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Averaging techniques yield reliable a posteriori finite element error control for obstacle problems

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Summary. The reliability of frequently applied averaging techniques for a posteriori error control has recently been established for a series of finite element methods in the context of second-order partial differential equations. This paper establishes related reliable and efficient a posteriori error estimates for the energy-norm error of an obstacle problem on unstructured grids as a model example for variational inequalities. The surprising main result asserts that the distance of the piecewise constant discrete gradient to *any* continuous piecewise affine approximation is a reliable upper error bound up to known higher order terms, consistency terms, and a multiplicative constant.

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1 Introduction

1.1 Averaging techniques for a posteriori error control

While a posteriori error control and adaptive mesh design is well established for (elliptic) partial differential equations [AO, BSt, EEHJ, V], their exploitation for variational inequalities started very recently [BSu, CN, LLT, V1, V2]. Amongst the a posteriori error estimation techniques are averaging schemes firstly justified by super-convergence properties on structured grids with symmetry properties. Their recent justification on unstructured grids in

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[BC,CA,CB,CF1,CF2,CF3] raises the question: How can averaging techniques be possibly reliable (i.e., be guaranteed upper bounds) for variational inequalities?

Our mathematical investigations recast this question into the design of a weak approximation operator that is compatible with the obstacle conditions and still enjoys local orthogonality properties to generate higher order terms. Utilising the operator J from [C1] and its dual J^* this paper provides an affirmative answer for a simple obstacle problem with affine obstacle and studies the nonconforming case.

1.2 Continuous and discrete obstacle problems

Given a bounded Lipschitz domain Ω in $\mathbb{R}^d$, $d = 2, 3$, $f \in H^1(\Omega)$, $g \in H^1(\Gamma_N)$, $u_D \in H^1(\Gamma_D)$, and $\chi \in H^1(\Omega)$ such that the closed and convex subset

$$K := \{v \in H^1(\Omega) : v = u_D \text{ on } \Gamma_D,\ \chi \le v \text{ almost everywhere in } \Omega\}$$

of $H^1(\Omega)$ is non-void, the obstacle problem under question reads: *Seek $u \in K$ such that*

$$(\nabla u; \nabla(u - v)) \le (f; u - v) + \int_{\Gamma_N} g(u - v)\, ds \quad \text{for all } v \in K. \tag{1.1}$$

Here, $(\cdot;\cdot)$ denotes the L^2-product and Γ_D is a closed subset of $\Gamma := \partial\Omega$ with positive surface measure; $\Gamma_N := \Gamma\backslash\Gamma_D$. It is known [R,GLT,K] that (1.1) has a unique solution. The finite element approximation employs a (closed and convex) discrete set K_h (i.e., a subset of a finite-dimensional subspace of $H^1(\Omega)$) and reads: *Seek $u_h \in K_h$ such that*

$$(\nabla u_h; \nabla(u_h - v_h)) \le (f; u_h - v_h) + \int_{\Gamma_N} g(u_h - v_h)\, ds \quad \text{for all } v_h \in K_h. \tag{1.2}$$

There exists a unique discrete solution u_h whose error $e := u - u_h$ is in some sense quasi-optimally small; we refer to [F,N] for a priori error estimates and focus on a posteriori estimates in this paper. The choice

$$K_h := \{v_h \in \mathcal{S}^1(\mathcal{T}) : v_h = u_{D,h} \text{ on } \Gamma_D,\ \chi_h \le v_h \text{ almost everywhere in } \Omega\} \tag{1.3}$$

can model a conforming (i.e., $K_h \subseteq K$) or non-conforming (i.e., $K_h \nsubseteq K$) discretisation. Here, $\mathcal{S}^1(\mathcal{T})$ is the P_1-finite element space defined through a regular triangulation $\mathcal{T}$ of Ω into triangles and tetrahedra if $d = 2$ and $d = 3$, respectively, [BSc,Ci]; $\chi_h \in \mathcal{S}^1(\mathcal{T})$ is an approximation to χ, $u_{D,h} \in \mathcal{S}^1(\mathcal{T})|_{\Gamma_D}$ is an approximation to u_D and we assume $K_h \neq \emptyset$.

