}$.

A modification of the standard residual-based *a posteriori* estimate [4,6,8,31,37, 38,40] with explicit constants is defined by

$$\eta_{\mathrm{R,C}}^2 := \sum_{z\in\mathcal{N}} \left(B_z \left\| \varphi_z^{1/2} \left[\frac{\partial u_h}{\partial n_{\mathcal{E}}} \right] \right\|_{L^2(\bigcup \mathcal{E}\cap\omega_z)} + C_z \|\varphi_z^{1/2} f\|_{L^2(\omega_z)} \right)^2.$$

Following [22], B_z and C_z are calculated from local eigenvalue problems, which is very costly. On the other hand, B_z and C_z depend on the shape and the size of the patch ω_z of the node z. Numerical evidence in [22] shows for the class of right isosceles that

$$\eta_{\mathrm{R,R}} = \left(\sum_{T\in\mathcal{T}} h_T^2 \|f\|_{L^2(T)}^2 \right)^{1/2} + \left(\sum_{E\in\mathcal{E}} h_E \left\| \left[\frac{\partial u_h}{\partial n_E} \right] \right\|_{L^2(E)}^2 \right)^{1/2} \tag{1.4}$$

(where $h_T := \operatorname{diam}(T)$ and $h_E := \operatorname{diam}(E)$) is reliable with explicit constant 1, i.e.,

$$\|\nabla e\|_{L^2(\Omega)} \leqslant \eta_{\mathrm{R,R}} \leqslant c_2 \|\nabla e\|_{L^2(\Omega)} + \text{h.o.t.} \tag{1.5}$$

The second estimate in (1.5) is shown in [40]. We estimate the constant c_2 and design an efficient error estimator $\eta_{\mathrm{R,E}}$ in section 3 with

$$\eta_{\mathrm{R,E}} \leqslant \|\nabla e\|_{L^2(\Omega)} + \text{h.o.t.} \tag{1.6}$$

The estimators $\eta_{\mathrm{R,E}}$ and $\eta_{\mathrm{R,R}}$ are very cheap, but the range between $\eta_{\mathrm{R,E}}$ and $\eta_{\mathrm{R,R}}$ is relatively large.

The overestimation by $\eta_{\mathrm{R,R}}$ is usually about a factor of ten and larger. Hence, sharper estimators, while possibly more expensive, are of interest; section 2 below illustrates why a cheaper estimator might be more expensive and vice versa.

Let $(\varphi_z\colon z \in \mathcal{K})$ denote the nodal basis of the finite element space $\mathcal{S}_0^1(\mathcal{T})$; $\omega_z := \{x \in \Omega\colon \varphi_z(x) > 0\}$ is the patch of $z \in \mathcal{K}$, and $a_z^2 := \int_{\omega_z} \varphi_z |\nabla w|^2 \,\mathrm{d}x$ for the local solution $w \in \mathcal{H}_z \subseteq H^1(\omega_z)$ of the weighted interface problem

$$\int_{\omega_z} \varphi_z \nabla w \cdot \nabla v \,\mathrm{d}x = \int_{\cup\mathcal{E}_\Omega} \varphi_z \left[\frac{\partial u_h}{\partial n_{\mathcal{E}}}\right] v \,\mathrm{d}s + \int_{\omega_z} \varphi_z v f \,\mathrm{d}x \tag{1.7}$$

for all $v \in \mathcal{H}_z$. The modification of Babuška–Rheinboldt's local estimator [7] then reads

$$\eta_{\mathrm{L}}^2 := \sum_{z \in \mathcal{K}} a_z^2. \tag{1.8}$$

For triangulations $\mathcal{T}$ into right isosceles triangles we quote from [22] (and only $c_2 = 2.36$ depends on $\mathcal{T}$)

$$\|\nabla e\|_{L^2(\Omega)} \leqslant \eta_{\mathrm{L}} \leqslant 2.36 \|\nabla e\|_{L^2(\Omega)}. \tag{1.9}$$

The second local problem solving estimator is an equilibrium estimator [1–4,9,32, 39]. Equilibrium estimators approximate the exact traction at element edges and, given an appropriate $\widetilde{\mu} \in L^2(\cup\mathcal{E})$, approximate the solution $\Phi|_T \in H^1(T)$ of

$$\int_T \nabla\Phi \cdot \nabla v \,\mathrm{d}x = \int_T f v \,\mathrm{d}x - \int_T \nabla u_h \cdot \nabla v \,\mathrm{d}x + \int_{\partial T} \sigma\, \widetilde{\mu} v \,\mathrm{d}s \quad \text{for all } v \in H^1(T) \tag{1.10}$$

($\sigma = \pm 1$ is a proper sign $n \cdot n_E$). This gives $\Phi|_T$, for each $T \in \mathcal{T}$, and so determines

$$\eta_{\mathrm{EQ}}^2 := \sum_{T \in \mathcal{T}} \|\nabla\Phi\|_{L^2(T)}^2. \tag{1.11}$$

The parameters $\widetilde{\mu}$ are chosen from the average of $\nabla u_h \cdot n_E$ such that the right-hand side vanishes for $v|_T = 1$. Any equilibrated choice yields

$$\|\nabla e\|_{L^2(\Omega)} \leqslant \eta_{\mathrm{EQ}} \tag{1.12}$$

and an optimal choice for $\widetilde{\mu}$ could even yield equality $\|\nabla e\|_{L^2(\Omega)} = \eta_{\mathrm{EQ}}$.

This paper presents a new proof of the reliability of averaging techniques where the discrete (piecewise constant) flux ∇u_h is approximated by a smoother approximation q_h, e.g., in the globally continuous and piecewise affine spline spaces $\mathcal{S}^1(\mathcal{T})^d$. There holds reliability for *all such* q_h, and so even for the *minimal* $\eta_{\mathrm{Z,M}}$,

$$\eta_{\mathrm{Z,M}} := \min_{q_h \in \mathcal{S}^1(\mathcal{T})^d} \|q_h - \nabla u_h\|_{L^2(\Omega)} \leqslant \eta_{\mathrm{Z,A}} := \|\nabla u_h - \mathcal{A}\nabla u_h\|_{L^2(\Omega)}.$$

Based on patchwise averaging, our second choice $\eta_{Z,A}$ is a cheap realization of the ZZ-estimator [41].

The estimator $\eta_{Z,M}$ is obviously efficient. A triangle inequality shows $\eta_{Z,M} \leqslant \|\nabla e\|_{L^2(\Omega)} + \|q_h - \nabla u\|$ for all $q_h \in \mathcal{S}^1(\mathcal{T})^d$, whence

$$\eta_{Z,M} \leqslant \|\nabla e\|_{L^2(\Omega)} + \min_{q_h \in \mathcal{S}^1(\mathcal{T})^d} \|q_h - \nabla u\|_{L^2(\Omega)},$$

and the last term is of higher order. We give a new proof of the second estimate in

$$\eta_{Z,M} - \text{h.o.t.} \leqslant \|\nabla e\|_{L^2(\Omega)} \leqslant c_2\, \eta_{Z,M} + \text{h.o.t.}, \tag{1.13}$$

where h.o.t. is of higher order if $f \in H^1(\Omega)$ [10,17,20,27]. The surprising experimental observation is that $\eta_{Z,M}$ and $\eta_{Z,A}$ are very accurate and can be recommended for an optimal error guess in practice.

To illustrate the necessity of a variety of estimators, we start with a discussion of a practical strategy for the termination of a mesh-refinement algorithm in section 2. The price for a sharper but more expensive estimator competes with the price of unnecessary refinements in an adaptive algorithm.

Details on the implementation and parts of the error analysis are provided for residual-based estimators $\eta_{R,R}$ from [22] and the new estimator $\eta_{R,E}$ in section 3. A new proof of the reliability for averaging estimators $\eta_{Z,M}$ and $\eta_{Z,A}$, recently established in [10,20,23–27], is given in section 4 which makes use of the H^1-stability of the L^2-projection [17,18]. An equilibrium estimator η_{EQ} according to [4] is discussed in section 5.

For comparisons, we added the local problem estimator η_L from [22] in section 6 which performs similar to η_{EQ}.

Section 7 introduces the adaptive mesh-refinements used in the numerical examples of section 8. Some observations and conclusions on the the experimental results for the three examples are drawn in section 9.

This paper concerns a simple model example and so compares various estimators in circumstances of their numerical performance. If coefficients in the partial differential equation vary with x, have jumps, and/or are anisotropic, this will seriously affect all estimators. Then, a ranking of estimators will possibly be different compared to our numerical conclusions. It appears, moreover, that special coefficients require special care. Averaging schemes of local problem solvers, for instance, should reflect the coefficients properly. This leaves important open cases which lie far beyond the aims of this paper.

2. *A posteriori* error control for the termination of an adaptive finite element program

Suppose the main interest is on the error in the energy norm, which is *not* true for many applications, but may serve as an illustrative example for the assessment of the estimators. For a uniform sequence of meshes $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ of the model example (1.1) with $f = 1$ on an L-shaped domain Ω, we computed the estimators

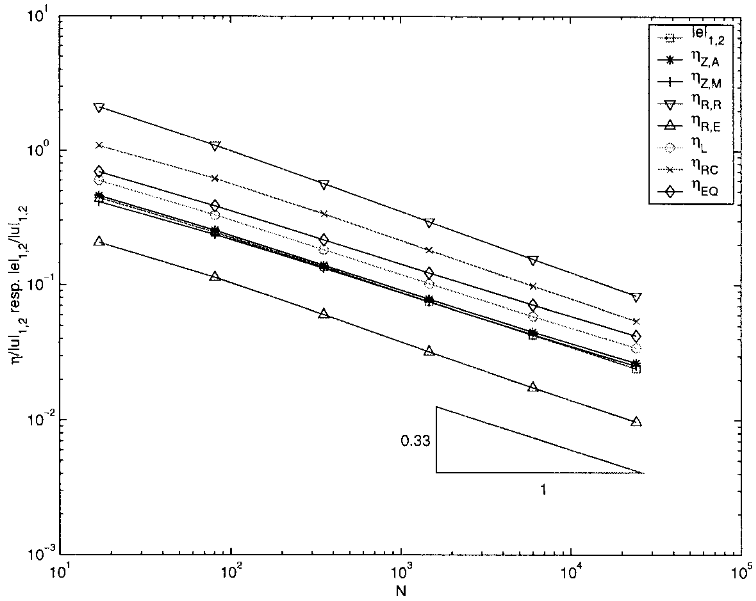

Figure 1. Relative error and error estimators for uniform mesh refinement.

