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Mathematical foundation of a posteriori error estimates and adaptive mesh-refining algorithms for boundary integral equations of the first kind

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Abstract

The article aims to provide a transparent introduction to and a state-of-the-art review on the mathematical theory of a posteriori error estimates for an operator equation $Au = f$ on a one- (or two-) dimensional boundary surface (piece) Γ. Symm's integral equation and a hypersingular equation serve as master examples for a boundary integral operator of the first kind. The non-local character of the involved pseudo-differential operator A and the non-local Sobolev spaces (of functions on Γ) cause difficulties in the mathematical derivation of computable lower and upper error bounds for a discrete (known) approximation u_h to the (unknown) exact solution u. If E denotes the norm of the error $u - u_h$ in a natural Sobolev norm, subtle localization arguments allow the derivation of reliable and/or efficient bounds $\eta = (\sum_{j=1}^{N} \eta_j^2)^{1/2}$. An error estimator η is called efficient if $C_1 \eta \le E$ and reliable if $E \le C_2 \eta$ holds with multiplicative constants C_1 and C_2, respectively, which are independent of underlying mesh-sizes, of data, or of the discrete and exact solution. The presented analysis of reliable and efficient estimates is merely based on elementary calculus such as integration by parts or interchange of the order of integration along the curve Γ.

Four examples of residual-based partly reliable and partly efficient computable error estimators η_j are discussed such as the weighted residuals on an element Γ_j, the localized residual norm on Γ_j, the norm of a solution of a certain local problem, or the correction in a multilevel method.

Since the error estimators can be evaluated elementwise, they motivate error indicators η_j (better be named refinement-indicators) in adaptive mesh-refining algorithms. Although they perform very efficiently in practice, not much is rigorously known on the convergence of those schemes.

Keywords: Boundary element method; Adaptive algorithm; A posteriori error estimate; Reliability; Efficiency

1. What is it all about?

Boundary integral equations on a one-dimensional arc Γ in the plane (or surface $\Gamma \subset \mathbb{R}^3$) read in abstract form: given a right-hand side f and a bounded linear operator A, seek u satisfying

$$Au = f. \tag{1.1}$$

In the context of boundary value problems for partial differential equations, the operator

$$A : H^{s+2\alpha}(\Gamma) \to H^s(\Gamma) \tag{1.2}$$

is a pseudo-differential operator of order $2\alpha \in \mathbb{R}$ between Sobolev–Slobodeckij spaces (examples of A will be given at the end of this section). These spaces $H^t(\Gamma)$ in Eq. (1.2) will be introduced in detail in Section 3; for the sake of the introduction and the presentation of the main results, it suffices to know that certain spline functions $\mathscr{H}(\mathscr{T})$ of (globally continuous or discontinuous, respectively) $\mathscr{T}$-piecewise polynomials based on a partition $\mathscr{T} = \{\Gamma_1, \ldots, \Gamma_N\}$ of Γ are included, i.e. $\mathscr{H}(\mathscr{T}) \subset H^{s+2\alpha}$, and that an integration of f multiplied with an $\mathscr{H}(\mathscr{T})$-function over Γ is meaningful. Then, the Galerkin discretization of Eq. (1.1) reads: seek the Galerkin solution $u_h \in \mathscr{H}(\mathscr{T})$ which satisfies

$$\int_\Gamma A u_h v_h \,\mathrm{d}s = \int_\Gamma f v_h \,\mathrm{d}s \qquad \text{for all } v_h \in \mathscr{H}(\mathscr{T}). \tag{1.3}$$

With f and u_h at hand, the residual $R := f - Au_h$ can (in principle) be evaluated to compute lower and upper bounds for the unknown error $e := u - u_h$. The following sections will be devoted to the design of *error indicators* $\eta_1, \ldots, \eta_N$ which are functions of the residual R, the mesh $\mathscr{T}$, and the index j which reflects an element Γ_j (or a neighborhood of

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it). Error indicators steer the mesh-refinement within adaptive algorithms such as in Step (iv) of Algorithm 1.

Algorithm 1.

(i) Start with an initial mesh $\mathscr{T}_0$ and set $m = 0$.
(ii) Compute Galerkin solution u_h for current mesh $\mathscr{T}_m = \{\Gamma_1, \ldots, \Gamma_N\}$.
(iii) Compute error indicators η_j and choose Θ with $0 \le \Theta \le 1$.
(iv) Refine Γ_j provided $\eta_j \ge \Theta \max\{\eta_1, \ldots, \eta_N\}$.
(v) Generate the new mesh $\mathscr{T}_{m+1}$ and update m. Stop or go to (ii).

Remarks 1.1.

(i) The usage of the phrase 'error indicator' (better named as refinement-indicator) is not specified to quantities which satisfy certain properties; $\eta_j := 1$ is an error indicator as well (and will uniformly refine a mesh when employed in the adaptive algorithm).
(ii) The intention in the design of error indicators, however, is that (a) η_j is a measure of the error near the element Γ_j and that (b) refining the element Γ_j reduces the error as the error indicator there.
(iii) Task (a) is discussed rigorously in a global form in the remainder of this paper; a local form is not available for refined meshes, but there is some hope owing to the pseudo-locality of the operators in Examples 1.1 and 1.2 below.
(iv) There is not even hope for task (b) at the moment! The corresponding question for the finite element method (and the local differential operators) has a positive answer [14]: under some additional assumptions the adaptive algorithm will generate a sequence of meshes whose discretization errors are eventually smaller than a given tolerance. The question remains open for boundary integral equations.
(v) Numerical experience predicts that the schemes will generate convergent discretizations which are far superior to uniform refinements when singularities occur.

To keep the discussion on a posteriori error estimators on the lowest level of technicality, we focus on two typical examples for integral operators of the first kind on one-dimensional arcs which equivalently describe solutions to the Laplace equation in a (bounded or unbounded) domain in the plane.

Example 1.1. The Dirichlet problem for the Laplace equation in the interior or exterior of Γ is equivalently related to Symm's integral equation (1.1), where $A = V$ reads

$$Vu(x) = -\frac{1}{2\pi}\int_\Gamma \log|x-y|u(y)\,\mathrm{d}s_y \qquad (x \in \Gamma); \tag{1.4}$$

($\mathrm{d}s_y$ denotes an integration along $y \in \Gamma$ with respect to the arc-length and the variable y). The single-layer potential operator V is of order $2\alpha = -1$ and $0 < s < 1$ in Eq. (1.2). If the domain is small enough, e.g. included in the unit disc, V is a continuous bijection (i.e. surjective and one-to-one) between $H^{s-1}(\Gamma)$ and $H^s(\Gamma)$. The discrete space $\mathscr{H}(\mathscr{T}) = \mathscr{L}^0(\mathscr{T})$ consists of the (globally possibly discontinuous) $\mathscr{T}$-piecewise constant functions and Eq. (1.3) has a unique solution. With $f \in H^1(\Gamma)$, the residual R belongs to $H^1(\Gamma)$ according to the mapping properties of V on closed Lipschitz boundaries [13]. Consequently, we may evaluate R and its derivative $\partial R/\partial s$ with respect to the arc-length along Γ to compute error indicators.

Example 1.2. The Neumann problem for the Laplace equation in the interior or exterior of Γ is equivalently related to Eq. (1.1), when $A = W$ is the hypersingular integral operator,

$$Wu(x) = \frac{1}{2\pi}\int_\Gamma \frac{\partial^2 \log|x-y|}{\partial n_x\,\partial n_y}u(y)\,\mathrm{d}s_y \qquad (x \in \Gamma), \tag{1.5}$$

(n_x is the unit normal on Γ at x) of order $2\alpha = 1$ and $-1 < s < 0$ in Eq. (1.2). Even for a domain of capacity smaller than 1 (e.g. Γ is included in the unit disc) the hypersingular integral operator W is not a bijection between $H^{s+1}(\Gamma)$ and $H^s(\Gamma)$ since $W1 = 0$. However, W is a continuous bijection between modified spaces $H_0^{s+1}(\Gamma)$ and $H_0^s(\Gamma)$ where functions have vanishing integral mean. Then, one can prove that discrete solutions exist, the residual R belongs to $L^2(\Gamma)$, and satisfies $\int_\Gamma R\varphi_j\,\mathrm{d}s = 0$ for each hat-function φ_j which forms the discrete space $\mathscr{H}(\mathscr{T})$. Hence, we may merely use R, but not its derivative, on Γ and possibly weaker seminorms to compute error indicators.

In the sequel we will refer to the two examples by writing $A = V$ or $A = W$, respectively, and then implicitly assume that α, s, $\mathscr{H}(\mathscr{T})$, and R are defined according to Examples 1.1 and 1.2, respectively.