1.3 Overview of new results

Our first result (Theorem 2) employs [BC, CB, C1, CV] and standard estimates for the proof of

$$\begin{aligned} \|\nabla(u-u_h)\| &\lesssim \eta_M + (\varrho_h;\, \chi - u_h - w) + \|\nabla w\| \\ &\quad + \|h_{\mathcal{T}}^2 \nabla f\| + \|h_{\mathcal{E}}^{3/2} \partial_{\mathcal{E}} g/\partial s\|_{L^2(\Gamma_N)}. \end{aligned} \tag{1.4}$$

Here, $\|\cdot\|$ denotes the $L^2(\Omega)$-norm and "$\lesssim$" substitutes "$\leq$ up to a multiplicative mesh-size-independent constant"; throughout this paper, this constant exclusively depends on the shape of the elements through the interior angles in the triangulation but not on the elements' lengths. Moreover, w is arbitrary in $H^1(\Omega)$ with $u_h + w \in K$, $w|_{\Gamma_D} = u_D - u_{D,h}$ vanishes at nodes on Γ_D, ϱ_h is a known discrete residual, $h_{\mathcal{T}}$ and $h_{\mathcal{E}}$ are local mesh sizes, and

$$\eta_M := \min\{\|p_h - \nabla u_h\| : p_h \in \mathcal{S}^1(\mathcal{T})^d,\ p_h \cdot n = g \text{ on } \mathcal{N} \cap \overline{\Gamma}_N\}; \tag{1.5}$$

where n denotes the outer unit normal on Γ_N and $\mathcal{N}$ is the set of nodes in $\mathcal{T}$ ($p_h \cdot n$ interpolates g at all nodes on $\overline{\Gamma}_N$). Consistency is included in the arbitrary choice of w to assess the error in $K_h \neq K$ and $u_{D,h} \neq u_D$; in the absence of contact near the boundary, with $(\cdot)_+ := \max\{\cdot, 0\}$ and $(\cdot)_- := \min\{\cdot, 0\}$,

$$\|\nabla w\| \lesssim \|h_{\mathcal{E}}^{3/2} \partial_{\mathcal{E}}^2 u_D/\partial s^2\|_{L^2(\Gamma_D)} + \|\nabla(\chi - u_h)_+\|. \tag{1.6}$$

The estimate (1.4) can be recommended for practical error control since $(\varrho_h;\, \chi - u_h - w)$ can be evaluated. Closer investigations reveal that this term can indeed be replaced by consistency, averaging, and higher order terms. Our main result (Theorem 3) implies

$$\begin{aligned} \|\nabla(u-u_h)\| &\lesssim \eta_M + \min_{q_h \in \mathcal{S}^1(\mathcal{T})^d} \|q_h - \nabla(\chi_h - u_h)\| + \|\nabla w\| \\ &\quad + \|h_{\mathcal{T}}^2 \nabla f\| + \|h_{\mathcal{T}} f\|_{L^2(\cup \mathcal{T}_D)} + \|h_{\mathcal{E}}^{3/2} \partial_{\mathcal{E}} g/\partial s\|_{L^2(\Gamma_N)} \\ &\quad + \|h_{\mathcal{T}}^{-1}(\chi - \chi_h - w)_-\| + (\|f\|\,\|(\chi - \chi_h - w)_-\|)^{1/2}. \end{aligned} \tag{1.7}$$

The term $\|h_{\mathcal{T}} f\|_{L^2(\cup \mathcal{T}_D)}$ is the L^2-norm over the shrinking domain $\cup \mathcal{T}_D$, a union of a few layers of elements near Γ_D; e.g., if $f \in L^\infty(\Omega)$ we have

$$\|h_{\mathcal{T}} f\|_{L^2(\cup \mathcal{T}_D)} \lesssim \|f\|_{L^\infty(\Omega)} \|h_{\mathcal{E}}^2\|_{L^2(\Gamma_D)}$$

and see that this term is of higher order (h.o.t.). In case $\chi = \chi_h$ and no contact near the boundary, the estimate reduces to

$$\|\nabla(u-u_h)\| \lesssim \eta_M + \min_{q_h \in \mathcal{S}^1(\mathcal{T})^d} \|q_h - \nabla(\chi_h - u_h)\| + \text{h.o.t.} \tag{1.8}$$

The finer estimate of Theorem 3 refines (1.7)-(1.8) in the substitution of $\|q_h - \nabla(\chi_h - u_h)\|_{L^2(\Omega)}$ by the refined norm $\|q_h - \nabla(\chi_h - u_h)\|_{L^2(\Omega_s)}$ on a smaller computable region Ω_s around the free boundary of the contact zone. Numerical examples convinced us that this refinement is necessary for efficient approximation and error control.