$\eta_{R,E}$, $\eta_{R,R}$, $\eta_{R,C}$, $\eta_{Z,A}$, $\eta_{Z,M}$, η_{EQ}, η_L, and the true energy error $|e|_{1,2} := \|\nabla e\|_{L^2(\Omega)}$. For any such estimator η, figure 1 displays an open polygon defined by the number of degrees of freedom and the corresponding estimator or error on the current mesh $\mathcal{T}_k$. In this way we obtain the coordinates of the kth entry of a polygon in figure 1. The legend shows the correspondence of symbols and entries for the 8 polygons. The logarithmic scaling used for both axes is responsible for a (nearly) constant horizontal distance of the entries in a uniform mesh-refining strategy and for a certain slope $\alpha/2$ which, in two dimensions, corresponds to an experimental convergence rate α as $h^\alpha \propto N^{-\alpha/2}$. As the domain has a re-entering corner with an interior angle $3\pi/2$, theoretical predictions of a convergence rate $2/3$ for uniform meshes suggest a slope $-1/3$ which is visible in figure 1.

When shall we terminate the calculation if the goal is a relative error $\leqslant 10\%$ (in the energy norm)? To decide this from figure 1, we divided all results by $\|\nabla u\|_{L^2(\Omega)} =: |u|_{1,2} \approx 0.462680$ for the exact solution u (since u is unknown in this example, $|u|_{1,2}$ is extrapolated from $|u_h|_{1,2}$ on a sequence of uniform meshes and then, $|e|_{1,2}$ is computed as $|e|_{1,2}^2 = \|\nabla u_h\|_{L^2(\Omega)}^2 - |u|_{1,2}^2$ by Galerkin orthogonality). Clearly, the error is below 10% for $\mathcal{T}_4$, but $|e|_{1,2}$ is unknown and so we have to rely on the estimators displayed.

In general, the lower bound $\eta_{R,E}$ holds in an asymptotic sense only. If we suppose this bound is strictly correct, the cheap information $10\% \, |u|_{1,2} \leqslant \eta_{R,E}$ is seen in figure 1 for $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2$. (The numerator $|u|_{1,2}$ is chosen here as 0.462680 but could be replaced by a value extrapolated from the discrete solutions on the previous meshes.)

Hence, utilizing the cheap estimators $\eta_{R,E}$ and $\eta_{R,R}$ we would refine up to the mesh $\mathcal{T}_3$. The overestimation of $|e|_{1,2}$ by $\eta_{R,R}$ is up to a factor 10 and if we base a termination criterion on $\eta_{R,R} \leqslant 10\% \, |u|_{1,2}$, we would refine until $\mathcal{T}_6$ with more than 20,000 unknowns (compare with 1473 unknowns for $\mathcal{T}_4$). Therefore, the use of a cheap estimator $\eta_{R,R}$ is very expensive.

It appears more efficient to spend more time on the calculation of a sharper estimate than on an overkill refinement $\mathcal{T}_6$. The second choice is $\eta_{Z,A}$ or even better $\eta_{Z,M}$ which is an asymptotic lower bound. We observe in figure 1 that $\eta_{Z,M} \leqslant 10\% \, |u|_{1,2}$ for $\mathcal{T}_4$. Motivated by the overall observation that $\eta_{Z,M}$ is a good error guess we would think that $\mathcal{T}_4$ is a good mesh to stop. This, however, does *not* guarantee $|e|_{1,2} \leqslant 10\% \, |u|_{1,2}$.

A guaranteed upper error bound requires c_2 in $c_2\eta_{Z,M} + \text{h.o.t.}$ (as well as control of higher order terms). This constant will presumably reflect a worst case scenario (as c_2 is universal and not adapted to our example) and so result in an overestimation, followed by unnecessary refinements.

It appears a better option to employ the more laborious estimators η_{EQ} or η_L which are good enough to pass $\eta_{EQ} \leqslant 10\% \, |u|_{1,2}$ or $\eta_L \leqslant 10\% \, |u|_{1,2}$ for the mesh $\mathcal{T}_5$. This yields a guaranteed error bound and justifies termination.

The history of estimators we employed shows that $\eta_{R,E}$ (and $\eta_{R,R}$) is evaluated for $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\eta_{Z,M}$ has been computed for $\mathcal{T}_3$, $\mathcal{T}_4$, and eventually η_{EQ} (or η_L) was employed once for $\mathcal{T}_4$. To summarize, the optimal decision in the refinement versus termination employs three different estimators and both, upper and lower error bounds.

The little example of this section clearly shows that the combination of various estimators performs better than each of them individually. The next sections provide details on the implementation and choice of constants in our experiments further reported on in the last sections of this paper.

Remark 2.1.

(a) Further references and applications are provided in [4,8,31].

(b) For other error norms or error functionals of interest we refer to [13].

(c) Nonhomogeneous Dirichlet boundary conditions are studied in [12].

(d) Multilevel estimators are excluded here which are certainly very useful if multigrid solvers are employed despite the problem with the saturation assumption [9,14,29, 30,35].

3. Residual-based *a posteriori* error estimates

A look in the book [40] gives an introduction to the standard residual-based error estimator. This section recalls the reliable modifications of [22] and presents the useful new efficient version $\eta_{R,E}$.

3.1. Reliable residual-based estimate

This subsection is devoted to the description of $\eta_{R,C}$.

Definition 3.1. Set $\mathcal{W}_z := H^1(\omega_z)/\mathbb{R}$ (e.g., $w \in \mathcal{W}_z$ satisfies $\int_{\omega_z} w\,\mathrm{d}x = 0$) if $z \in \mathcal{K}$ and $\mathcal{W}_z := H^1_D(\omega_z) := \{w \in H^1(\omega_z)\colon\ w|_{\partial\Omega} = 0\}$ if $z \in \mathcal{N}\backslash\mathcal{K}$. Set $R \in L^2(\Omega)$ and $J \in L^2(\bigcup\mathcal{E})$ by

$$R := f \quad \text{and} \quad J|_E := [\nabla u_h \cdot n_E] \quad \text{on } E \in \mathcal{E}_\Omega,$$

where $[\nabla u_h \cdot n_E] = (\nabla u_h|_{T_2} - \nabla u_h|_{T_1}) \cdot n_E$ if $E = T_1 \cap T_2$ for $T_1, T_2 \in \mathcal{T}$ and n_E points from T_1 into T_2. Set $J|_E := 0$ for $E \in \mathcal{E} \setminus \mathcal{E}_\Omega$.

Definition 3.2. Define, for $z \in \mathcal{N}$,

$$B_z^2 := \sup_{w\in\mathcal{W}_z\backslash\{0\}} \frac{\int_{\bigcup\mathcal{E}\cap\omega_z} \varphi_z(w - \Pi_z w)^2\,\mathrm{d}s}{\|\varphi_z^{1/2}\nabla w\|^2_{L^2(\omega_z)}},$$
$$C_z^2 := \sup_{w\in\mathcal{W}_z\backslash\{0\}} \frac{\int_{\omega_z} \varphi_z(w - \Pi_z w)^2\,\mathrm{d}x}{\|\varphi_z^{1/2}\nabla w\|^2_{L^2(\omega_z)}},$$

where for $w \in H^1(\omega_z)$, $\Pi_z w := \int_{\omega_z} w\,\mathrm{d}x/|\omega_z|$ if $z \in \mathcal{K}$ and $\Pi_z w := 0$ if $z \in \mathcal{N} \setminus \mathcal{K}$.

Theorem 3.1 [22]. We have $B_z, C_z < \infty$ and

$$\|\nabla e\|^2_{L^2(\Omega)} \leqslant \eta^2_{\mathrm{RC}} := \sum_{z\in\mathcal{N}} \big(B_z \|\varphi_z^{1/2} J\|_{L^2(\bigcup\mathcal{E}\cap\omega_z)} + C_z \|\varphi_z^{1/2} R\|_{L^2(\omega_z)}\big)^2.$$

Proof. We refer to [22] for a proof of $B_z, C_z < \infty$ and give the idea of a proof for the reliability estimate. An integration by parts and utilising that $(\varphi_z\colon\ z \in \mathcal{N})$ is a partition of unity, we infer

$$\begin{aligned}\int_\Omega |\nabla e|^2\,\mathrm{d}x &= \int_\Omega Re\,\mathrm{d}x + \int_{\bigcup\mathcal{E}} Je\,\mathrm{d}s \\ &= \sum_{z\in\mathcal{N}} \left(\int_{\bigcup\mathcal{E}\cap\omega_z} Je\varphi_z\,\mathrm{d}s + \int_{\omega_z} Re\varphi_z\,\mathrm{d}x\right) \\ &\leqslant \sum_{z\in\mathcal{N}} \sup_{v\in\mathcal{W}_z\backslash\{0\}} \frac{\int_{\cup\mathcal{E}\bigcap\omega_z} Jv\varphi_z\,\mathrm{d}s + \int_{\omega_z} Rv\varphi_z\,\mathrm{d}x}{\|\varphi_z^{1/2}\nabla v\|_{L^2(\omega_z)}} \|\varphi_z^{1/2}\nabla e\|_{L^2(\omega_z)}. \qquad (3.1)\end{aligned}$$