The plan for the remainder of this paper is as follows. Section 2 reviews some notation on elements and patches and then defines the four error indicators which are analyzed in the remaining sections. Difficulties arising within mathematics illustrate in Section 3 why we should be very careful (in dealing with different but essentially equivalent notions of fractional-order Sobolev spaces required to define errors and residuals). This entire paper appears to be the first where a thorough analysis is based on a single explicit definition of those spaces by the explicit Sobolev–Slobodeckij norm (cf. Eq. (3.2) below for its definition) based on the second author's earlier work [17–19].

Consequently, every estimate proven in Sections 4–7 is an implication of integration by parts (merely in one dimension) and interchanging the order of integration (Fubini's lemma)! Technicalities are kept at the lowest level possible and so the analysis is often restricted to real intervals in order to make it, in principle, easy to read.

Section 4 concentrates on the localization property of the non-local Sobolev–Slobodeckij norm as the main auxiliary result. The subsequent sections concern local double-integral seminorms as error estimators (Section 5), Babuška–Rheinboldt-type error estimates (Section 6), weighted-residual error estimators (Section 7), and multilevel error estimators (Section 8). Connections of related efficient and/or reliable error estimates are drawn for the master situations of Examples 1.1 and 1.2 and links are provided to more general cases and higher dimensions.

2. Reliable and efficient a posteriori error estimators

This section introduces various error indicators whose reliability and efficiency is then studied in the remainder of this paper. Although focus is on Galerkin boundary element schemes for Symm's or the hypersingular integral equation (cf. Examples 1.1 and 1.2), some error estimates are available for collocation and qualocation.

2.1. Elements and neighborhoods

Let $\gamma : [0, L] \to \Gamma$ be the arc-length parameterization of an open or closed Lipschitz curve $\Gamma \subset \mathbb{R}^2$ and let $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_N = L$ be a partition of the parameter interval. Then, $\Gamma_j := \{\gamma(s) : \xi_{j-1} \le s \le \xi_j\}$ and $x_j := \gamma(\xi_j)$ define the mesh, i.e. a collection of elements, $\mathcal{T} = \{\Gamma_1, \ldots, \Gamma_N\}$ and the nodes $\mathcal{N} := \{x_0, x_1, \ldots, x_N\}$. If Γ is closed, $x_0 = x_N$ and Γ_1 and Γ_N are neighbors. In any case, $\gamma : (0, L) \to \Gamma$ is an injective Lipschitz mapping with $|\gamma'| = 1$ almost everywhere.

For a node x_j and an element Γ_j we define their neighborhoods

$$\omega_j := \cup\{\Gamma_k : x_j \in \Gamma_k\} \quad \text{and } \hat{\Gamma}_j := \cup\{\Gamma_k : \Gamma_j \cap \Gamma_k \neq \emptyset\}. \tag{2.1}$$

2.2. Weighted-residual error indicators

Given a residual $R \in H^k(\Gamma)$ we could calculate the derivative $\partial^k R/\partial s^k$ and its L^2-norm $\|\partial^k R/\partial s^k\|_{L^2(\Gamma_j)}$ over one element Γ_j of length $h_j = |\Gamma_j|$ and so the error indicators

$$\eta_{R,j} := h_j^s \|\partial^k R/\partial s^k\|_{L^2(\Gamma_j)}. \tag{2.2}$$

The weighted-residual error indicators (2.2) are applicable for Example 1.1 with $k = 1$ and $0 < s < 1$ for error control in H^{s-1} and in Example 1.2 with $k = 0$ and $0 < s < 1$ for error control in H^s.

The error indicators (2.2) were established in Refs. [4,5,7], for the hp-method in Refs. [1,8], and for three dimensions recently in Refs. [9,12]. In practice, a (e.g. Simpson's) quadrature rule is required to approximate the $L^2(\Gamma_j)$-norm [7].

2.3. Local double-integral seminorm error indicators

Local double-integral seminorms arose in Refs. [18,19] as error indicators for Example 1.1. Their computation requires a double integration over the parameter domain $(\xi_{j-1}, \xi_{j+1}) \times (\xi_{j-1}, \xi_{j+1})$,

$$\eta_{F,j}^2 := |R|_{s,\omega_j}^2 := \int_{\xi_{j-1}}^{\xi_{j+1}} \int_{\xi_{j-1}}^{\xi_{j+1}} \frac{|R(\gamma(x)) - R(\gamma(y))|^2}{|\gamma(x) - \gamma(y)|^{1+2s}} \, dx \, dy, \tag{2.3}$$

(γ is the parameterization of Γ) approximated, e.g. by Simpson's tensor product rule.

2.4. Babuška–Rheinboldt-type error indicators

The Babuška–Rheinboldt-type error indicators, based on an idea in the finite element literature [2], are suggested in Refs. [15–17], read (with hat-function φ_j of node x_j)

$$\eta_{B,j} = \sup_{\substack{v \in H^\alpha(\Gamma) \\ \varphi_j v \neq 0}} \frac{|\int_\Gamma R\varphi_j v \, ds|}{\|\varphi_j v\|_{\alpha,\Gamma}} \tag{2.4}$$

for Example 1.2 and, besides numerical integrations, require further numerical computations (cf. Remark 6.1 (i) for details).

2.5. Multilevel error indicators

Given Symm's integral equation (1.1) with $A = V$ and a mesh $\mathcal{T}$ we introduce new finer ansatz functions χ_j with support on Γ_j which are $+1$ along the first and equal to -1 on the second half of Γ_j. Then, the multilevel error estimator reads

$$\eta_{M,j} := \frac{\left|\int_{\Gamma_j} R\chi_j \, ds\right|}{\left(\int_{\Gamma_j} (V\chi_j)\chi_j \, ds\right)^{1/2}}, \tag{2.5}$$

for $j = 1, \ldots, N$. The reliability of this estimator depends very much on the saturation condition (cf. Eq. (8.5) for a definition). The indicator was introduced in Refs. [23,25] and performs very efficiently in practice.

2.6. Other suggested error indicators

At least three other error indicators are certainly worth mentioning which are, for different reasons, not analyzed in this paper.

2.6.1. Rank–Wendland–Yu error indicators

The earliest suggestions for error indicators and adaptive boundary element methods in the engineering [27,28] and mathematical literature [33–35], respectively, were based on the notion of an influence index and utilize strengthened Cauchy inequalities. They are known to be equivalent to the error indicators $\eta_{B,j}$ of Section 6 but are more complicated to define and to compute. We refer to Refs. [16,17] for proofs and details.

2.6.2. Feistauer–Hsiao–Kleinmann error indicators

The error indicator in Ref. [20] for Example 1.1 treats the outer integration in the Sobolev–Slobodeckij norm of the residual for each element and the inner integration on the entire boundary. Thereby, the computational costs are doubled and those indicators are not fully local. Since we regard the error indicators $\eta_{F,j}$ of Section 5 as a more local and cheaper improvement, we omit a detailed presentation of the Feistauer–Hsiao–Kleinmann error indicators.

2.6.3. Recovering techniques for error indication

Superconvergence properties or gradient recovery techniques for the purpose of adaptive mesh-refinements are employed in Ref. [31] and suggested for error indication. However, corners and lack of smoothness of the exact solution deserve further investigations [29].

2.6.4. Correction methods for error indication

An integral equation of the first kind provides another identity for the residual and characterizes the error $u - u_h$. Based on a Neumann series, a sequence of improved error approximations can be computed [24,30,32]. The norms of those error terms might serve as error indicators. Since this new and promising approach is completely different from the other more residual-based estimates and works for the direct method (where the aforementioned identities are known), we omit a detailed presentation in this paper.

2.7. Efficiency and reliability

Up to this point, error indicators such as η_j were defined, which are based on the information of the discrete solution and the right-hand side plus some numerical quadrature or other local computations. We speak of error indicators η_j in the context of adaptive algorithms.

The sum of all of those error indicators $\eta = (\sum_{j=1}^{N} \eta_j^2)^{1/2}$ might be regarded as an approximation of the norm of the unknown error $u - u_h$ in which context we speak of error estimators.

If there exists a constant c_1 such that

$$\|u - u_h\|_{H^\alpha(\Gamma)} \le c_1 \eta \tag{2.6}$$

then the error estimator η is called reliable and if there exists a constant c_2 such that

$$c_2 \eta \le \|u - u_h\|_{H^\alpha(\Gamma)} \tag{2.7}$$

the error estimator η is called efficient. In this context the constants should depend neither on the mesh-sizes nor on u, u_h, or f but possibly on Γ and α. They might depend in a mild and computable form on $\mathscr{T}$ such as through $\kappa(\mathscr{T})$ defined below.