1.4 Affine obstacles

If the obstacle $\chi \equiv \chi_h$ is globally affine, then $\nabla\chi_h = A$ is constant and $q_h = p_h + A$ in (1.8) provides

$$\|\nabla(u - u_h)\| \lesssim \eta_M + \text{h.o.t.} \tag{1.9}$$

Hence, the averaging estimator η_M (from the variational equality) is indeed reliable for the obstacle problem up to a multiplicative constant and up to known higher order terms. It is stressed that the averaging estimator η_M is efficient; the proof is provided by a triangle inequality

$$\begin{aligned} \eta_M \le \|\nabla(u_h - u)\| &+ \min \\ &\times\{\|\nabla u - p_h\| : p_h \in \mathcal{S}^1(\mathcal{T})^d,\ p_h \cdot n = g \text{ on } \mathcal{N} \cap \overline{\Gamma}_N\}; \end{aligned} \tag{1.10}$$

in case u is sufficiently smooth (e.g. $u \in H^{2+\varepsilon}(\Omega)$), the minimum in the right-hand side is of higher order. In the practical examples of this paper, the estimator $\eta_A \ge \eta_M$ is employed with a local averaging. Owing to [C2], $\eta_A \lesssim \eta_M$ and hence η_A is reliable and efficient.

It appears to us that the reliability of averaging techniques is always related to smooth data (u_D, g, and f) and hence rough obstacles might be excluded from the assumptions; this is seen in our analysis by consistency terms which are not always of higher order and may dominate the error estimate. Consequently, this paper does not focus on coarse approximation of rough data.

1.5 Plan of the paper

The rest of this paper is organised as follows. Preliminaries and notation is introduced in Section 2 where we recall a few results and state some basic estimates. Section 3 is devoted to the a posteriori error estimates and their proofs. Section 4 outlines the numerical realisation of an adaptive mesh refinement strategy based on our a posteriori error estimates. Section 5 reports on three examples where the estimate of the error in the energy norm is extremely accurate.

2 Preliminaries

This section firstly recalls notation on the triangulation and recalls the approximation operator J and some of its established and adapted properties in Lemma 1–4. The section closes with the concept of the continuous and discrete residuals and nodewise Kuhn-Tucker conditions in Lemma 6.

Throughout this paper, $u \in K$ solves (1.1) and $u_h \in K_h$ solves (1.2). The aim is to prove reliability of the aforementioned estimators. We let $H^1_D(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ and define $\mathcal{S}^1_D(\mathcal{T}) := \mathcal{S}^1(\mathcal{T}) \cap H^1_D(\Omega)$.

Let $(\varphi_z : z \in \mathcal{N})$ be the nodal basis of $\mathcal{S}^1(\mathcal{T})$. Note that $(\varphi_z : z \in \mathcal{N})$ is a partition of unity and the open patches

$$\omega_z := \{x \in \Omega : 0 < \varphi_z(x)\} \tag{2.1}$$

form an open cover $(\omega_z : z \in \mathcal{N})$ of Ω with finite overlap.

Let $\mathcal{K} := \mathcal{N} \setminus \Gamma_D$ denote the set of free nodes and let $\mathcal{E}$ denote the set of all edges ($d = 2$) or faces ($d = 3$) appearing for some T in $\mathcal{T}$. In order to define a weak interpolation operator $J : H^1_D(\Omega) \to \mathcal{S}^1_D(\mathcal{T})$ we modify $(\varphi_z : z \in \mathcal{K})$ to a partition of unity $(\psi_z : z \in \mathcal{K})$. For each fixed node $z \in \mathcal{N} \setminus \mathcal{K}$, we choose a neighboring node $\zeta(z) \in \mathcal{K}$ and let $\zeta(z) := z$ if $z \in \mathcal{K}$. In this way, we define a partition of $\mathcal{N}$ into card$(\mathcal{K})$ classes $I(z) := \{\tilde{z} \in \mathcal{N} : \zeta(\tilde{z}) = z\}$, $z \in \mathcal{K}$. For each $z \in \mathcal{K}$ set

$$\psi_z := \sum_{\zeta \in I(z)} \varphi_\zeta \quad \text{and} \quad \Omega_z := \{x \in \Omega : 0 < \psi_z(x)\} \tag{2.2}$$

and notice that $(\psi_z : z \in \mathcal{K})$ is a partition of unity. It is required that Ω_z is connected and that $\psi_z \neq \varphi_z$ implies that $\Gamma_D \cap \partial\Omega_z$ has a positive surface measure.