Galerkin's orthogonality implies

$$\begin{aligned}\|\nabla e\|^2_{L^2(\Omega)} \leqslant{}& \sum_{z\in\mathcal{N}} \|\varphi_z^{1/2}\nabla e\|_{L^2(\omega_z)} \\ &\times \sup_{v\in\mathcal{W}_z\backslash\{0\}} \frac{\int_{\bigcup\mathcal{E}\cap\omega_z} J(v - \Pi_z v)\varphi_z\,\mathrm{d}s + \int_{\omega_z} R(v - \Pi_z v)\varphi_z\,\mathrm{d}x}{\|\varphi_z^{1/2}\nabla v\|_{L^2(\omega_z)}},\end{aligned}$$

and Cauchy's inequality yields

$$
\begin{aligned}
\|\nabla e\|^2_{L^2(\Omega)} \leqslant \sum_{z\in\mathcal{N}} \big\|\varphi_z^{1/2}\nabla e\big\|_{L^2(\omega_z)} \\
\times\Bigg(\big\|\varphi_z^{1/2}J\big\|_{L^2(\cup\mathcal{E}\cap\omega_z)} \sup_{v\in\mathcal{W}_z\setminus\{0\}} \frac{\|\varphi_z^{1/2}(v-\Pi_z v)\|_{L^2(\cup\mathcal{E}\cap\omega_z)}}{\|\varphi_z^{1/2}\nabla v\|_{L^2(\omega_z)}} \\
+\big\|\varphi_z^{1/2}R\big\|_{L^2(\omega_z)} \sup_{v\in\mathcal{W}_z\setminus\{0\}} \frac{\|\varphi_z^{1/2}(v-\Pi_z v)|_{L^2(\omega_z)}}{\|\varphi_z^{1/2}\nabla v\|_{L^2(\omega_z)}}\Bigg).
\end{aligned}
\tag{3.2}
$$

Definition 3.2 and a discrete Cauchy inequality show

$$
\begin{aligned}
\|\nabla e\|^2_{L^2(\Omega)} \leqslant \bigg(\sum_{z\in\mathcal{N}}\big(B_z\big\|\varphi_z^{1/2}J\big\|_{L^2(\cup\mathcal{E}\cap\omega_z)} + C_z\big\|\varphi_z^{1/2}R\big\|_{L^2(\omega_z)}\big)^2\bigg)^{1/2} \\
\times\bigg(\sum_{z\in\mathcal{N}}\int_{\omega_z}\varphi_z|\nabla e|^2\,\mathrm{d}x\bigg)^{1/2}.
\end{aligned}
\tag{3.3}
$$

Since $\sum_{z\in\mathcal{N}}\varphi_z = 1$, this and a division by $\|\nabla e\|_{L^2(\Omega)}$ conclude the proof. □

Remark 3.1. The constants B_z and C_z are determined as (Rayleigh quotients and so as analytical) local eigenvalues. They are approximated by the p-version of the finite element method on the patch (with respect to the mesh $\mathcal{T}_z := \{T \in \mathcal{T}\colon z \in T\}$). The numbers displayed below in section 8 are obtained with fourth order polynomials. We refer to [22] for details and examples on the algorithms.

3.2. *Simplified version for triangulations into right isosceles triangles*

This subsection is devoted to the description of $\eta_{\mathrm{R,R}}$.

Definition 3.3. For $T \in \mathcal{T}$ define

$$
\eta_{\mathrm{R,R}}(T)^2 := h_T^2\,\|R\|^2_{L^2(T)} + \sum_{E\in\mathcal{E}_\Omega;\, E\subseteq\partial T} h_E\,\|J\|^2_{L^2(E)},
\tag{3.4}
$$

$$
\eta_{\mathrm{R,R}} := \bigg(\sum_{T\in\mathcal{T}} h_T^2\,\|R\|^2_{L^2(T)}\bigg)^{1/2} + \bigg(\sum_{E\in\mathcal{E}_\Omega} h_E\,\|J\|^2_{L^2(E)}\bigg)^{1/2}.
\tag{3.5}
$$

The error estimator $\eta_{\mathrm{R,R}}$ is reliable with known constants.

Theorem 3.2 [22]. If $d = 2$ and $\mathcal{T}$ consists of right isosceles triangles, we have

$$
\|\nabla e\|_{L^2(\Omega)} \leqslant \eta_{\mathrm{R,R}}.
$$

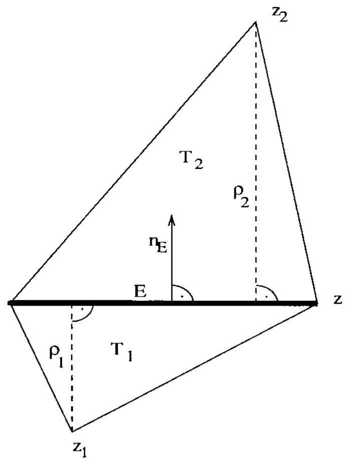

Figure 2. Patch ω_E of an edge $E \in \mathcal{E}_\Omega$.

Remark 3.2.

(a) The point is that the constant 1 in (3.5) yields a guaranteed upper bound.

(b) The proof in [22] utilizes numerical values for certain eigenvalue problems on a finite number of patches. Related analytic estimates are given in [21].

3.3. *Efficient residual-based error estimates*

This subsection is devoted to the presentation of $\eta_{\mathrm{R,E}}$. We confine ourselves to $d = 2$.

Definition 3.4. For $E \in \mathcal{E}_\Omega$ and $E = T_1 \cap T_2$, $T_1, T_2 \in \mathcal{T}$, let $\omega_E := T_1 \cup T_2$,

$$c_E^2 := \frac{1}{18(h_E/\rho_1 + h_E/\rho_2)} \times \left(\frac{2(\rho_1 - \rho_2)^2(\rho_1 + \rho_2)^2}{(\rho_1 + \rho_2)^4 - 4\rho_1\rho_2(\alpha_1\rho_2 + \alpha_2\rho_1)((1-\alpha_1)\rho_2 + (1-\alpha_2)\rho_1)} + 1 \right),$$

$$\rho_j := \frac{2|T_j|}{h_E}, \qquad \alpha_j^2 := \frac{|z - z_j|^2 - \rho_j^2}{h_E}, \quad j = 1, 2,$$

where z_j is the node of T_j (of area $|T_j|$) not on E and $z \in E \cap \mathcal{K}$, cf. figure 2. Moreover, let n_E be a unit vector perpendicular to E.

Definition 3.5. For $T \in \mathcal{T}$ with edges $E_1, E_2, E_3 \in \mathcal{E}$, $\partial T = E_1 \cup E_2 \cup E_3$, and area $|T|$, let

$$c_T := \frac{|T|}{h_T\sqrt{5}(h_{E_1}^2 + h_{E_2}^2 + h_{E_3}^2)^{1/2}}.$$

Definition 3.6. Set $\theta := 0$ if $f = 0$ and $\theta := 1/2$ otherwise. Define

$$\eta_{\mathrm{R,E}} := \theta\left(\sum_{T\in\mathcal{T}} c_T^2 h_T^2 \|R\|^2_{L^2(T)}\right)^{1/2} + (1-\theta)\left(\sum_{E\in\mathcal{E}} c_E^2 h_E \|J\|^2_{L^2(E)}\right)^{1/2} \tag{3.6}$$

and

$$\eta_{\mathrm{R,E}}(T)^2 := \theta^2 c_T^2 h_T^2 \|R\|^2_{L^2(T)} + (1-\theta)^2 \sum_{E\in\mathcal{E};\, E\subseteq\partial T} c_E^2 h_E \|J\|^2_{L^2(E)} \tag{3.7}$$

for $T \in \mathcal{T}$.

Up to higher order terms the error estimator $\eta_{\mathrm{R,E}}$ is efficient with known constants. For $\omega \subseteq \mathbb{R}^2$ we denote by $\mathcal{P}_1(\omega)$ the space of polynomials on ω with total degree less or equal 1.

Theorem 3.3. Assume $f|_T \in H^1(T)$ for all $T \in \mathcal{T}$ and let $h_{\mathcal{T}} \in L^\infty(\Omega)$ satisfy $h_{\mathcal{T}}|_T = h_T$ for all $T \in \mathcal{T}$. Then there exists a constant $c_1 > 0$ such that

$$\eta_{\mathrm{R,E}} \leqslant \|\nabla e\| + \theta\frac{c_1}{\pi}\left\|h_{\mathcal{T}}^2 \nabla f\right\|_{L^2(\Omega)} + (1-\theta)\left(\frac{1}{3}\sum_{E\in\mathcal{E}_\Omega} \min_{q_E\in\mathcal{P}_1(\omega_E)^2} \|\nabla u - q_E\|^2_{L^2(\omega_E)}\right)^{1/2}.$$

The following lemmas are needed for the proof of theorem 3.3.

Lemma 3.1. For each $E \in \mathcal{E}_\Omega$ and $p_h \in \mathcal{L}^0(\mathcal{T}|\omega_E)^2$ ($p_h \in L^\infty(\omega_E)^2$ and $p_h|_T$ is constant for each $T \subseteq \overline{\omega}_E$), we have

$$\frac{1}{\sqrt{3}} \min_{q_E\in\mathcal{P}_1(\omega_E)^2} \|p_h - q_E\|_{L^2(\omega_E)} = c_E h_E^{1/2}\, \|[p_h]\|_{L^2(E)}.$$

Proof. Let $E \in \mathcal{E}_\Omega$ and $E = T_1 \cap T_2$ for $T_1, T_2 \in \mathcal{T}$. It suffices to prove the assertion for each component of $p_h \in \mathcal{L}^0(\mathcal{T}|\omega_E)^2$ separately, i.e., it suffices to prove, for $r_h \in \mathcal{L}^0(\mathcal{T}|\omega_E)$,

$$\frac{1}{\sqrt{3}} \min_{q_E\in\mathcal{P}_1(\omega_E)} \|r_h - q_E\|_{L^2(\omega_E)} = c_E h_E^{1/2} \|[r_h]\|_{L^2(E)}.$$

We may assume $r_h|_{T_1} = 0$ and $r_h|_{T_2} = s$ for some $s \in \mathbb{R}$. Then, $h_E^{1/2}\|[r_h]\|_{L^2(E)} = h_E|s|$. The evaluation of $\|r_h - q_E\|_{L^2(\omega_E)}$ for

$$q_E(x, y) = ax + by + c$$

and the minimization of the resulting expression over $a, b, c \in \mathbb{R}$ eventually lead to the formulae for c_E. □