2.8. Comparison of the error indicators

Throughout this paper, we will prove that η_F is efficient and reliable under mild conditions on the mesh and that, even in a more local form,

$$c_3 \eta_M \le \eta_F \le c_4 \eta_R \tag{2.8}$$

for Symm's integral equation $A = V$. The efficiency of η_R is widely open and the reliability of η_M is not guaranteed a priori.

2.9. Assumptions on the mesh

Although highly graded meshes are considered, i.e. the mesh-sizes $h_j := |\Gamma_j| := \xi_j - \xi_{j-1}$, $|\cdot| :=$ length$(\cdot)$, may vary for different j, some assertions require that the difference in size of two neighboring elements is not too big.

Definition 2.1. The local mesh ratio $\kappa(\mathscr{T})$ of a mesh $\mathscr{T}$ is the smallest number κ that satisfies

$$\kappa^{-1} \le \frac{|\Gamma_k|}{|\Gamma_j|} \le \kappa \tag{2.9}$$

for all $j, k \in \{1, \dots, N\}$ such that Γ_j and Γ_k are neighbors.

Example 2.1. Note that even a geometric mesh-refinement $\xi_0 = 0$ and $\xi_j = Lq^{N-j}$ for $j = 1, \dots, N$ and $0 < q < 1$ leads to a local mesh ratio $\kappa(\mathscr{T}) = 1/q$ if Γ is an open arc (while otherwise $\kappa(\mathscr{T}) = \max\{1/q, (1-q)q^{1-N}, q^{N-1}/(1-q)\}$ is unbounded as $N \to \infty$).

3. Good reasons for caution

Partial differential equations and therefore related (essentially equivalent) integral equations have weak solutions exactly in Sobolev spaces. Second-order elliptic differential equations have solutions in Sobolev spaces of integer order $k \ge 0$ such as $H^k(\Gamma)$ on a domain Γ which consists of all functions $u : \Gamma \to \mathbb{R}$ with weak derivatives u, $\mathrm{D}u, \dots, \mathrm{D}^k u$, which are square integrable over Γ; the norm in $H^k(\Gamma)$ is then given by

$$\|u\|_{H^k(\Gamma)} := \left(\sum_{j=0}^{k} \|D^j u\|_{L^2(\Gamma)}^2 \right)^{1/2}, \tag{3.1}$$

and $\mathrm{D}^j u$ denotes the matrix of all partial derivatives of exact order j. The weak solutions of related integral equations are usually the traces (i.e. the values on the boundary or an interface of the domain) of Sobolev functions of integer order and so they naturally belong to a much more complicated trace space $H^{k-1/2}(\Gamma)$, $k \geq 1$. One direct way to define the most prominent trace space $H^{1/2}(\Gamma)$ utilizes the Sobolev–Slobodeckij norm (also denoted as double-integral norm in view of Eq. (3.3))

$$\|u\|_{s,\Gamma} := (\|u\|^2_{L^2(\Gamma)} + |u|^2_{s,\Gamma})^{1/2} \tag{3.2}$$

of a function $u : \Gamma \to \mathbb{R}$ with $0 < s < 1$ (instead of only $s = 1/2$ for $H^{1/2}(\Gamma)$) and the seminorm

$$|u|^2_{s,\Gamma} := \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} \, \mathrm{d}s_x \, \mathrm{d}s_y. \tag{3.3}$$

(The notation $\int_\Gamma \mathrm{d}s_x$ means that we integrate over Γ with respect to the variable x.) The seminorm $|u|_{s,\Gamma}$ is defined for a domain $\Gamma \subseteq \mathbb{R}^d$ and for a d-dimensional manifold $\Gamma \subseteq \mathbb{R}^{d+1}$, respectively. The elements in $H^s(\Gamma)$ are all measurable functions $u : \Gamma \to \mathbb{R}$ (such as piecewise continuous functions) with $\|u\|_{s,\Gamma} < \infty$.

The reader is assured that typical solutions of integral equations on open surface pieces, e.g. $\Gamma = [0,1]^d \times \{0\}$ (which is not the boundary of a bounded domain), fail to belong to $H^k(\Gamma)$ for $k = 0$ or $k = 1$ but they belong to $H^s(\Gamma)$ for all $0 < s < 1$ which may be regarded as spaces between $H^0(\Gamma) \equiv L^2(\Gamma)$ and $H^1(\Gamma)$.

As a consequence we need to estimate the residual $R := f - Au_h$ in spaces like $H^s(\Gamma)$. One difficulty is the non-locality of those spaces which we illustrate with the simple observation that

$$\|v\|^2_{H^k(\Gamma)} = \sum_{j=1}^N \|v\|^2_{H^k(\Gamma_j)} \qquad \text{for any } v \in H^k(\Gamma), \tag{3.4}$$

whenever $\mathcal{T} = \{\Gamma_1, \ldots, \Gamma_N\}$ is a partition of Γ and k is a non-negative integer. For the double-integral norm, we still have

$$\sum_{j=1}^N \|v\|^2_{s,\Gamma_j} \leq \|v\|^2_{s,\Gamma}, \tag{3.5}$$

but the reverse inequality does not even hold in the more general form

$$\|v\|^2_{s,\Gamma} \leq C \sum_{j=1}^N \|v\|^2_{s,\Gamma_j}. \tag{3.6}$$

This section is concluded with two results to show that Eq. (3.6) fails for general functions and meshes. In the subsequent sections we study sufficient conditions on v and $\mathcal{T}$, which guarantee an assertion similar to Eq. (3.6).

The first negative result indicates that Eq. (3.6) may fail even for $\mathcal{T}$-piecewise functions with respect to a uniform mesh $\mathcal{T}$.

Theorem 3.1. *Let $\mathcal{T}_N$ be the uniform mesh on $\Gamma = [0,1]$ with $2N$ elements, i.e. $\mathcal{T}_N = \{\Gamma_{j,N} : j = 1, \ldots, 2N\}$ with $\Gamma_{j,N} := [(j-1)h, jh]$, and with mesh-size $h := 1/(2N)$. For $0 < s < 1/2$ there exists a sequence of $\mathcal{T}$-piecewise constant functions $v_N \in H^s(\Gamma)$ with*

$$\lim_{N\to\infty} \frac{\|v_N\|^2_{s,\Gamma}}{\sum_{j=1}^{2N} \|v_N\|^2_{s,\Gamma_{j,N}}} = \infty. \tag{3.7}$$

Proof. The discontinuous piecewise constant function

$$v_N(x) := \begin{cases} 0 & \text{for } x \in \Gamma_{j,N} \text{ with } j \text{ even}, \\ 1 & \text{for } x \in \Gamma_{j,N} \text{ with } j \text{ odd}, \end{cases}$$

satisfies $v_N \in H^s(\Gamma)$ for $0 < s < 1/2$ and the denominator in Eq. (3.7) is given by

$$\sum_{j=1}^{2N} \|v_N\|^2_{s,\Gamma_{j,N}} = \sum_{j=1}^{2N} \|v_N\|^2_{L^2(\Gamma_{j,N})} = \sum_{\substack{j=1 \\ j \text{ odd}}}^{2N} \int_{(j-1)h}^{jh} 1 \, \mathrm{d}x = 1/2. \tag{3.8}$$

The double-integral norm $\|\cdot\|_{s,\Gamma}$ is equivalent to the Fourier norm defined by

$$\|v\|^2_{H^s_{\text{four}}(\Gamma)} := |\mathcal{F}_0(v)|^2 + \sum_{\ell \in \mathbb{Z}\setminus\{0\}} |\mathcal{F}_\ell(v)|^2 |\ell|^{2s}$$

with the Fourier coefficients $\mathcal{F}_\ell(v) := \int_0^1 v(x)\, \mathrm{e}^{-2\pi i \ell x} \, \mathrm{d}x$ (see, e.g. Ref. [21, Corollary 8.6]). By explicit computations (see Ref. [17, Satz 3.1] for more details) one can prove

$$\|v_N\|^2_{H^s_{\text{four}}(\Gamma)} \geq N^{2s} \frac{1}{\pi^2} \sum_{\ell=1}^\infty (2\ell + 1)^{2(s-1)}.$$

Hence, $\|v_N\|_{H^s_{\text{four}}(\Gamma)}$ as the equivalent norm $\|v_N\|_{s,\Gamma}$ tend with N to ∞. □

The second negative result indicates that Eq. (3.6) may fail even for $\kappa(\mathcal{T}) \to \infty$.