For $g \in L^1(\Omega)$ define

$$Jg := \sum_{z \in \mathcal{K}} g_z \varphi_z \in \mathcal{S}^1_D(\mathcal{T}) \quad \text{where} \quad g_z := (g; \psi_z)/(1; \varphi_z) \in \mathbb{R}. \tag{2.3}$$

The local mesh-sizes are denoted by $h_\mathcal{T}$ and $h_\mathcal{E}$ where $h_\mathcal{T} \in \mathcal{L}^0(\mathcal{T})$ denotes the element size, $h_\mathcal{T}|_T := h_T := \operatorname{diam}(T)$ for $T \in \mathcal{T}$, and the edge size $h_\mathcal{E} \in L^\infty(\cup\mathcal{E})$ is defined on the union or skeleton $\cup\mathcal{E}$ of all edges E in $\mathcal{E}$ by $h_\mathcal{E}|_E := h_E := \operatorname{diam}(E)$. The patch size $h_z := \operatorname{diam}(\Omega_z)$ is defined for each node $z \in \mathcal{K}$ separately. For $z \in \mathcal{N} \setminus \mathcal{K}$ set $h_z := \operatorname{diam}(\omega_z)$ and for $T \in \mathcal{T}$ let $\omega_T := \cup_{z \in T \cap \mathcal{N}} \Omega_{\zeta(z)}$. Note that the sets of patches $(\omega_T : T \in \mathcal{T})$ and $(\Omega_z : z \in \mathcal{K})$ have a finite overlap.

In the following we write $\|\cdot\|_{p,A}$ instead of $\|\cdot\|_{L^p(A)}$ and $\|\cdot\|$ abbreviates $\|\cdot\|_{2,\Omega}$. Similarly, we denote by $|\cdot|_{1,2,A} := \|\nabla \cdot\|_{2,A}$ the semi-norm in $H^1(A)$ and $|\cdot|_{1,2}$ abbreviates $|\cdot|_{1,2,\Omega}$.

Theorem 1 ([C1,Cl,CV,CB]). *The operator J is H^1-stable and first-order convergent, i.e.,*

$$\|h_\mathcal{T}^{-1}(g - Jg)\| + \|h_\mathcal{E}^{-1/2}(g - Jg)\|_{2,\Gamma_N} + |g - Jg|_{1,2} \lesssim |g|_{1,2} \tag{2.4}$$

for $g \in H^1_D(\Omega)$. Moreover, for $f \in L^2(\Omega)$, there holds

$$(f; g - Jg) \lesssim |g|_{1,2}\Big(\sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathbb{R}} \|f - f_z\|^2_{2,\Omega_z}\Big)^{1/2}. \quad \square \tag{2.5}$$

Lemma 1. *We have, for all $v \in H^1_D(\Omega)$,*

$$(f; v - Jv) + \int_{\Gamma_N} g(v - Jv)\,dx - (\nabla u_h; \nabla(v - Jv)) \\ \lesssim |v|_{1,2} \min_{p_h \in \mathcal{S}^1(\mathcal{T})^d} (\|\nabla u_h - p_h\| + \|h_{\mathcal{E}}^{1/2}(g - p_h \cdot n)\|_{2,\Gamma_N} + \|h_{\mathcal{T}}^2 \nabla f\|).$$

Proof. The lemma is, at least implicitly, included in [CB] (and also in [BC, CF1, CF2]) and so we merely sketch its proof. From (2.5) we have by Poincaré's inequality

$$(f; v - Jv) \lesssim |v|_{1,2} \|h_{\mathcal{T}}^2 \nabla f\|.$$

An integration by parts of $-(p_h; \nabla(v - Jv))$ and $\mathrm{div}_{\mathcal{T}} \nabla u_h = 0$ reveal that the last two terms in the left-hand side of the asserted inequality equal