Lemma 3.2. There exists an h_T-independent constant $c_1 > 0$ such that, for each $T \in \mathcal{T}$,

$$c_T\, h_T \|R\|_{L^2(T)} \leqslant \|\nabla e\|_{L^2(T)} + c_1 h_T \|f - f_T\|_{L^2(T)},$$

where $f_T = 1/|T| \int_T f \,\mathrm{d}x$. For each $E \in \mathcal{E}_\Omega$, we have

$$\sqrt{3} c_E\, h_E^{1/2} \|J\|_{L^2(E)} \leqslant \|\nabla e\|_{L^2(\omega_E)} + \min_{q_E \in \mathcal{P}_1(\omega_E)^2} \|\nabla u - q_E\|_{L^2(\omega_E)}.$$

Proof. Let $T \in \mathcal{T}$ with $T = \operatorname{conv}\{z_1, z_2, z_3\}$, $z_1, z_2, z_3 \in \mathcal{K}$, and let $b_T = \varphi_{z_1}\varphi_{z_2}\varphi_{z_3}$ denote the bubble function on T. A direct calculation shows

$$\big\|b_T^{1/2}\big\|_{L^2(T)}^2 = \frac{|T|}{60} \quad \text{and} \quad \|\nabla b_T\|_{L^2(T)}^2 = \frac{h_{E_1}^2 + h_{E_2}^2 + h_{E_3}^2}{720\,|T|}.$$

Following the arguments in [40] we start with

$$h_T \|f\|_{L^2(T)} \leqslant h_T \|f - f_T\|_{L^2(T)} + \frac{|T|^{1/2}}{\|b_T^{1/2}\|_{L^2(T)}} h_T \big\|b_T^{1/2} f_T\big\|_{L^2(T)}. \tag{3.8}$$

Writing

$$\big\|b_T^{1/2} f_T\big\|_{L^2(T)}^2 = f_T\bigg(\int_T b_T (f_T - f)\,\mathrm{d}x + \int_T f b_T\,\mathrm{d}x\bigg)$$

we find

$$\big\|b_T^{1/2} f_T\big\|_{L^2(T)} \leqslant \frac{\|b_T\|_{L^2(T)} \|f - f_T\|_{L^2(T)} + \big|\int_T f b_T\,\mathrm{d}x\big|}{\|b_T^{1/2}\|_{L^2(T)}}. \tag{3.9}$$

Since $\operatorname{div} \nabla u_h|_T = 0$ and $b_T|_{\partial T} = 0$ we have

$$\int_T f b_T\,\mathrm{d}x = \int_T \nabla e \cdot \nabla b_T\,\mathrm{d}x, \tag{3.10}$$

and the combination of (3.8)–(3.10) yields the first assertion,

$$\|f\|_{L^2(T)} \leqslant \bigg(1 + \frac{|T|^{1/2}\|b_T\|_{L^2(T)}}{\|b_T^{1/2}\|_{L^2(T)}^2}\bigg)\|f - f_T\|_{L^2(T)} + \frac{|T|^{1/2}\|\nabla b_T\|_{L^2(T)}}{\|b_T^{1/2}\|_{L^2(T)}^2}\|\nabla e\|_{L^2(T)},$$

with the h_T-independent constant

$$c_3 = \max_{T \in \mathcal{T}} \bigg(1 + \frac{|T|^{1/2}\|b_T\|_{L^2(T)}}{\|b_T^{1/2}\|_{L^2(T)}^2}\bigg).$$

The second estimate follows from lemma 3.1 with a triangle inequality and

$$\|[\nabla u_h \cdot n_E]\|_{L^2(E)} \leqslant \|[\nabla u_h]\|_{L^2(E)}$$

(which is an equality for conforming elements since $[\nabla u_h] \cdot t_E = 0$). $\square$

Proof of theorem 3.3. Lemma 3.2 and a Poincaré inequality [36] show, for arbitrary $\gamma > 0$,

$$\sum_{T \in \mathcal{T}} c_T^2 h_T^2 \|R\|_{L^2(T)}^2 \leqslant (1+\gamma)\|\nabla e\|_{L^2(\Omega)}^2 + \bigg(1 + \frac{1}{\gamma}\bigg)\frac{c_1^2}{\pi^2}\big\|h_T^2 \nabla f\big\|_{L^2(\Omega)}^2.$$

A minimization of the right-hand side with respect to $\gamma > 0$ yields $\gamma = \|h_{\mathcal{T}}^2 \nabla f\|_{L^2(\Omega)} / \|\nabla e\|_{L^2(\Omega)}$ and

$$\left(\sum_{T\in\mathcal{T}} c_T^2 h_T^2 \|R\|_{L^2(T)}^2\right)^{1/2} \leqslant \|\nabla e\|_{L^2(\Omega)} + \frac{c_1}{\pi}\left\|h_{\mathcal{T}}^2 \nabla f\right\|_{L^2(\Omega)}. \tag{3.11}$$

Moreover, from lemma 3.2 we obtain, for $\gamma > 0$,

$$\begin{aligned}
&3\sum_{E\in\mathcal{E}_\Omega} c_E^2 h_E \|J\|_{L^2(E)}^2\\
&\quad\leqslant \sum_{E\in\mathcal{E}_\Omega}\big(\|\nabla e\|_{L^2(\omega_E)} + \min_{q_E\in\mathcal{P}_1(\omega_E)^2}\|\nabla u - q_E\|_{L^2(\omega_E)}\big)^2\\
&\quad\leqslant \sum_{E\in\mathcal{E}_\Omega}(1+\gamma)\|\nabla e\|_{L^2(\omega_E)}^2 + \sum_{E\in\mathcal{E}_\Omega}\left(1+\frac{1}{\gamma}\right)\min_{q_E\in\mathcal{P}_1(\omega_E)^2}\|\nabla u - q_E\|_{L^2(\omega_E)}^2\\
&\quad\leqslant 3(1+\gamma)\|\nabla e\|_{L^2(\Omega)}^2 + \left(1+\frac{1}{\gamma}\right)\sum_{E\in\mathcal{E}_\Omega}\min_{q_E\in\mathcal{P}_1(\omega_E)^2}\|\nabla u - q_E\|_{L^2(\omega_E)}^2
\end{aligned}$$

($T \subseteq \omega_E$ holds for at most three distinct $E \in \mathcal{E}_\Omega$). The optimal $\gamma > 0$ shows

$$\left(\sum_{E\in\mathcal{E}_\Omega} c_E^2 h_E \|J\|_{L^2(E)}^2\right)^{1/2} \leqslant \|\nabla e\|_{L^2(\Omega)} + \left(\frac{1}{3}\sum_{E\in\mathcal{E}_\Omega}\min_{q_E\in\mathcal{P}_1(\omega_E)^2}\|\nabla u - q_E\|_{L^2(\omega_E)}^2\right)^{1/2}.$$

Adding (3.11) and the last estimate concludes the proof. □

4. Averaging error estimators

This section is devoted to the description of the averaging error estimators $\eta_{Z,M}$ and $\eta_{Z,A}$ and a new proof of their reliability.

Definition 4.1. Let $q_h^* \in \mathcal{S}^1(\mathcal{T})^d$ denote the minimizer in (4.1) and set, for $T \in \mathcal{T}$,

$$\eta_{Z,M} := \min_{q_h\in\mathcal{S}^1(\mathcal{T})^d}\|\nabla u_h - q_h\|_{L^2(\Omega)} = \left\|\nabla u_h - q_h^*\right\|_{L^2(\Omega)}, \tag{4.1}$$

$$\eta_{Z,M}(T) := \left\|\nabla u_h - q_h^*\right\|_{L^2(T)}. \tag{4.2}$$

An application of the triangle inequality shows efficiency of $\eta_{Z,M}$ up to a term that depends on the smoothness of the exact solution,

$$\eta_{Z,M} \leqslant \|\nabla e\|_{L^2(\Omega)} + \min_{q_h\in\mathcal{S}^1(\mathcal{T})^d}\|\nabla u - q_h\|_{L^2(\Omega)}. \tag{4.3}$$

The following error estimator $\eta_{Z,A}$ is a local version of $\eta_{Z,M}$.

Definition 4.2. Define $\mathcal{A}: L^2(\Omega)^d \to \mathcal{S}^1(\mathcal{T})^d$ by $\mathcal{A}p := \sum_{z\in\mathcal{N}} p_z\varphi_z$ for

$$p_z := \frac{1}{|\omega_z|}\int_{\omega_z} p\,\mathrm{d}x \in \mathbb{R}^d, \quad z \in \mathcal{N}$$

and, for $T \in \mathcal{T}$, set

$$\begin{aligned} \eta_{\mathrm{Z,A}} &:= \|\nabla u_h - \mathcal{A}\nabla u_h\|_{L^2(\Omega)} \quad \text{and} \\ \eta_{\mathrm{Z,A}}(T) &:= \|\nabla u_h - \mathcal{A}\nabla u_h\|_{L^2(T)}. \end{aligned} \tag{4.4}$$

Reliability of $\eta_{\mathrm{Z,M}} \leqslant \eta_{\mathrm{Z,A}}$ is given up to a multiplicative constant factor and a known higher order term. The new proof employs the L^2-projection.

Definition 4.3. Let $\Pi : L^2(\Omega) \to \mathcal{S}_0^1(\mathcal{T})$ denote the L^2-projection onto $\mathcal{S}_0^1(\mathcal{T})$. The L^2-projection is called H^1-stable if there exists an $h_\mathcal{T}$-independent constant $c_4 > 0$ such that, for all $g \in H_0^1(\Omega)$, we have

$$\big\|\nabla(\Pi g)\big\|_{L^2(\Omega)} \leqslant c_4\|\nabla g\|_{L^2(\Omega)}.$$

Red-green-blue-refinements of a triangle are explained below in section 7 and needed in the following theorem.

Theorem 4.1 [18]. If $d = 2$ and $\mathcal{T}$ is a triangulation into triangles obtained by successive red-green-blue-refinements, then Π is H^1-stable.

The H^1-stability of Π is thus given in all our test examples.