Theorem 3.2. Let $0 < \varepsilon < 1$ *and set* $v_\varepsilon \in H^1(-1,1)$ *by*

$$v_\varepsilon(x) := \begin{cases} 0 & \text{for } x \in \Gamma_1 := [-1, 0], \\ x/\varepsilon & \text{for } x \in \Gamma_2 := [0, \varepsilon], \\ 1 & \text{for } x \in \Gamma_3 := [\varepsilon, 1]. \end{cases}$$

Then we have

$$\lim_{\varepsilon\to 0}\frac{\|v_\varepsilon\|^2_{1/2,(-1,1)}}{\sum_{j=1}^{3}\|v_\varepsilon\|^2_{1/2,\Gamma_j}}=\infty. \tag{3.9}$$

Proof. The proof is given by explicit calculations and elementary estimations in Ref. [17, Appendix] where $\sum_{j=1}^3\|v_\varepsilon\|^2_{H^{1/2}(\Gamma_j)}\le 2$ while $\|v_\varepsilon\|^2_{H^{1/2}(-1,1)}\ge -\log(2\varepsilon)+2\log(1+\varepsilon)-\log(\varepsilon)$. □

4. Heart of the matter: localization of Sobolev–Slobodeckij norms

In view of Theorems 3.1 and 3.2 we need further assumptions on the local domains and on the functions to establish a valid modification of Eq. (3.6).

Theorem 4.1. *Suppose* $v\in H^s(\Gamma)$ *satisfies* $\int_{\Gamma_j} v\,\mathrm{d}s=0$ *for all* $j=1,\dots,N$. *Then,*

$$\|v\|^2_{s,\Gamma}\le c_5\sum_{j=1}^{N}|v|^2_{s,\omega_j} \tag{4.1}$$

with a (v,N)*-independent constant* c_5 *that depends only on* $0<s<1$, Γ, *and* $\kappa(\mathcal{T})$.

The proof of Theorem 4.1 is divided into three lemmas. Set $B_{d_j}(y):=(y-d_j,y+d_j)$ for $y\in\mathbb{R}$ and

$$d_j:=\operatorname{dist}(\Gamma_j,\Gamma\backslash\hat\Gamma_j). \tag{4.2}$$

Lemma 4.2. *For any function* $v\in H^s(\Gamma)$ *and all meshes* $\mathcal{T}$, *we have*

$$\|v\|^2_{s,\Gamma}\le c_6\sum_{j=1}^{N}(|v|^2_{s,\omega_j}+d_j^{-2s}\|v\|^2_{L^2(\Gamma_j)}) \tag{4.3}$$

with a $(v,\mathcal{T})$*-independent constant* c_6 *that depends only on* $0<s<1$ *and* Γ.

Proof. For simplicity, we focus on $\Gamma=[0,1]$ (the proof for a general boundary $\Gamma=\partial\Omega$ with a domain $\Omega\subseteq\mathbb{R}^{d+1}$ can be found in Refs. [18,19]). Abbreviate $D_j:=\Gamma\backslash\hat\Gamma_j$ and

$$\int_{\Gamma'}\int_{\Gamma''}:=\int_{\Gamma'}\int_{\Gamma''}\frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \quad\text{and}$$

$$\int\int_{\Gamma'}:=\int_{\Gamma'}\int_{\Gamma'}\qquad\text{for }\Gamma',\Gamma''\subseteq\Gamma.$$

Then

$$\begin{aligned}|v|^2_{s,\Gamma}&=\int_\Gamma\int_\Gamma=\sum_{j=1}^{N}\int_{\Gamma_j}\int_\Gamma\\&=\sum_{j=1}^{N}\Bigg[\underbrace{\int\int_{\Gamma_j}+\int_{\Gamma_j}\int_{\Gamma_{j-1}}+\int_{\Gamma_j}\int_{\Gamma_{j+1}}}_{\le(1/2)\int\int_{\omega_{j-1}}+(1/2)\int\int_{\omega_j}}+\int_{\Gamma_j}\int_{D_j}\Bigg]\\&\le\sum_{j=1}^{N}|v|^2_{s,\omega_j}+\sum_{j=1}^{N}\int_{\Gamma_j}\int_{D_j}\frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{4.4}$$

Note that $D_j\subseteq\mathbb{R}\backslash B_{d_j}(y)$ for $y\in\Gamma_j$ (since $d_j=\operatorname{dist}(\Gamma_j,D_j)$) and so

$$\begin{aligned}\int_{D_j}|x-y|^{-1-2s}\,\mathrm{d}x&\le\int_{\mathbb{R}\backslash B_{d_j}(y)}|x-y|^{-1-2s}\,\mathrm{d}x\\&=2\int_{d_j}^{\infty}x^{-1-2s}\,\mathrm{d}x=d_j^{-2s}/s.\end{aligned} \tag{4.5}$$

Therefore,

$$J_{j,1}:=\int_{\Gamma_j}|v(y)|^2\left(\int_{D_j}|x-y|^{-1-2s}\,\mathrm{d}x\right)\mathrm{d}y\le s^{-1}d_j^{-2s}\|v\|^2_{L^2(\Gamma_j)}. \tag{4.6}$$

We infer with the characteristic function $\chi_j:=\chi_{D_j}$ of D_j

$$\begin{aligned}\sum_{j=1}^{N}J_{j,2}&:=\sum_{j=1}^{N}\int_{D_j}|v(x)|^2\left(\int_{\Gamma_j}|x-y|^{-1-2s}\,\mathrm{d}y\right)\mathrm{d}x\\&=\sum_{j=1}^{N}\int_\Gamma\chi_j(x)|v(x)|^2\left(\int_{\Gamma_j}|x-y|^{-1-2s}\,\mathrm{d}y\right)\mathrm{d}x\\&=\int_\Gamma|v(x)|^2\Bigg(\underbrace{\sum_{j=1}^{N}\chi_j(x)\int_{\Gamma_j}|x-y|^{-1-2s}\,\mathrm{d}y}_{=:f(x)}\Bigg)\mathrm{d}x.\end{aligned}$$

Let $k,j\in\{1,\dots,N\}$ and $x\in\Gamma_k$. Then, $\chi_j(x)=1$ is equivalent to $x\in D_j=\Gamma\backslash[\Gamma_{j-1}\cup\Gamma_j\cup\Gamma_{j+1}]$, and so to $j\notin\{k-1,k,k+1\}$. Consequently,

$$\begin{aligned}f(x)&=\sum_{\substack{j=1\\ j\notin\{k-1,k,k+1\}}}^{N}\int_{\Gamma_j}|x-y|^{-1-2s}\,\mathrm{d}y\\&=\int_{D_k}|x-y|^{-1-2s}\,\mathrm{d}y\overset{(4.5)}{\le}d_k^{-2s}/s\end{aligned}$$

and so

$$\sum_{j=1}^{N}J_{j,2}\le\frac{1}{s}\sum_{k=1}^{N}d_k^{-2s}\|v\|^2_{L^2(\Gamma_k)}. \tag{4.7}$$

Young's inequality in Eq. (4.4) (i.e. $|v(x) - v(y)|^2 \le 2(|v(x)|^2 + |v(y)|^2)$) and Eqs. (4.6) and (4.7) show that

$$|v|^2_{s,\Gamma} \le \sum_{j=1}^{N} |v|^2_{s,\omega_j} + \frac{4}{s}\sum_{j=1}^{N} d_j^{-2s}\|v\|^2_{L^2(\Gamma_j)}. \qquad \square$$

For $s = 1$ (and $|v|_{s,(a,b)}$ replaced by $|v|_{H^1(a,b)}$), the following estimate is known as a Poincaré inequality (cf. e.g. Ref. [26], Theorem 1.3).

Lemma 4.3. *For $a < b$, $0 < s < 1$, and $v \in H^s(a,b)$ we have*

$$\|v\|^2_{L^2(a,b)} \le \frac{1}{2}(b-a)^{2s}|v|^2_{s,(a,b)} + \frac{1}{b-a}\left(\int_a^b v(x)\,\mathrm{d}x\right)^2. \quad (4.8)$$

Proof. Elementary calculations with the binomial theorem yield

$$\begin{aligned}
&2(b-a)\|v\|^2_{L^2(a,b)} - 2\left(\int_a^b v(x)\,\mathrm{d}x\right)^2 \\
&\quad = \int_a^b\int_a^b v(x)^2\,\mathrm{d}x\,\mathrm{d}y + \int_a^b\int_a^b v(y)^2\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad - 2\int_a^b\int_a^b v(x)v(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_a^b\int_a^b (v(x)-v(y))^2\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \le (b-a)^{1+2s}\int_a^b\int_a^b \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = (b-a)^{1+2s}|v|^2_{s,(a,b)}.
\end{aligned}$$

$\square$

Lemma 4.4. *Suppose $v \in H^s(\Gamma)$ satisfies $\int_{\Gamma_j} v\,\mathrm{d}s = 0$ for all $j = 1,\dots,N$. Then,*

$$\sum_{j=1}^{N} d_j^{-2s}\|v\|^2_{L^2(\Gamma_j)} \le c_7\sum_{j=1}^{N}|v|^2_{s,\Gamma_j}. \quad (4.9)$$

with a (v,N)-independent constant c_7 that depends only on $0 < s < 1$, Γ, and $\kappa(\mathscr{T})$.