$$\begin{aligned} &\int_{\Gamma_N} (g - p_h \cdot n)(v - Jv)\,ds + (p_h - \nabla u_h; \nabla(v - Jv)) \\ &\quad + (\mathrm{div}_{\mathcal{T}}(p_h - \nabla u_h); v - Jv) \\ &\lesssim |v|_{1,2} \Big(\|h_{\mathcal{E}}^{1/2}(g - p_h \cdot n)\|_{2,\Gamma_N} + \|p_h - \nabla u_h\| \\ &\quad + \|h_{\mathcal{T}} \,\mathrm{div}_{\mathcal{T}}(p_h - \nabla u_h)\|\Big) \end{aligned} \tag{2.6}$$

by (2.4) and Cauchy inequalities. This and a $\mathcal{T}$-elementwise inverse estimate of the form $h_T \|\,\mathrm{div}_{\mathcal{T}}(p_h - \nabla u_h)\|_{2,T} \lesssim \|p_h - \nabla u_h\|_{2,T}$ conclude the proof. □

Lemma 2 ([BCD]). *Assume $u_D \in H^1(\Gamma_D) \cap C(\Gamma_D)$, $u_D|_E \in H^2(E)$ for all $E \in \mathcal{E}$ such that $E \subseteq \Gamma_D$, and let $\partial_{\mathcal{E}}^2 u_D/\partial s^2$ denote the edgewise second derivative of u_D along Γ_D. Suppose $u_{D,h}$ is the nodal interpolant of u_D, i.e., $u_{D,h}(z) = u_D(z)$ for all $z \in \mathcal{N} \cap \Gamma_D$. Then there exists $w_D \in H^1(\Omega)$ such that $w_D|_{\Gamma_D} = u_D - u_{D,h}$,* $\mathrm{supp}\, w_D \subseteq \bigcup_{T \in \mathcal{T}, T \cap \Gamma_D \neq \emptyset} T$,

$$\|w_D\|_\infty = \|u_D - u_{D,h}\|_{\infty,\Gamma_D}, \quad |w_D|_{1,2} \lesssim \|h_{\mathcal{E}}^{3/2} \partial_{\mathcal{E}}^2 u_D/\partial s^2\|_{2,\Gamma_D}. \quad \square$$

Definition 1. *Define $\mathcal{T}_D := \{T \in \mathcal{T} : T \cap \Gamma_D \neq \emptyset\}$ and*

$$\mathcal{T}_c := \{T \in \mathcal{T} \setminus \mathcal{T}_D : (\chi_h - u_h)|_T = 0\}.$$

The following lemma shows (1.6) and estimates the terms which include w in (1.7).

Lemma 3. *Suppose that u_D satisfies the conditions of Lemma 2, that $\chi|_{\Gamma_D} \leq u_{D,h}$, and that $(\chi - u_h)_- \leq w_D$ in $\bigcup \mathcal{T}_D$ with w_D from Lemma 2. Then we* *have*

$$\min_{\substack{w\in H^1(\Omega)\\ u_h+w\in K}} |w|_{1,2} \lesssim \|h_{\mathcal{E}}^{3/2}\partial_{\mathcal{E}}^2 u_D/\partial s^2\|_{2,\Gamma_D} + |(\chi-u_h)_+|_{1,2} \qquad \textit{and}$$

$$\begin{aligned}\min_{\substack{w\in H^1(\Omega)\\ u_h+w\in K}} \Big(|w|_{1,2}^2 &+ \sum_{T\in\mathcal{T}\setminus\mathcal{T}_D} \|h_T^{-1}(\chi-\chi_h-w)_-\|_{2,T}^2 \\ &+ \sum_{T\in\mathcal{T}_c}\|f\|_{2,\omega_T}\|(\chi-\chi_h-w)_-\|_{2,T}\Big)\\ \lesssim \|h_{\mathcal{E}}^{3/2}&\partial_{\mathcal{E}}^2 u_D/\partial s^2\|_{2,\Gamma_D}^2 + |(\chi-u_h)_+|_{1,2}^2\\ &+\sum_{T\in\mathcal{T}\setminus\mathcal{T}_D}\|h_{\mathcal{T}}^{-1}(\chi-\chi_h)_-\|_{2,T}^2 + \sum_{T\in\mathcal{T}_c}\|f\|_{2,\omega_T}\|(\chi-\chi_h)_-\|_{2,T}.\end{aligned}$$