Theorem 4.2. If Π is H^1-stable then there exists an $h_\mathcal{T}$-independent constant $c_5 > 0$ such that

$$\|\nabla e\|_{L^2(\Omega)} \leqslant c_5\Big(\eta_{\mathrm{Z,M}} + \inf_{f_h\in\mathcal{S}_0^1(\mathcal{T})} \big\|h_\mathcal{T}(f - f_h)\big\|_{L^2(\Omega)}\Big).$$

The following lemma is needed for the proof of theorem 4.2.

Lemma 4.1. If Π is H^1-stable then there exists a constant $c_6 > 0$ such that, for all $g \in H_0^1(\Omega)$,

$$\big\|h_\mathcal{T}^{-1}(g - \Pi g)\big\|_{L^2(\Omega)} \leqslant c_6\|\nabla g\|_{L^2(\Omega)}.$$

Remark 4.1.

(a) Since $\eta_{\mathrm{Z,M}} \leqslant \eta_{\mathrm{Z,A}}$, theorem 4.2 yields reliability up to a multiplicative constant factor and a known higher order term for $\eta_{\mathrm{Z,A}}$ as well.

(b) The higher order contribution is the L^2-norm of $h_\mathcal{T}(f - \Pi f)$. Notice carefully that f may have non homogeneous values at the boundary (e.g., $f = 1$ does not belong to $H_0^1(\Omega)$). Thus, the h.o.t. are not of quadratic order (but of higher order).

(c) The first order approximation operator from [16,27] has an additional orthogonality property that still allows us to verify that the norm of $h_{\mathcal{T}} f$ can be replaced by a higher order term. We refer to [10,20,23–26] for alternative proofs which also reflect nonhomogeneous and mixed boundary conditions.

(d) Averaging techniques where proposed by engineers [41]; their general reliability was first indicated by [37,38] by dominating edge contributions [16,27].

(e) The observation that *all averaging estimators are reliable* is due to [20] and studied in [10] for higher order finite element schemes, in [23–25,25,26] in elasticity and the Stokes equations, and eventually in [11,19] for variational inequalities.

(f) The argument of efficiency (4.3) employs higher order terms that depend on the smoothness of the exact solution. Utilizing the inverse estimates technique from [40], one can prove that efficiency holds up to higher order terms which depend on $f \in H^1(\Omega)$. In the latter case, however, the constant in front of the error in the upper bound is larger than one.

(g) The lemma is (essentially) known from [27]; we give a proof at the end of this section for completeness.

Proof of theorem 4.2. For arbitrary $q_h \in \mathcal{S}^1(\mathcal{T})^d$ the Galerkin orthogonality shows

$$\|\nabla e\|^2_{L^2(\Omega)} = \int_\Omega (\nabla u - q_h) \cdot \nabla(e - \Pi e)\,\mathrm{d}x + \int_\Omega (q_h - \nabla u_h) \cdot \nabla(e - \Pi e)\,\mathrm{d}x.$$

A Cauchy inequality in the latter term is combined with the H^1-stability of Π to show

$$\begin{aligned}\int_\Omega (q_h - \nabla u_h) \cdot \nabla(e - \Pi e)\,\mathrm{d}x &\leqslant \|q_h - \nabla u_h\|_{L^2(\Omega)} \big\|\nabla(e - \Pi e)\big\|_{L^2(\Omega)} \\ &\leqslant (1 + c_4)\|\nabla e\|_{L^2(\Omega)} \|q_h - \nabla u_h\|_{L^2(\Omega)}.\end{aligned}$$

An integration by parts in the second term shows

$$\int_\Omega (\nabla u - q_h) \cdot \nabla(e - \Pi e)\,\mathrm{d}x = \int_\Omega (f + \operatorname{div} q_h)(e - \Pi e)\,\mathrm{d}x.$$

Since $e - \Pi e$ is L^2-orthogonal onto $f_h := \Pi f \in \mathcal{S}^1_0(\mathcal{T})$, this leads to

$$\begin{aligned}&\textstyle\int_\Omega (\nabla u - q_h) \cdot \nabla(e - \Pi e)\,\mathrm{d}x \\ &\quad= \textstyle\int_\Omega (f - f_h + \operatorname{div} q_h)(e - \Pi e)\,\mathrm{d}x \\ &\quad\leqslant \big\|h_{\mathcal{T}}^{-1}(e - \Pi e)\big\|_{L^2(\Omega)} \big(\big\|h_{\mathcal{T}}(f - \Pi f)\big\|_{L^2(\Omega)} + \|h_{\mathcal{T}} \operatorname{div}_{\mathcal{T}} q_h\|_{L^2(\Omega)}\big).\end{aligned}$$

By $\operatorname{div}_{\mathcal{T}}$ we denote the elementwise application of the divergence operator and so have $\operatorname{div}_{\mathcal{T}} \nabla u_h = 0$. An elementwise inverse estimate of the form

$$\big\|\operatorname{div}_{\mathcal{T}}(q_h - \nabla u_h)\big\|_{L^2(T)} \leqslant c_7 / h_T\, \|q_h - \nabla u_h\|_{L^2(T)}$$

(notice that $q_h|_T$ is a polynomial) yields

$$\|h_{\mathcal{T}} \operatorname{div}_{\mathcal{T}} q_h\|_{L^2(\Omega)} \leqslant c_7 \|q_h - \nabla u_h\|_{L^2(\Omega)}.$$

Combining the above arguments with lemma 4.1, we deduce

$$\begin{aligned}\|\nabla e\|^2_{L^2(\Omega)} &\leqslant (1 + c_4)\|\nabla e\|_{L^2(\Omega)}\|q_h - \nabla u_h\|_{L^2(\Omega)} \\ &\quad + c_6 \|\nabla e\|_{L^2(\Omega)}\big(\big\|h_{\mathcal{T}}(f - \Pi f)\big\|_{L^2(\Omega)} + c_7\|q_h - \nabla u_h\|_{L^2(\Omega)}\big).\end{aligned}$$

Proof of lemma 4.1. There exists a shape-depending constant $c_8 > 0$ such that, for all $T \in \mathcal{T}$ and $z \in \mathcal{N}$ with $T \subseteq \overline{\omega}_z$, we have $h_z := \operatorname{diam}(\omega_z) \leqslant c_8 h_T$. Therefore, and since $\sum_{z\in\mathcal{N}} \varphi_z = 1$, we infer

$$\begin{aligned}c_8^{-2}\big\|h_{\mathcal{T}}^{-1}(g - \Pi g)\big\|^2_{L^2(\Omega)} &\leqslant \sum_{z\in\mathcal{N}} h_z^{-2} \int_{\omega_z} \varphi_z (g - \Pi g)^2 \,\mathrm{d}x \\ &= \sum_{z\in\mathcal{N}} h_z^{-2} \int_{\omega_z} \varphi_z (g - \Pi g - g_z)(g - \Pi g)\,\mathrm{d}x,\end{aligned}$$

where we used $\int_\Omega g_z \varphi_z (g - \Pi g)\,\mathrm{d}x = 0$ for $g_z \in \mathbb{R}$ and $z \in \mathcal{K}$ and set $g_z := 0$ if $z \in \mathcal{N} \setminus \mathcal{K}$. Hence, with $\gamma > 0$,

$$\begin{aligned}&c_8^{-2}\big\|h_{\mathcal{T}}^{-1}(g - \Pi g)\big\|^2_{L^2(\Omega)} \\ &\quad\leqslant \textstyle\sum_{z\in\mathcal{N}} h_z^{-2}\|g - \Pi g - g_z\|_{L^2(\omega_z)}\big\|\varphi_z(g - \Pi g)\big\|_{L^2(\omega_z)} \\ &\quad\leqslant \frac{\gamma}{4}\textstyle\sum_{z\in\mathcal{N}} h_z^{-2}\|g - \Pi g - g_z\|^2_{L^2(\omega_z)} + \frac{1}{\gamma}\sum_{z\in\mathcal{N}} h_z^{-2}\big\|\varphi_z(g - \Pi g)\big\|^2_{L^2(\omega_z)}.\end{aligned}$$

A Poincaré respective Friedrichs inequality shows for g_z suitably chosen

$$h_z^{-2}\|g - \Pi g - g_z\|^2_{L^2(\omega_z)} \leqslant c_9\big\|\nabla(g - \Pi g)\big\|^2_{L^2(\omega_z)}.$$

This and the partition of unity property of the nodal basis functions again show

$$\begin{aligned}&\big\|h_{\mathcal{T}}^{-1}(g - \Pi g)\big\|^2_{L^2(\Omega)} \\ &\quad\leqslant \frac{c_8^2 c_9 \gamma}{4}\textstyle\sum_{z\in\mathcal{K}}\big\|\nabla(g - \Pi g)\big\|^2_{L^2(\omega_z)} + \frac{c_8^2}{\gamma}\big\|h_{\mathcal{T}}^{-1}(g - \Pi g)\big\|^2_{L^2(\Omega)}.\end{aligned}$$

Since at most $d + 1$ patches contribute to one element, we deduce with $\gamma = 2c_8^2$,

$$\big\|h_{\mathcal{T}}^{-1}(g - \Pi g)\big\|^2_{L^2(\Omega)} \leqslant (d + 1)c_8^2 c_9\big\|\nabla(g - \Pi g)\big\|^2_{L^2(\Omega)}.$$

This and the H^1-stability of Π conclude the proof with $c_6^2 := (d + 1)c_8^4 c_9 (1 + c_4)^2$. $\square$

5. Equilibration error estimators

This section is devoted to the equilibration estimator due to Ladeveze and Leguillon [1–3,9,32,39] in the implementation of [4].