Proof. For simplicity, we focus on $\Gamma = [0,1]$ and obtain from Lemma 4.3 that

$$\sum_{j=1}^{N} d_j^{-2s}\|v\|^2_{L^2(\Gamma_j)} \overset{(4.8)}{\le} \frac{1}{2}\sum_{j=1}^{N} d_j^{-2s}|\Gamma_j|^{2s}|v|^2_{s,\Gamma_j} \overset{(2.9)}{\le} \frac{\kappa^{2s}}{2}\sum_{j=1}^{N}|v|^2_{s,\Gamma_j}.$$

$\square$

Proof of Theorem 4.1. Combine Lemmas 4.2 and 4.4. $\square$

Remarks 4.1.

(i) If Γ is (part of) a Lipschitz boundary, we have for sufficiently close $x = \gamma(a)$ and $y = \gamma(b)$ on Γ that

$$C|a-b| \le |x-y| \le |a-b|. \quad (4.10)$$

Therefore, our calculations with Sobolev–Slobodeckij norms can essentially be reduced (by cut-off, transformations and resolution of periodicity for closed arcs) to calculations on the parameter interval. As a consequence, we may and will illustrate our proofs for the case $\Gamma \subset \mathbb{R}$.
(ii) Theorems 4.1 and 4.4 are formulated for functions $v \in H^s(\Gamma)$ which are L^2-orthogonal to $\mathscr{L}^0(\mathscr{T})$. The results also hold for functions which are L^2-orthogonal to quite arbitrary finite element spaces $\mathscr{H}(\mathscr{T})$ [18,19].

5. Local double-integral seminorms as error estimators

Theorem 4.1 establishes that local Sobolev–Slobodeckij seminorms $\eta_{F,j} := |R|_{s,\omega_j}$ of the residual $R := f - Vu_h \in H^1(\Gamma)$ serve as efficient and reliable local error indicators for Symm's integral equation (1.1) where $A = V$, $u \in H^{s-1}(\Gamma)$ and $u_h \in \mathscr{L}^0(\mathscr{T})$ satisfy Eqs. (1.1) and (1.3), respectively.

Theorem 5.1. *For some (N,u,f,u_h)-independent positive constants c_8 and c_9 we have*

$$c_8\sum_{j=1}^{N}|R|^2_{s,\omega_j} \le \|u-u_h\|^2_{s-1,\Gamma} \le c_9\sum_{j=1}^{N}|R|^2_{s,\omega_j}. \quad (5.1)$$

The constant c_8 depends on $0 < s < 1$, Γ, but not on $\mathscr{T}$, while c_9 depends also on $\kappa(\mathscr{T})$.

Proof. Since $\{\omega_j$: j even$\}$ is (almost) pairwise disjoint, we conclude as in Eq. (3.5) that

$$\sum_{j\ \text{even}}\|R\|^2_{s,\omega_j} \le \|R\|^2_{s,\cup\{\omega_j: j\ \text{even}\}} \le \|R\|^2_{s,\Gamma}. \quad (5.2)$$

A similar estimate holds for $\sum_{j\ \mathrm{odd}} |R|^2_{s,\omega_j}$ and their sum yields

$$\sum_{j=1}^{N} \|R\|^2_{s,\omega_j} = \sum_{j\ \mathrm{even}} \|R\|^2_{s,\omega_j} + \sum_{j\ \mathrm{odd}} \|R\|^2_{s,\omega_j} \le 2\|R\|^2_{s,\Gamma}. \tag{5.3}$$

By continuity of V, we have a bounded operator norm $c_{10} := \|V\|_{L(H^{s-1}(\Gamma);H^s(\Gamma))} < \infty$ with

$$\|R\|_{s,\Gamma} = \|V(u-u_h)\|_{s,\Gamma} \le c_{10}\|u-u_h\|_{s-1,\Gamma}. \tag{5.4}$$

The combination of Eqs. (5.3) and (5.4) proves the efficiency estimate in Eq. (5.1).

The linear and bounded operator $V : H^{s-1}(\Gamma) \to H^s(\Gamma)$ is a bijection and so its inverse $V^{-1} : H^s(\Gamma) \to H^{s-1}(\Gamma)$ exists and furthermore is bounded owing to the open mapping theorem in functional analysis. By continuity of V^{-1}, we have a bounded operator norm $c_{11} := \|V^{-1}\|_{L(H^s(\Gamma);H^{s-1}(\Gamma))} < \infty$ and deduce, similarly to Eq. (5.4),

$$\|u-u_h\|_{s-1,\Gamma} = \|V^{-1}V(u-u_h)\|_{s-1,\Gamma} \le c_{11}\|V(u-u_h)\|_{s,\Gamma} = c_{11}\|R\|_{s,\Gamma}. \tag{5.5}$$

Since $v_h = 1$ on Γ_j and $v_h = 0$ elsewhere defines an admissible test function in Eq. (1.3), we have

$$\int_{\Gamma_j} R\,\mathrm{d}s = \int_{\Gamma_j} (f - Vu_h)\,\mathrm{d}s = \int_{\Gamma} (f - Vu_h)v_h\,\mathrm{d}s = 0. \tag{5.6}$$

Therefore, Theorem 4.1 shows that

$$\|R\|^2_{s,\Gamma} \le c_5 \sum_{j=1}^{N} |R|^2_{s,\omega_j}. \tag{5.7}$$

The combination of Eqs. (5.5)–(5.7) proves the reliability estimate in Eq. (5.1).

Remarks 5.1.

(i) If a residual R is generally lacking any kind of L^2-orthogonal condition (because it is generated by a method totally different from a Galerkin scheme), then

$$\hat{\eta}_{F,j} := |R|_{s,\omega_j} + d_j^s\|R\|_{L^2(\Gamma_j)} \tag{5.8}$$

still defines a reliable error estimator. (The proof is analogous to that given for Theorem 5.1, but merely employs Lemma 4.2.)
(ii) In case of a Galerkin scheme, Eq. (5.8) defines a reliable and efficient error estimate. (The latter follows from Lemma 4.4.)
(iii) The results carry over to any bounded, linear, and bijective operator $A : H^{s+2\alpha}(\Gamma) \to H^s(\Gamma)$ for $0 < s < 1$ provided $s + 2\alpha < 1/2$.
(iv) The results of Theorem 5.1 can be generalized to Galerkin discretizations where $\mathscr{H}(\mathscr{T})$ consists of quite arbitrary finite element functions [18,19].

6. Babuška–Rheinboldt-type error estimates

This section is devoted to the analysis of the error indicators $\eta_{B,j}$ for the errors in the hypersingular equation (1.1) with $A = W$. Given a mesh $\mathscr{T}$ and the nodes $\mathscr{N}$ we define the hat-functions $\varphi_1, \ldots, \varphi_N \in \mathscr{S}^1(\mathscr{T})$ as $\mathscr{T}$-piecewise affine and continuous mappings (with respect to the parameterization γ) characterized by support ω_j

$$\varphi_j(x_k) = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \ne k, \end{cases} \tag{6.1}$$

which form a partition of unity,

$$\sum_{j=1}^{N} \varphi_j = 1 \qquad \text{on } \Gamma. \tag{6.2}$$

Depending on Γ open or closed, we define Sobolev–Slobodeckij spaces and discrete subspaces thereof. If $f \in H^s(\Gamma)$ is continuous, then $f \in H_0^s(\Gamma)$ if and only if $f(x_0) = 0 = f(x_N)$ and Γ is open or $\int_\Gamma f\,\mathrm{d}s = 0$ and Γ is closed. (For a function $f \in H^s(\Gamma)$ that is discontinuous at the endpoints of an open arc the conditions for $f \in H_0^s(\Gamma)$ are complicated if $s = 1/2$, they are as above for $1/2 < s < 1$ and there are none if $0 < s < 1/2$ [22]). In any case, hat-functions are continuous and so it makes sense to define

$$\mathscr{S}_0^1(\mathscr{T}) := \{v_h \in \mathscr{S}^1(\mathscr{T}) : v_h \in H_0^1(\Gamma)\}. \tag{6.3}$$