Proof. Set $w := (\chi-u_h)_+ + w_D$ and notice $u_h + w \in K$. Then $|w|_{1,2} \le |(\chi-u_h)_+|_{1,2} + |w_D|_{1,2}$. Utilising $w_D|_T = 0$ and $\chi_h \le u_h$ on each $T\in\mathcal{T}\setminus\mathcal{T}_D$ we have on each $T\in\mathcal{T}\setminus\mathcal{T}_D$

$$\|(\chi-\chi_h-w)_-\|_{2,T} = \|(\chi-\chi_h-(\chi-u_h)_+)_-\|_{2,T} \le \|(\chi-\chi_h)_-\|_{2,T}.$$

Then, Lemma 2 proves the assertions. □

Remark 1. Since $\|w_D\|_\infty = \|u_D-u_{D,h}\|_{\infty,\Gamma_D}$ by Lemma 2, the assumption $(\chi-u_h)_- \le -\|u_D-u_{D,h}\|_{\infty,\Gamma_D}$ in $\bigcup\mathcal{T}_D$ implies $(\chi-u_h)_- \le w_D$ in $\bigcup\mathcal{T}_D$.

Lemma 4 ([BC,CB]). *Let $g|_E \in H^1(E)\cap C(E)$ for all $E\in\mathcal{E}$ such that $E\subseteq\overline{\Gamma}_N$ and, for each node $z\in\mathcal{N}\cap\overline{\Gamma}_N$ where the outer unit normal n on Γ_N is continuous (hence constant in a neighbourhood of z as Γ_N is a polygon), let g be continuous. Assume that the set*

$$\mathcal{S}_N^1(\mathcal{T},g) := \{p_h\in\mathcal{S}^1(\mathcal{T})^d : \forall E\in\mathcal{E}, z\in E\subseteq\overline{\Gamma}_N,\ p_h(z)\cdot n_E = g(z)\}$$

is non-void. Then ($\partial_{\mathcal{E}} g/\partial s$ denotes the edgewise surface gradient of g on Γ_N)

$$\begin{aligned}\min_{p_h\in\mathcal{S}^1(\mathcal{T})^d} &\Big(\|\nabla u_h - p_h\| + \|h_{\mathcal{E}}^{1/2}(g-p_h\cdot n)\|_{2,\Gamma_N}\Big)\\ &\lesssim \min_{q_h\in\mathcal{S}_N^1(\mathcal{T},g)}\|\nabla u_h - q_h\| + \|h_{\mathcal{E}}^{3/2}\partial_{\mathcal{E}} g/\partial s\|_{2,\Gamma_N}. \quad \square\end{aligned}$$

Remark 2. For $d=2$ the conditions on g in Lemma 4 suffice for $\mathcal{S}_N^1(\mathcal{T},g)\neq\emptyset$ [CB].

Definition 2. *Define $\varrho\in(H_D^1(\Omega))^*$ and $\varrho_h\in\mathcal{S}^1(\mathcal{T})$, for $v\in H_D^1(\Omega)$, by*

(2.7) $$\varrho(v) := (f;v) + \int_{\Gamma_N} gv\,ds - (\nabla u;\nabla v),$$

(2.8) $$\varrho_h := \sum_{z\in\mathcal{K}}\Big((f;\varphi_z) + \int_{\Gamma_N} g\varphi_z\,ds - (\nabla u_h;\nabla\varphi_z)\Big)\psi_z/(1;\varphi_z).$$

Remark 3. Note that $0 \leq \varrho(e-w)$ for $w \in H^1(\Omega)$ satisfying $w|_{\Gamma_D} = u_D - u_{D,h}$ and $\chi - u_h \leq w$ (since $u_h + w \in K$). If $u_h \in K$ we may choose $w = 0$. If not, let, e.g., $P_K u_h$ be the projection of u_h onto K with respect to $|\cdot|_{1,2}$ and $w := P_K u_h - u_h$ minimises $|w|_{1,2}$ among all w with $u_h + w \in K$.