Definition 5.1. For $\mu : \bigcup \mathcal{E} \to \mathbb{R}$ and each $T \in \mathcal{T}$ define (n is the outer normal on ∂T)

$$R_T(v) := \int_T f v \,\mathrm{d}x - \int_T \nabla u_h \cdot \nabla v \,\mathrm{d}x + \int_{\partial T} \left(\left\langle \frac{\partial u_h}{\partial n} \right\rangle + \sigma \mu \right) v \,\mathrm{d}s,$$
$$B_T(u, v) := \int_T \nabla u \cdot \nabla v \,\mathrm{d}x,$$

with $\langle \partial u_h / \partial n \rangle = n \cdot (\nabla u_h|_{T_1} + \nabla u_h|_{T_2})/2$ on $E = T_1 \cap T_2$, $T_1, T_2 \in \mathcal{T}$ and $\sigma := n \cdot n_E$, while $\langle \partial u_h / \partial n \rangle + \sigma \mu =: g$ on Γ_N. Define local spaces $H^1_D(T)$ as $H^1(T)/\mathbb{R}$ if $\partial T \cap \partial \Omega$ has surface measure zero and otherwise as $\{v \in H^1(T)\colon\ v = 0 \text{ on } \partial\Omega \cap \partial T\}$. Set

$$V(\mathcal{T}) := \bigl\{ v \in L^2(\Omega)\colon\ \forall T \in \mathcal{T}, v|_T \in H^1_D(T) \bigr\}.$$

Theorem 5.1. Suppose $R_T(1) = 0$ for any $T \in \mathcal{T}$ with a positive distance to the boundary. Then, for all $T \in \mathcal{T}$, there exists a unique $\Phi_T \in H^1_D(T)$ with

$$B_T(\Phi_T, v) = R_T(v) \quad \text{for all } v \in H^1_D(T). \tag{5.1}$$

Moreover (Φ_T denotes the solution of (5.1)),

$$|e|_{H^1(\Omega)} \leqslant \eta_{\mathrm{EQ}} := \left(\sum_{T \in \mathcal{T}} \|\nabla \Phi_T\|^2_{L^2(T)} \right)^{1/2}. \tag{5.2}$$

Remark 5.1.

(a) The construction of μ guarantees $R_T(1) = 0$ for all $T \in \mathcal{T}$ and follows the algorithm in [4]. The idea is a split into a constant part and two affine contributions $2 - 3s$ and $3s - 1$ over each edge $0 \leqslant s \leqslant 1$. Those affine functions decouple when their product with φ_z is integrated over one edge. By solving local systems of equations, we can then achieve $R_T(\varphi_z) = 0$ for all $T \subset \Omega$.

(b) Our η_{EQ}-steered adaptive algorithms are based on the refinement indicator $\eta_{\mathrm{EQ}}(T)$,

$$\eta_{\mathrm{EQ}}(T) := \|\nabla \Phi_T\|_{L^2(T)} \quad \text{for } T \in \mathcal{T}. \tag{5.3}$$

(c) The iterative improvement of μ suggested in [8,15] performs very accurate in numerical experiments; a convergence theory, however, appears missing for the higher dimensional case.

Proof of theorem 5.1. The solubility of the local problems (5.1) is well known. The construction of the function μ is described in [4]. The Cauchy inequality and the discrete Cauchy inequality show

$$
\begin{aligned}
|e|_{H^1(\Omega)} &= \sup_{v\in H_0^1(\Omega)} \frac{R(v)}{|v|_{1,2}} \\
&= \sup_{v\in H_0^1(\Omega)} \frac{\sum_{T\in\mathcal{T}} R_T(v)}{|v|_{H^1(\Omega)}} \\
&\leqslant \sup_{v\in V(\mathcal{T})} \frac{\sum_{T\in\mathcal{T}} R_T(v)}{\big(\sum_{T\in\mathcal{T}} |v|^2_{H^1(T)}\big)^{1/2}} \\
&= \sup_{v\in V(\mathcal{T})} \frac{\sum_{T\in\mathcal{T}} B_T(\Phi_T, v)}{\big(\sum_{T\in\mathcal{T}} |v|^2_{H^1(T)}\big)^{1/2}} \\
&\leqslant \sup_{v\in V(\mathcal{T})} \frac{\sum_{T\in\mathcal{T}} B_T(\Phi_T,\Phi_T)^{1/2} B_T(v,v)^{1/2}}{\big(\sum_{T\in\mathcal{T}} |v|^2_{H^1(T)}\big)^{1/2}} \\
&\leqslant \sup_{v\in V(\mathcal{T})} \frac{\big(\sum_{T\in\mathcal{T}} B_T(\Phi_T,\Phi_T)\big)^{1/2}\big(\sum_{T\in\mathcal{T}} B_T(v,v)\big)^{1/2}}{\big(\sum_{T\in\mathcal{T}} |v|^2_{H^1(T)}\big)^{1/2}} \\
&= \bigg(\sum_{T\in\mathcal{T}} B_T(\Phi_T,\Phi_T)\bigg)^{1/2}.
\end{aligned}
\tag{5.4}
$$

□

6. Error estimation by local transmission problems

This subsection is devoted to the description of η_{L}.

Definition 6.1. Adopt notation from definition 3.1 and define, for $z \in \mathcal{N}$,

$$A_z := \sup_{w\in\mathcal{W}_z\setminus\{0\}} \frac{\int_{w_z} Rw\varphi_z\,\mathrm{d}x + \int_{\cup\mathcal{E}} Jw\varphi_z\,\mathrm{d}s}{\|\varphi_z^{1/2}\nabla w\|_{L^2(\omega_z)}}.$$

Theorem 6.1 [22]. We have

$$\|\nabla e\|^2_{L^2(\Omega)} \leqslant \eta_{\mathrm{L}}^2 := \sum_{z\in\mathcal{N}} A_z^2. \tag{6.1}$$

Proof. Integration by parts shows

$$\int_\Omega |\nabla e|^2\,\mathrm{d}x = \int_\Omega Re\,\mathrm{d}x + \int_{\cup\mathcal{E}} Je\,\mathrm{d}s. \tag{6.2}$$

Since $\sum_{z\in\mathcal{N}} \varphi_z$ is a partition of unity we deduce

$$\begin{aligned}\|\nabla e\|_{L^2(\Omega)} &= \sum_{z\in\mathcal{N}}\left(\int_{\cup\mathcal{E}\cap\omega_z} Je\varphi_z\,\mathrm{d}s + \int_{\omega_z} Re\varphi_z\,\mathrm{d}x\right)\\ &\leqslant \sum_{z\in\mathcal{N}} A_z\big\|\varphi_z^{1/2}\nabla e\big\|_{L^2(\omega_z)}\\ &\leqslant \left(\sum_{z\in\mathcal{N}} A_z^2\right)^{1/2}\left(\sum_{z\in\mathcal{N}}\int_{\omega_z}\varphi_z|\nabla e|^2\,\mathrm{d}x\right)^{1/2}\\ &= \left(\sum_{z\in\mathcal{N}} A_z^2\right)^{1/2}\|\nabla e\|_{L^2(\Omega)}.\end{aligned}$$

□

Theorem 6.2 [22]. If $d = 2$ and $\mathcal{T}$ consists of right isosceles triangles, we have

$$\eta_{\mathrm{L}} \leqslant 2.36\|\nabla e\|_{L^2(\Omega)}.$$

Remark 6.1.

(a) The factor A_z can be obtained from the solution of a local problem [22], namely, as $A_z = \|\varphi_z^{1/2}\nabla w\|_{L^2(\omega_z)}$ for the unique solution $w \in \mathcal{W}_z$ of

$$\int_{\omega_z}\varphi_z\nabla w\cdot v\,\mathrm{d}x = \int_{\cup\mathcal{E}}\varphi_z J v\,\mathrm{d}s + \int_{\omega_z}\varphi_z v R\,\mathrm{d}x \quad \text{for all } v\in\mathcal{W}. \tag{6.3}$$

In the numerical experiments we replaced w by w_h computed by the p-version of the finite element method on the patch (with respect to the mesh $\mathcal{T}_z := \{T \in \mathcal{T}\colon z \in T\}$). The numbers displayed below in section 8 are obtained with fourth order polynomials. We refer to [22] for details and examples on the algorithms.

(b) The paper [34] suggests a realization of A_z with special quadratic ansatz functions on $\mathcal{T}_z$ to compute an approximation $\widetilde{w}_h$ to w with (6.3). It is shown in [34] that their discrete approximation is reliable,

$$\|\nabla e\|^2_{L^2(\Omega)} \leqslant c_{10}\sum_{z\in\mathcal{N}}\big\|\varphi_z^{1/2}\nabla w\big\|^2_{L^2(\omega_z)} =: c_{10}\,\widetilde{\eta}^2_{\mathrm{L}},$$

with an undetermined constant c_{10}. Their numerical experiments show a surprisingly accurate agreement of $\|\nabla e\|_{L^2(\Omega)}$ and $\widetilde{\eta}_{\mathrm{L}}$. (Note carefully that this estimate is neither reliable nor efficient as constants such as c_{10} are not involved.)

(c) Our η_{L}-steered adaptive algorithms are based on the refinement indicator $\eta_{\mathrm{L}}(T)$,

$$\eta^2_{\mathrm{L}}(T) := \frac{1}{d+1}\sum_{z\in\mathcal{N};\, z\in T} A_z^2 \quad \text{for each } T\in\mathcal{T}. \tag{6.4}$$

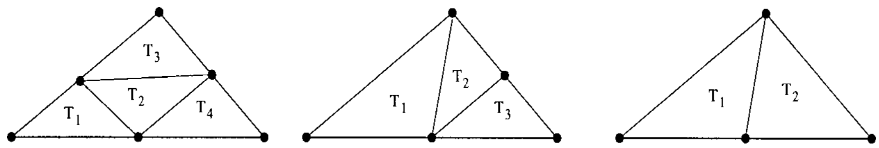

Figure 3. Red-, blue-, and green-refinement of a triangle.

7. Adaptive mesh refinement

Automatic mesh refinement generates a sequence of meshes $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \dots$ by marking and refining elements according to a refinement rule (7.1).

Algorithm (A_Θ)**.**

(a) Start with a coarse mesh $\mathcal{T}_0$, $k = 0$.

(b) Compute the discrete solution u_h on the actual mesh $\mathcal{T}_k$ with N degrees of freedom.