It is known that the discrete problem (1.3) has exactly one solution $u_h \in \mathscr{S}_0^1(\mathscr{T})$ and so the residual $R := f - Wu_h$ is a well-defined function in $L^2(\Gamma)$. Recall the mapping property $W : H^{1-\alpha}(\Gamma) \to H^{-\alpha}(\Gamma)$ for $0 < \alpha < 1$ and that the Babuška–Rheinboldt-type error indicators are defined by

$$\eta_{B,j} = \sup_{\substack{v \in H^\alpha(\Gamma) \\ \varphi_j v \ne 0}} \frac{\int_{\omega_j} R\varphi_j v\,\mathrm{d}s}{\|\varphi_j v\|_{\alpha,\Gamma}}. \tag{6.4}$$

Theorem 6.1 ([15,16]). *For some (N, u, f, u_h)-independent positive constants c_{12} and c_{13} we have*

$$c_{12}\sum_{j=1}^{N} \eta^2_{B,j} \le \|u-u_h\|^2_{1-\alpha,\Gamma} \le c_{13}\sum_{j=1}^{N} \eta^2_{B,j}. \tag{6.5}$$

The constant c_{12} depends on $0 < \alpha < 1$, Γ, but not on $\mathscr{T}$, while in addition c_{13} depends on $\kappa(\mathscr{T})$.

Remarks 6.1.

(i) The values of $\eta_{B,j}$ cannot be obtained by an analytical formula, they need to be calculated. If $\mathscr{P}_k$ denotes the space of algebraic polynomials of order ($\leq k$, we could solve a $k \times k$-system of linear equations and compute

$$\eta_{B,j}^{(k)} := \max_{v_k \in \mathscr{P}_k} \frac{\int_{\omega_j} R\varphi_j v_k \, ds}{\|\varphi_j v_k\|_{\alpha,\Gamma}} \leq \eta_{B,j}. \tag{6.6}$$

In case of the energy norm $\alpha = 1/2$ we could replace $\|\varphi_j v_k\|_{\alpha,\Gamma}$ by $\int_\Gamma W(\varphi_j v_k)\varphi_j v_k \, ds$. Note that Eq. (6.6) provides lower bounds of the exact value. Upper bounds are available with Theorem 7.1 below.

(ii) In practice, all (double) integrals are approximated with proper quadrature rules.

The proof of Theorem 6.1 is split into four lemmas.

Lemma 6.2. *For $v \in H^s(\omega_j)$ we have*

$$|\varphi_j v|_{s,\Gamma} \leq c_{14}(|v|_{s,\omega_j} + h_j^{-s}\|v\|_{L^2(\omega_j)}) \tag{6.7}$$

with an (N,v)-independent positive constant c_{14} which depends on $0 < s < 1$, Γ, and $\kappa(\mathscr{T})$.

Proof. Suppose, for simplicity, $\Gamma = [0,1]$ as the general case will follow with similar arguments. Set $\Omega_k := \cup\{\Gamma_\ell : \Gamma_k \cap \hat{\Gamma}_\ell \neq \emptyset\}$ and notice that $|\Omega_k| \leq 5\kappa(\mathscr{T})^2 h_k$.

Splitting domains of integration we infer

$$|\varphi_j v|_{s,\Gamma}^2 - |\varphi_j v|_{s,\omega_j}^2 = 2\int_{\omega_j}\int_{\Gamma\setminus\omega_j} \frac{|\varphi_j v(x)|^2}{|x-y|^{1+2s}} \, dy \, dx. \tag{6.8}$$

For $x \in \Gamma_k \subset \omega_j$ and with $\varphi_j(x) \leq \mathrm{Lip}(\varphi_j)|x-y|$ for $y \in \Gamma\setminus\omega_j$ we infer as in Eq. (4.5) that

$$\int_{\Gamma\setminus\omega_j} \frac{|\varphi_j(x)|^2}{|x-y|^{1+2s}} \, dy = \int_{\Omega_k\setminus\omega_j} \frac{|\varphi_j(x)|^2}{|x-y|^{1+2s}} \, dy + \int_{\Gamma\setminus\Omega_k} \frac{|\varphi_j(x)|^2}{|x-y|^{1+2s}} \, dy$$

$$\leq \mathrm{Lip}(\varphi_j)^2 \int_{\Omega_k\setminus\omega_j} |x-y|^{1-2s} \, dy + d_k^{-2s}/s$$

$$\leq \mathrm{Lip}(\varphi_j)^2 |\Omega_k|^{2(1-s)}/(1-s) + d_k^{-2s}/s. \tag{6.9}$$

Young's inequality with the additive split $\varphi_j(x)v(x) - \varphi_j(y)v(y) = \varphi_j(x)(v(x) - v(y)) + v(y)(\varphi_j(x) - \varphi_j(y))$ shows that

$$\frac{1}{2}|\varphi_j v|_{s,\omega_j}^2 \leq \int_{\omega_j}\int_{\omega_j} \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}} \, dx \, dy + \int_{\omega_j}\int_{\omega_j} v(y)^2 \frac{|\varphi_j(x)-\varphi_j(y)|^2}{|x-y|^{1+2s}} \, dx \, dy$$

$$\leq |v|_{s,\omega_j}^2 + \mathrm{Lip}(\varphi_j)^2 \int_{\omega_j}\int_{\omega_j} v(y)^2 |x-y|^{1-2s} \, dx \, dy$$

$$\leq |v|_{s,\omega_j}^2 + \mathrm{Lip}(\varphi_j)^2 |\omega_j|^{2(1-s)}/(1-s)\|v\|_{L^2(\omega_j)}^2. \tag{6.10}$$

Combining Eqs. (6.9) and (6.10) with estimates of neighboring element-sizes (e.g. $h_j \leq \kappa(\mathscr{T})^2 d_k$ or $\mathrm{Lip}(\varphi_j) \leq \kappa(\mathscr{T})/h_j$) verifies Eq. (6.7). □

Lemma 6.3. *For some (N,u,f,u_h)-independent positive constant c_{13} we have*

$$\|u - u_h\|_{1-\alpha,\Gamma}^2 \leq c_{13} \sum_{j=1}^N \eta_{B,j}^2. \tag{6.11}$$

The constant c_{13} depends on $0 < \alpha < 1$, Γ, and $\kappa(\mathscr{T})$.

Proof. Notice that W^{-1} exists as a linear bounded operator between the Hilbert spaces $H_0^{-\alpha}(\Gamma)$ and $H_0^{1-\alpha}(\Gamma)$ with the α-depending operator norm c_{15}. Hence

$$\|u - u_h\|_{H_0^{1-\alpha}(\Gamma)} = \|W^{-1}R\|_{H_0^{1-\alpha}(\Gamma)} \leq c_{15}\|R\|_{H_0^{-\alpha}(\Gamma)}. \tag{6.12}$$

The definition of the negative norm in $H^{-\alpha}(\Gamma)$ by duality on a closed arc Γ reads

$$\|R\|_{-\alpha,\Gamma} := \sup_{\|v\|_{\alpha,\Gamma}=1} \int_\Gamma Rv \, ds. \tag{6.13}$$

Since $\int_\Gamma R\varphi_j \, ds = 0$ we may choose the integral mean $v_j := \int_{\omega_j} v \, ds$ of v and obtain with Eqs. (6.2), (6.4), and Cauchy's inequality that

$$\int_\Gamma Rv \, ds = \sum_{j=1}^N \int_\Gamma R(v - v_j)\varphi_j \, ds \leq \sum_{j=1}^N \eta_{B,j}\|(v-v_j)\varphi_j\|_{\alpha,\Gamma}$$

$$\leq \left(\sum_{j=1}^N \eta_{B,j}^2\right)^{1/2} \left(\sum_{j=1}^N \|(v-v_j)\varphi_j\|_{\alpha,\Gamma}^2\right)^{1/2}. \tag{6.14}$$

With Lemma 6.2 (with v replaced by $v - v_j$ which has the same seminorm) and Lemma 4.3 (to estimate the L^2-norm of $v - v_j$) we infer

$$\|(v-v_j)\varphi_j\|_{\alpha,\Gamma} \leq c_{14}|v|_{\alpha,\omega_j} + (c_{14}h_j^{-\alpha} + 1)\|v - v_j\|_{L^2(\omega_j)}$$

$$\leq c_{16}|v|_{\alpha,\omega_j}. \tag{6.15}$$

Arguing as in Eq. (5.3) we deduce from Eqs. (6.14) and (6.15)

$$\int_\Gamma Rv\,\mathrm{d}s \le \left(\sum_{j=1}^N \eta_{B,j}^2\right)^{1/2} \sqrt{2}c_{16}|v|_{\alpha,\Gamma}. \tag{6.16}$$

Since $\|v\|_{\alpha,\Gamma} = 1$ in Eq. (6.13), the assertion follows from Eqs. (6.12) and (6.16). □

To show efficiency, we require a modification of Eq. (3.6).