(c) Compute error estimators η_N and their local contributions η_T such that $\eta_N^2 = \sum_{T\in\mathcal{T}} \eta_T^2$ for $\eta_N = \eta_{\mathrm{R,R}}$ in (3.4) and (3.5), for $\eta_N = \eta_{\mathrm{Z,M}}$ in (3.6) and (3.7), for $\eta_N = \eta_{\mathrm{Z,A}}$ in (4.4), for $\eta_N = \eta_{\mathrm{EQ}}$ in (5.2) and (5.3), for $\eta_N = \eta_{\mathrm{L}}$ in (6.1) and (6.4).

(d) Mark the element T for *red*-refinement provided

$$\eta_T \geqslant \Theta \max_{K\in\mathcal{T}_k} \eta_K. \tag{7.1}$$

(e) Mark further elements (*red-blue-green*-refinement) to avoid hanging nodes. Generate a new triangulation $\mathcal{T}_{k+1}$. Update k and go to (b).

Definition 7.1.

(i) A *red-refinement* of $T \in \mathcal{T}$ is performed by dividing T into four congruent sub-triangles which are obtained by connecting the midpoints of the edges $E_1, E_2, E_3 \in \mathcal{E}$, $\partial T = E_1 \cup E_2 \cup E_3$, cf. figure 3.

(ii) A *blue-refinement* of $T \in \mathcal{T}$ is performed by dividing T into three sub-triangles which are obtained by connecting the midpoint of the longest edge $E_1 \in \mathcal{E}$, $E_1 \subseteq \partial T$, with the mid-point of another edge $E_2 \in \mathcal{E} \setminus E_1$, $E_2 \subseteq \partial T$, and with the node opposite to it, cf. figure 3.

(iii) A *green-refinement* of $T \in \mathcal{T}$ is performed by dividing T into two sub-triangles which are obtained by connecting the midpoint of the longest edge $E \in \mathcal{E}$, $E \subseteq \partial T$, with the node opposite to it, cf. figure 3.

Remark 7.1.

(a) The parameter Θ allows adaptive mesh refinement for $\Theta = 1/2$ and uniform mesh refinement for $\Theta = 0$.

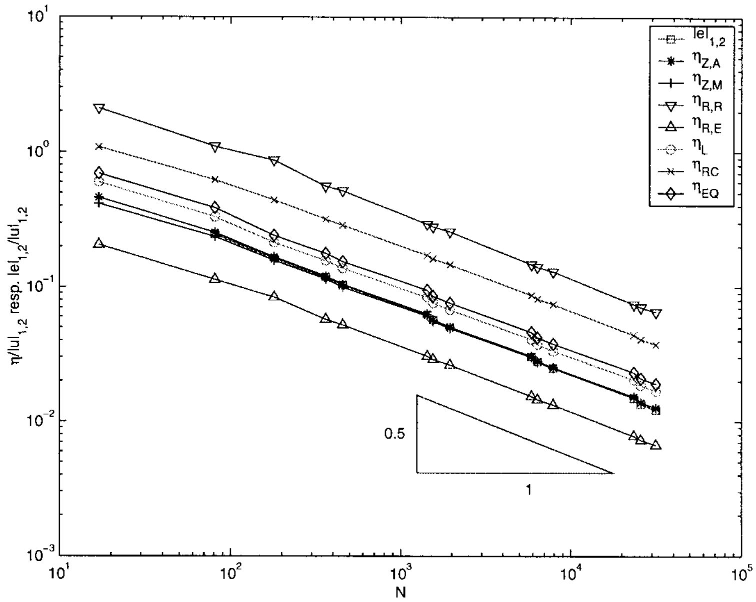

Figure 4. Relative error and error estimators for $\eta_{R,R}$ adaptive mesh refinement in example 8.1.

(b) There is remarkably little literature on *a priori* properties of adaptive algorithms [29, 30,33,34]; but their practical performance is actually very good. The convergence rates are usually reasonably improved.

(c) The finite element scheme and the adaptive algorithms were implemented in Matlab based on [5] with direct solution of all linear systems of equations.

8. Numerical examples

Three examples are reported in this section; the first is already discussed in section 2.

Example 8.1 [22]. Let $f := 1$ on the L-shaped domain

$$\Omega := (-1, 1)^2 \setminus [0, 1] \times [-1, 0],$$

$u = 0$ on $\partial\Omega$. The coarsest triangulation $\mathcal{T}_0$ consists of 12 triangles obtained by dividing each of the three squares into four congruent triangles.

The exact solution u of (1.1) is not known (the calculation of $|e|_{1,2}$ is explained in section 2). The solution has a typical corner singularity at the origin. In this example, the right-hand sides are smooth, but the solution is not.

Figure 4 shows the relative energy error and the error estimators divided by the energy of the exact solution on a sequence of meshes generated by algorithm $(A_{1/2})$

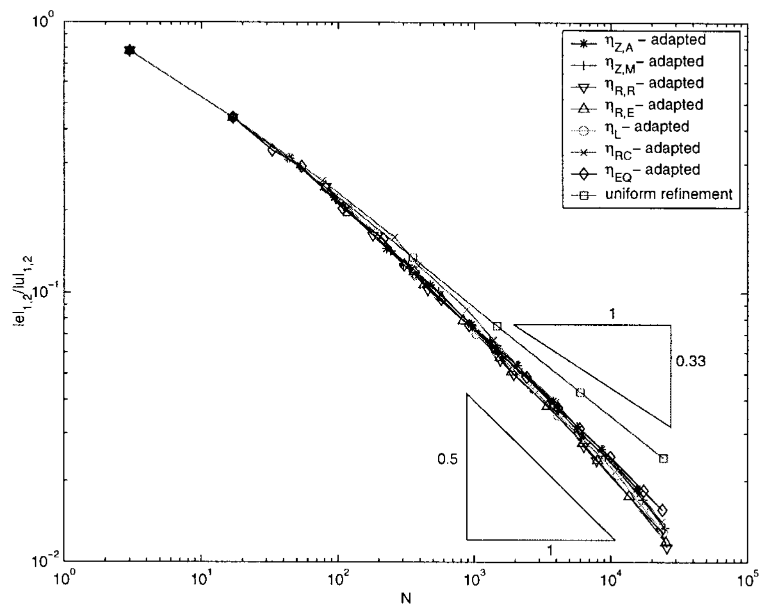

Figure 5. Relative error on a uniform and on adaptively refined meshes in example 8.1.

with local refinement indicators related to $\eta_{R,R}$. The experimental convergence rate $2/3$ for uniform mesh refinement, cf. figure 1, is improved to the optimal value 1. A relative error $\leqslant 10\%$ is achieved with about 400 degrees of freedom in comparison to about 800 for uniform mesh refinement. Moreover, the reliable error estimator η_L guarantees that the relative error is $\leqslant 10\%$ for about 1000 unknowns.

To discuss the performance as refinement indicators, figure 5 displays the relative energy error on a uniformly and on the various adaptively refined meshes. The residual based error estimators generate the best meshes while the averaging estimators produce the worst. All adaptive refinement strategies improve the experimental convergence rate $2/3$ of algorithm (A_0) to the optimal value 1.

Example 8.2 [22]. Let $f := -\Delta u$ for the function $u(x, y) := x(1-x)y(1-y) \times \arctan(60(r-1))$, $r^2 := (x-1.25)^2 + (y+0.25)^2$ on the unit square $\Omega := (0,1)^2$. The solution u to (1.1) is H^2-regular but f (although theoretically smooth) has large gradients on the circle with radius 1 around $(1.25, -0.25)$. The energy of the solution is $\|\nabla u\|_{L^2(\Omega)} = 0.4839$. The coarsest triangulation $\mathcal{T}_0$ consists of 16 congruent squares halved by diagonals parallel to the vector $(1, 1)$ with $N = 9$.

Figure 6 shows the relative energy error and estimators on a sequence of meshes generated by algorithm (A_0). Since u is H^2-regular, we obtain the experimental convergence rate 1, but the relative error remains larger than 10% up to 10,000 degrees of freedom. Although the adaptive refinement strategy does not improve the experimental convergence rate, figure 7 shows that it leads to a significant error reduction. A mesh

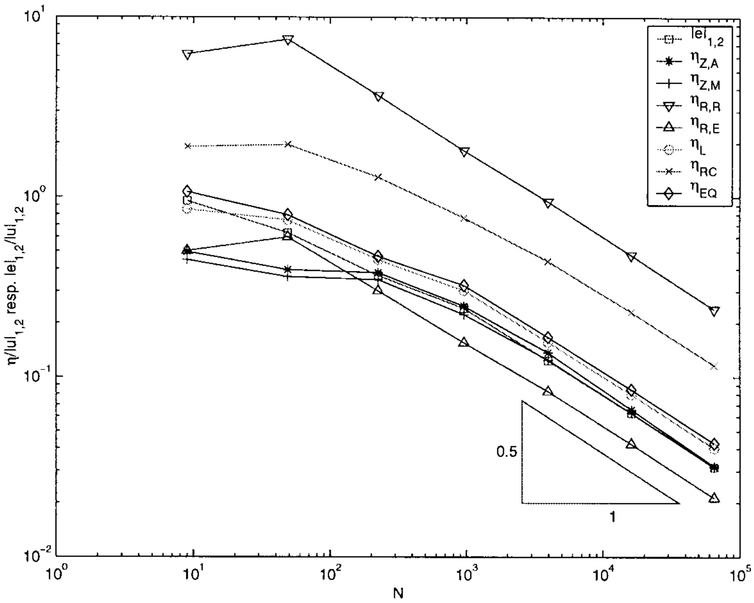

Figure 6. Relative error and error estimators for uniform mesh refinement in example 8.2.