Lemma 6.4. *Let $s > 0$. Then, any functions $v_1, \ldots, v_M \in H^s(\Gamma)$ with pairwise disjoint support satisfy*

$$\left|\sum_{j=1}^M v_j\right|_{s,\Gamma}^2 \le 5/2 \sum_{j=1}^M |v_j|_{s,\Gamma}^2. \tag{6.17}$$

Proof. The arguments are similar as in the proof of Lemma 4.2; cf. Ref. [17, Satz 3.26]. □

Remark 6.2. The reverse inequality of Eq. (6.17) is false: let $0 < \varepsilon < 1$ and set $v_\varepsilon, w_\varepsilon \in H^1(-1,1)$ as in Theorem 3.2 by $w_\varepsilon(x) := v_\varepsilon(-x)$ for $-1 \le x \le 1$. Then we have $v_\varepsilon w_\varepsilon = 0$ and

$$\lim_{\varepsilon\to 0} \frac{\|v_\varepsilon + w_\varepsilon\|_{1/2,(-1,1)}^2}{\|v_\varepsilon\|_{1/2,(-1,1)}^2 + \|w_\varepsilon\|_{1/2,(-1,1)}^2} = 0. \tag{6.18}$$

(The proof is similar to that of Theorem 3.2 and can be found in Ref. [17, Appendix].)

The point in this example is that a hat-function with values 0, 1, 0 at the positions $-a$, 0, b (for positive a, b), respectively, has an $H^{1/2}(\mathbb{R})$-seminorm which depends on b/a but not on $\max\{a,b\}$ and tends logarithmically to ∞ as $\max\{b/a, a/b\} \to \infty$. The situation is different on the smaller domain $(-a,b)$ and the $H^{1/2}(-a,b)$-seminorm of the hat-function stays bounded. We refer to Ref. [7] for more details.

Lemma 6.5. *For some (N,u,f,u_h)-independent positive constant c_{12} we have*

$$c_{12}\sum_{j=1}^N \eta_{B,j}^2 \le \|u - u_h\|_{\alpha-1,\Gamma}^2. \tag{6.19}$$

The constant c_{12} depends on $0 < \alpha < 1$ and Γ, but not on $\mathscr{T}$.

Proof. Notice that the supremum in Eq. (6.4) is attained and so we find $v_j \in H^\alpha(\Gamma)$ with positive norm $\|\varphi_j v_j\|_{\alpha,\Gamma}$ and

$$\eta_{B,j}\|\varphi_j v_j\|_{\alpha,\Gamma} = \int_{\omega_j} R\varphi_j v_j\,\mathrm{d}s. \tag{6.20}$$

A different scaling guarantees $\|\varphi_j v_j\|_{\alpha,\Gamma} = \eta_{B,j}$ (the case when this is zero is excluded at the moment).

Set $I_\ell := \{\ell + 3k : k = 1,2,3,\ldots \text{ with } \ell + 3k \le N\}$ for $\ell = 1,2,3$. Since $(\omega_j : j \in I_\ell)$ are pairwise disjoint with a positive distance from each other,

$$\sum_{j\in I_\ell} \varphi_j v_j = \psi_\ell \in H^\alpha(\Gamma) \qquad \text{and } \varphi_j v_j = \psi_\ell \text{ on } \omega_j. \tag{6.21}$$

Combining $\|\varphi_j v_j\|_{\alpha,\Gamma} = \eta_{B,j}$ with Eqs. (6.20) and (6.21) we infer that

$$\sum_{j\in I_\ell} \eta_{B,j}^2 = \sum_{j\in I_\ell} \int_{\omega_j} R\varphi_j v_j\,\mathrm{d}s = \int_\Gamma R\psi_\ell\,\mathrm{d}s \tag{6.22}$$

With the α-depending operator norm c_{17} of $W : H^{1-\alpha}(\Gamma) \to H^{-\alpha}(\Gamma)$ we deduce

$$\sum_{j\in I_\ell} \eta_{B,j}^2 = \int_\Gamma W(u-u_h)\psi_\ell\,\mathrm{d}s \le c_{17}\|u-u_h\|_{1-\alpha,\Gamma}\|\psi_\ell\|_{\alpha,\Gamma}. \tag{6.23}$$

Lemma 6.4 and Eq. (6.21) show that

$$\|\psi_\ell\|_{\alpha,\Gamma}^2 \le 5/2 \sum_{j\in I_\ell} \|\varphi_j v_j\|_{\alpha,\Gamma}^2 = 5/2 \sum_{j\in I_\ell} \eta_{B,j}^2. \tag{6.24}$$

Combining Eqs. (6.23) and (6.24) and summing for $\ell = 1,2,3$ we conclude the assertion. □

Proof of Theorem 6.1. Combine Lemmas 6.3 and 6.5. □

7. Weighted-residual error estimators

This section is devoted to the analysis of the weighted-residual error estimators η_R. The first estimate concerns Example 1.2 and provides an upper estimate of the Babuška–Rheinboldt-type error estimates together with Theorem 6.1, the following estimate shows the reliability of η_R.

Theorem 7.1. *For some (N,u,f,u_h)-independent positive constant c_{18} and $j = 1,\ldots,N$ we have*

$$\eta_{B,j} \le c_{18}|\omega_j|^\alpha \|R\|_{L^2(\omega_j)}. \tag{7.1}$$

The constant c_{18} depends on $0 < \alpha < 1$ and Γ, but not on $\mathscr{T}$ or ω_j.

In Example 1.1, we have $R \in H^1(\Gamma)$ and obtain the subsequent result which, together with Theorem 5.1, shows the reliability of η_R.

Theorem 7.2. *For some (N,u,f,u_h)-independent positive constant c_{19} and $j = 1,\ldots,N$ we have*

$$\eta_{F,j} \le c_{19}|\omega_j|^{1-s}\|\partial R/\partial s\|_{L^2(\omega_j)}. \tag{7.2}$$

The constant c_{19} depends on $0 < s < 1$ and Γ, but not on $\mathscr{T}$ or ω_j.

Remarks 7.1.

(i) The weighted-residual error estimators were established by using the interpolation spaces [3] and employed an interpolation estimate in Refs. [5,7–12]. The new proofs given below are based on η_F and η_B.
(ii) The efficiency of the weighted-residual error estimators is mainly an open and important question. Its only answer is positive for quasiuniform meshes on closed boundaries [6].
(iii) Practical experience underlines that the weighted-residual error estimators, the simplest estimators known, perform very efficiently in adaptive algorithms and a posteriori error control.

The proof of Theorem 7.1 requires the following Friedrichs-type inequality.

Lemma 7.3. *For $v \in H^s(\Gamma)$ which vanishes outside a subarc ω of length $h \le |\Gamma|/3$ of Γ we have*

$$\|v\|_{L^2(\omega)} \le c_{18} h^s |v|_{s,\Gamma} \tag{7.3}$$

with a (v, ω)-independent positive constant c_{18} which depends on $0 < s < 1$ and Γ.

Proof. We prove the lemma for $\Gamma \subset \mathbb{R}$ of length L. Because of the side condition on the length of ω there is either space h to the left or to the right of $\omega \subset \Gamma$. Without loss of generality we suppose that $\omega = (-h, 0)$ and $(-h, h) \subset \Gamma$. Given v with support in ω, we define $w(x) := -v(-x)$ for $-h < x < h$ and extend w by zero outside. Then, v and w are just reflections of each other and consequently, their $L^2(\Gamma)$-norms coincide while their $H^s(\Gamma)$-seminorms might differ because the contributions from the integration on $\Gamma \backslash (0, h)$ and $\Gamma \backslash (-h, 0)$ might differ. However, their $H^s(-h, h)$-seminorms coincide and we know $w \in H^s(-h, h)$.