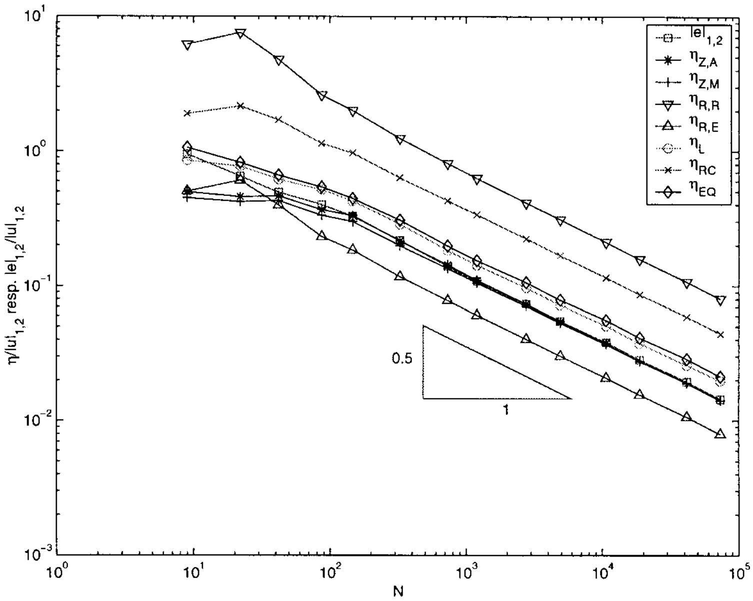

Figure 7. Relative error and error estimators for $\eta_{R,R}$ adaptive mesh refinement in example 8.2.

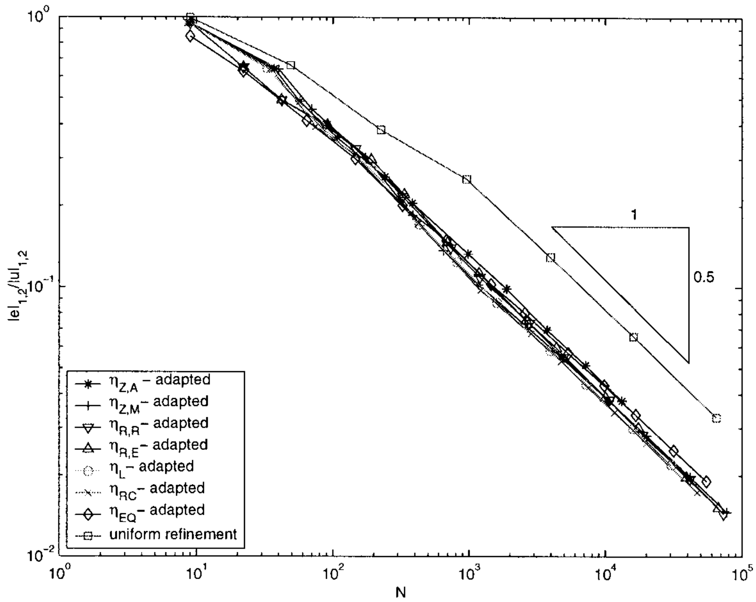

Figure 8. Relative error on a uniform and on adaptively refined meshes in example 8.2.

generated by algorithm $(A_{1/2})$ with local refinement indicators related to $\eta_{\mathrm{R,R}}$ and with only about 1000 unknowns gives a solution with relative error $\leqslant 10\%$.

To discuss the performance as refinement indicators, figure 8 displays the relative energy error on a uniformly and on various adaptively refined meshes. The residual based error estimators generate the best meshes compared to the remaining error estimators.

Example 8.3 [34]. The exact solution of (1.1) on the domain $\Omega = \{(x, y) \in \mathbb{R}: |x| + |y| < 1\}\backslash[0, 1] \times \{0\}$ with $f = 1$ is given (in polar coordinates) by $u(r, \varphi) = r^{1/2} \sin\varphi/2 - 1/2\, r^2 \sin^2\varphi$. The energy norm of the solution is $|u|_{1,2} = 0.9908$. The coarsest triangulation $\mathcal{T}_0$ consists of 16 triangles obtained by red-refining each of the four triangles in Ω minus the x- and y-axis. The solution has a typical corner or crack singularity [34].

Figure 9 shows the relative energy error and estimators on a sequence of meshes generated by algorithm (A_0) with an expected convergence rate $1/2$. Figure 10 displays the results of algorithm $(A_{1/2})$ with local refinement indicator $\eta_{\mathrm{R,R}}$.

It is remarkable that the estimator η_{L} is sharper than η_{EQ} for the adapted meshes while η_{EQ} is sharper than η_{L} for uniform meshes. Hence, we cannot say that η_{L} is always sharper than η_{EQ}.

Figure 11 displays the relative energy error on a uniformly and on the various adaptively refined meshes. All adaptive refinement strategies improve the experimental convergence rate $1/2$ of algorithm (A_0) to the optimal value 1.

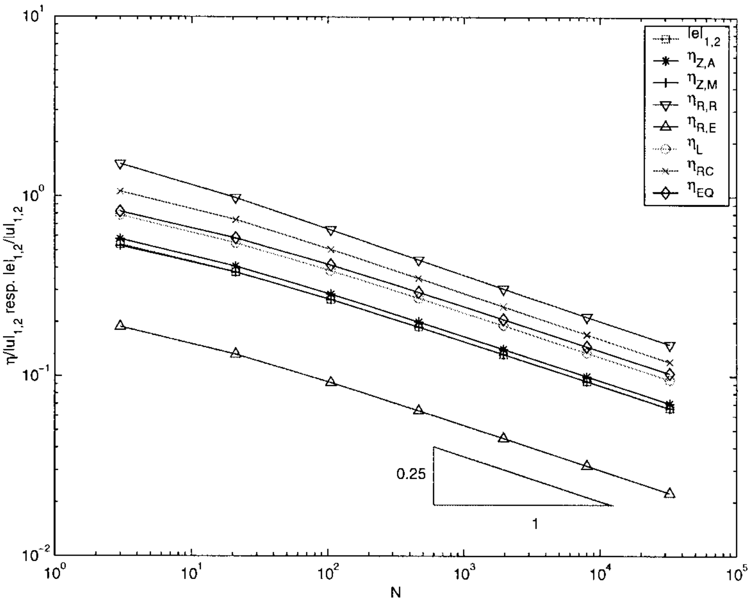

Figure 9. Relative error and error estimators for uniform mesh refinement in example 8.3.

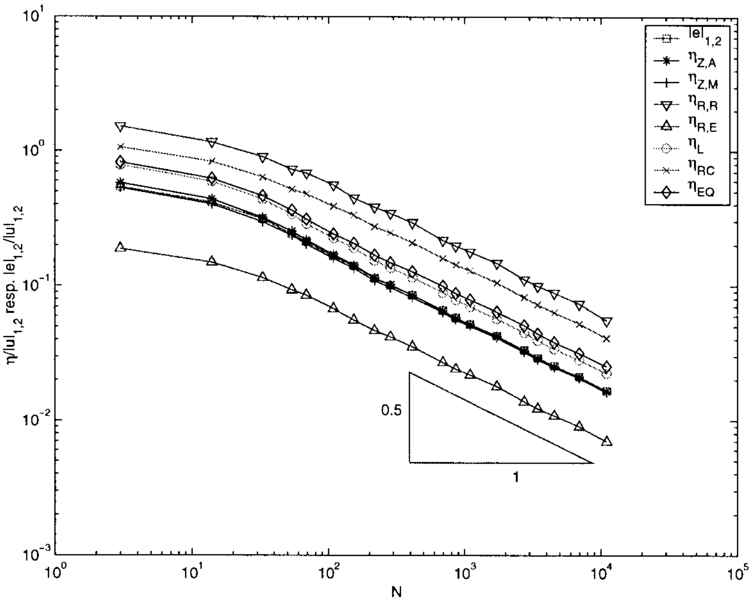

Figure 10. Relative error and error estimators for $\eta_{R,R}$ adaptive mesh refinement in example 8.3.

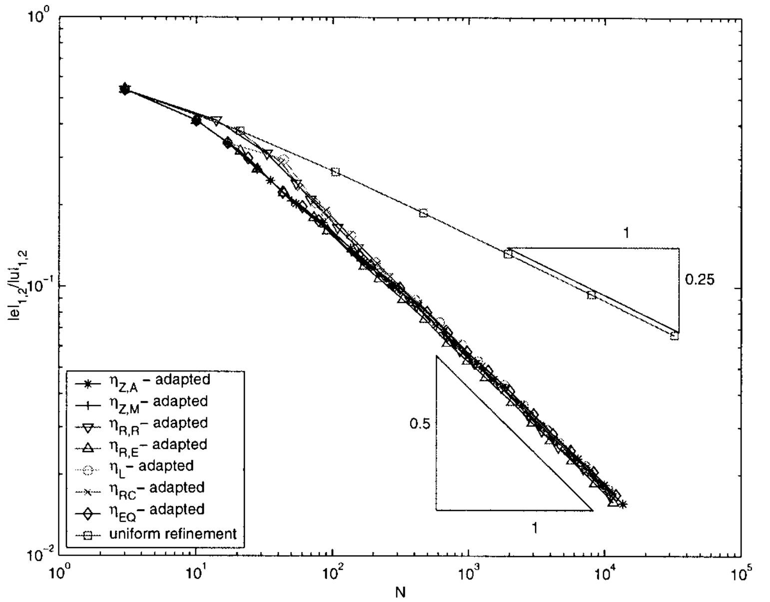

Figure 11. Relative error on a uniform and on adaptively refined meshes in example 8.3.

9. Comparisons and concluding remarks

The theoretical and practical results of this paper support the following observations.

(i) Adaptive mesh refinement may be steered by simple $\eta_{R,R}$-based refinement rules. It does not appear favorable to spend more computer time for more laborious refinement rules if the data are (relatively) smooth.

(ii) There is a need for cheaper and coarser estimators but also for finer and more costly error control. A combination of several estimators is favorable; cf. the example in section 2.

(iii) The residual-based error estimators $\eta_{R,C}$, $\eta_{R,E}$, and $\eta_{R,R}$ are too coarse and not appropriate as a termination criterion for guaranteed error control.

(iv) The ZZ-estimator is a very accurate error guess although $\eta_{Z,M}$ is reliable and efficient only up to multiplicative factors and higher order terms. It is not recommended for guaranteed error control, but may certainly serve as a practical tool for a good error guess.

(v) The ZZ-estimator is recommended as a criterion for a decision either to refine or to employ a fine error estimator for guaranteed error control.

(vi) The error estimators η_L and η_{EQ} behave similarly and are recommended as a termination criterion for guaranteed error control.

(vii) We found that fourth order polynomials are sufficient enough to provide accurate approximations of the guaranteed upper bounds.

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