On the other hand, the integral mean of $v + w$ vanishes by design and so Lemma 4.3 (on $(-h, h)$) yields

$$2\|v\|^2_{L^2(\Gamma)} = \|v + w\|^2_{L^2(-h,h)} \le 1/2(2h)^{2s}|v + w|^2_{s,(-h,h)}. \tag{7.4}$$

This and a triangle inequality show that

$$\begin{aligned} 2\|v\|_{L^2(\Gamma)} &\le (2h)^s(|w|_{s,(-h,h)} + |v|_{s,(-h,h)}) = 2(2h)^s|v|_{s,(-h,h)} \\ &\le 2(2h)^s|v|_{s,\Gamma}. \end{aligned} \tag{7.5}$$

□

Proof of Theorem 7.1. For any $v \in H^\alpha(\Gamma)$ and $j = 1, \dots, N$, a Cauchy inequality and Lemma 7.3 lead to

$$\int_{\omega_j} R\varphi_j v \, ds \le \|R\|_{L^2(\omega_j)} \|\varphi_j v\|_{L^2(\omega_j)} \le c_{18}|\omega_j|^\alpha \|R\|_{L^2(\omega_j)} |\varphi_j v|_{\alpha,\Gamma}. \tag{7.6}$$

The theorem then follows by the definition of $\eta_{B,j}$. □

The proof of Theorem 7.2 follows from the subsequent imbedding estimate (where we omit the case of a curved boundary).

Lemma 7.4. *For $v \in H^1(0, h)$ we have*

$$|v|_{s,(0,h)} \le c_{19} h^{1-s} |\partial v/\partial s|_{L^2(0,h)} \tag{7.7}$$

with a (v, h)-independent positive constant c_{19} which depends on $0 < s < 1$.

Proof. It suffices to prove Eq. (7.7) for $h = 1$ and employ a scaling argument afterward to deduce the right power of h. We will suppose in the sequel that $1/2 < s < 1$ as this is the hardest case. There are merely few modifications necessary for $0 < s < 1/2$ while the integration below results in logarithmic terms which are similarly adapted. We omit the calculations for the other cases (cf. Ref. [17] for them) and focus on $1/2 < s < 1$.

The main theorem on calculus and a Cauchy inequality show, for $g(z) := (v'(z))^2$ and $x < y$,

$$|v(y) - v(x)|^2 = \left(\int_x^y v'(z)\, dz\right)^2 \le (y - x)\int_x^y g(z)\, dz. \tag{7.8}$$

The symmetry in x and y in Eq. (3.3) and the estimate (7.8) yield

$$1/2|v|^2_{s,(0,1)} \le \int_0^1 \int_0^y \int_x^y g(z)\, dz (y - x)^{-2s}\, dx\, dy. \tag{7.9}$$

Note that $(2s - 1)(y - x)^{-2s} = \partial(y - x)^{1-2s}/\partial x$ and integrate by parts in Eq. (7.9) with respect to the variable x. The boundary term involves $\int_x^y g(z)\, dz (y - x)^{1-2s}$ for $x = y$ and $x = 0$. Since the integral mean $\int_x^y g(z)\, dz/(y - x)$ converges for almost all y for $x \to y$ toward $g(y)$, we infer that the limit of $\int_x^y g(z)\, dz (y - x)^{1-2s}$ vanishes for $x \to y$ since $s < 1$.

Therefore, we obtain

$$\begin{aligned} (s - 1/2)|v|^2_{s,(0,1)} \le &- \int_0^1 \int_0^y g(z)\, dz y^{1-2s}\, dy \\ &+ \int_0^1 \int_0^y g(x)(y - x)^{1-2s}\, dx\, dy. \end{aligned} \tag{7.10}$$

The first term on the right-hand side of Eq. (7.10) is non-positive and the second is integrated with respect to the variable y after an exchange of the differentiation order to verify

$$(s-1/2)|v|^2_{s,(0,1)} \le \int_0^1 g(x)\int_x^1 (y-x)^{1-2s}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 g(x)(1-x)^{2-2s}\,\mathrm{d}x/(2-2s).$$

Since $\max_{0\le x\le 1}(1-x)^{2-2s}=1$, the last estimate implies Eq. (7.7) with $c_{19}=2^{-1}(1-s)^{-1}(s-1/2)^{-1}$. □

8. Multilevel error estimators

Given Symm's integral equation (1.1) with $A=V$ and a mesh $\mathscr{T}_h := \mathscr{T}$ we compute a (coarse grid) Galerkin solution $u_h \in \mathscr{H}_h := \mathscr{L}^0(\mathscr{T})$ and a residual $R\in H^1(\Gamma)$. A finer mesh $\mathscr{T}_{h/2}$ is defined by halving (some or) all elements. The additional fine-grid ansatz and test functions are $\chi_j = \partial\psi_j/\partial s$ with support Γ_j defined as the derivatives of the hat-functions ψ_j with respect to the arc-lengths; the fine hat-functions $\psi_j(x)=\psi_j(\gamma(s))$ are defined as affine functions in s on $(\xi_{j-1},\xi_{j-1/2})$ and on $(\xi_{j-1/2},\xi_j)$ with values 0, 1, and 0 at ξ_{j-1}, $\xi_{j-1/2}$, and ξ_j, respectively, where ξ_{j-1} and ξ_j mark the parameter interval of Γ_j and $\xi_{j-1/2} := (\xi_{j-1/2}+\xi_j)/2$ is their midpoint.

The multilevel error estimator (unaffected by a different scaling of χ_j) reads

$$\eta_{M,j} := \frac{\left|\int_{\Gamma_j} R\chi_j\,\mathrm{d}s\right|}{\left(\int_{\Gamma_j}(V\chi_j)\chi_j\,\mathrm{d}s\right)^{1/2}} \qquad \text{for } j=1,\dots,N. \tag{8.1}$$

It is stressed that this solution on a finer mesh is not needed in the calculation of the error estimator (8.1) which is efficient.

Theorem 8.1. *With some $(\mathscr{T},u,f,u_h)$-independent positive constant c_{20} (which merely depends on Γ) we have*

$$c_{20}\eta_{M,j} \le |R|_{1/2,\Gamma_j} \le \eta_{F,j}. \tag{8.2}$$

Proof. For simplicity, we focus on $\Gamma_j=[-h,h]$ such that $\chi_j(x)=-\mathrm{sign}(x)/h$ and so

$$h^2\left|\int_{\Gamma_j} R\chi_j\,\mathrm{d}s\right| = h\left|\int_{-h}^0 R(x)\,\mathrm{d}x - \int_0^h R(y)\,\mathrm{d}y\right|$$

$$= \left|\int_0^h\int_{-h}^0 (R(x)-R(y))\,\mathrm{d}x\,\mathrm{d}y\right|$$

$$\le \int_{\Gamma_j}\int_{\Gamma_j} |R(x)-R(y)|\,\mathrm{d}x\,\mathrm{d}y. \tag{8.3}$$

A Cauchy inequality on $\Gamma_j\times\Gamma_j$ with $|R(x)-R(y)|/|x-y|$ and $|x-y|$ shows

$$\int_{\Gamma_j}\int_{\Gamma_j}|R(x)-R(y)|\,\mathrm{d}x\,\mathrm{d}y$$

$$\le \left(\int_{\Gamma_j}\int_{\Gamma_j}\frac{|R(x)-R(y)|^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}$$

$$\times\left(\int_{\Gamma_j}\int_{\Gamma_j}|x-y|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} = \sqrt{8/3}h^2|R|_{1/2,\Gamma_j}. \tag{8.4}$$

A maple calculation shows $\int_{\Gamma_j}(V\chi_j)\chi_j\,\mathrm{d}s = 2/\pi\log(2)$ and so Eqs. (8.3) and (8.4) prove the first inequality of the theorem. The second follows from the definition of $\eta_{F,j}$ and $\Gamma_j\subset\omega_j$. □

Remarks 8.1.

(i) Theorem 8.1 is a modification of a result in Ref. [9] with an elementary proof.
(ii) The proof of reliability of η_M requires the strong saturation assumption that a fine-grid solution $u_{h/2}$ (with respect to the discrete space $\mathscr{H}_{h/2} := \mathscr{L}^0(\mathscr{T}_{h/2}) = \mathscr{H}_h \oplus \mathrm{span}\{\chi_1,\dots,\chi_N\}$) is a much better approximation to the (unknown) solution than the known discrete solution u_h :

$$\|u-u_{h/2}\|_V \le \beta\|u-u_h\|_V \tag{8.5}$$

for a constant $0<\beta<1$. On a quasiuniform mesh (the proof requires an inverse estimate) this guarantees [23,25]

$$\|u-u_h\|_V^2 \le c_{21}\sum_{j=1}^N \eta_{M,j}^2 \tag{8.6}$$

with a constant c_{21} which tends to infinity as β tends to 1.
(iii) Since the saturation condition could be crucial in particular for coarse meshes, we recommend handling this estimator with care in practice, as the reliability depends crucially on $\beta<1$. It is therefore recommended to employ $\eta_{M,j}$ as an error indicator in adaptive schemes but not necessarily as a reliable error estimate (especially not for coarse meshes).

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