Lemma 5. *We have, for all $w \in H^1(\Omega)$ satisfying $w|_{\Gamma_D} = u_D - u_{D,h}$,*

$$\begin{aligned}\frac{1}{2}|e-w|_{1,2}^2 + \frac{1}{2}|e|_{1,2}^2 &= (f; e-w-J(e-w)) \\ &\quad -(\nabla u_h; \nabla(e-w-J(e-w))) + \int_{\Gamma_N} g(e-w-J(e-w))\,ds \\ &\quad +\frac{1}{2}|w|_{1,2}^2 + (\varrho_h; e-w) - \varrho(e-w).\end{aligned} \tag{2.9}$$

Proof. Note that $e - w \in H_D^1(\Omega)$. The definition of $J(e-w)$ yields, e.g.,

$$\sum_{z\in\mathcal{K}} (\nabla u_h; \nabla\varphi_z)(\psi_z; e-w)/(1;\varphi_z) = (\nabla u_h; \nabla J(e-w))$$

and eventually leads to

$$(\varrho_h; e-w) = (f; J(e-w)) - (\nabla u_h; \nabla J(e-w)) + \int_{\Gamma_N} gJ(e-w)\,ds.$$

This and some elementary calculations show

$$\begin{aligned}&\varrho(e-w) - (\varrho_h; e-w) = (f; e-w-J(e-w)) \\ &\quad + \int_{\Gamma_N} g(e-w-J(e-w))\,ds - (\nabla u; \nabla(e-w)) \\ &\quad +(\nabla u_h; \nabla J(e-w)) = (f; e-w-J(e-w)) - (\nabla e; \nabla(e-w)) \\ &\quad + \int_{\Gamma_N} g(e-w-J(e-w))\,ds - (\nabla u_h; \nabla(e-w-J(e-w))).\end{aligned}$$

Since $2(\nabla e; \nabla(e-w)) = |e-w|_{1,2}^2 + |e|_{1,2}^2 - |w|_{1,2}^2$ we deduce (2.9). □

Our motivation for the definition of ϱ_h is that its nodal values reflect Kuhn-Tucker conditions.

Lemma 6. *We have $\varrho_h \leq 0 \leq u_h - \chi_h$ almost everywhere in Ω and, for $z \in \mathcal{K}$,*

$$\varrho_h(z)(\chi_h(z) - u_h(z)) = 0.$$

Proof. Given $z \in \mathcal{K}$ and a real number w define $v_h \in \mathcal{S}^1(\mathcal{T})$ by $v_h(z) := w$ and $v_h(\zeta) = u_h(\zeta)$ for $\zeta \in \mathcal{N}\backslash\{z\}$. If $\chi_h(z) \leq w$ we have $v_h \in K_h$ and calculate with (1.2)

$$\begin{aligned}(u_h(z) - w)(\nabla u_h; \nabla\varphi_z) &= (\nabla u_h; \nabla(u_h - v_h))\\ &\leq (f; u_h - v_h) + \int_{\Gamma_N} g(u_h - v_h)\, ds\\ &= (u_h(z) - w)\big((f; \varphi_z) + \int_{\Gamma_N} g\varphi_z\, ds\big).\end{aligned}$$

According to (2.8) this gives (after a division by $(1; \varphi_z) > 0$)

$$0 \leq (u_h(z) - w)\varrho_h(z).$$

A discussion of $w \in \mathbb{R}$ under the restriction $\chi_h(z) \leq w$ yields the assertions. □

3 A posteriori estimates

This section is devoted to the precise statement and the proof of (1.4) and (1.7)–(1.8) of the introduction.

The combination of the next result with Lemma 4 provides a proof of (1.4).

Theorem 2 (A posteriori estimate I). *If $w \in H^1(\Omega)$ is such that $u_h + w \in K$, i.e., $w|_{\Gamma_D} = u_D - u_{D,h}$ and $\chi - u_h \leq w$, then*

$$\begin{aligned}|e - w|_{1,2} + |e|_{1,2} \lesssim &\min_{p_h \in \mathcal{S}^1(\mathcal{T})^d} (\|\nabla u_h - p_h\| + \|h_{\mathcal{E}}^{1/2}(g - p_h \cdot n)\|_{2,\Gamma_N})\\ &+\|h_{\mathcal{T}}^2 \nabla f\| + |w|_{1,2} + (\varrho_h; \chi - u_h - w).\end{aligned}$$

Proof. Since $u_h + w \in K$ we have $\varrho(e - w) \geq 0$, cf. Remark 3. Moreover, $\varrho_h \leq 0 \leq u - \chi$ almost everywhere in Ω by Lemma 6 so that $(\varrho_h; u - \chi) \leq 0$ and hence

$$\begin{aligned}(\varrho_h; e - w) - \varrho(e - w) &\leq (\varrho_h; e - w)\\ &= (\varrho_h; u - \chi) + (\varrho_h; \chi - u_h - w) \leq (\varrho_h; \chi - u_h - w).\end{aligned}$$

Utilising this estimate and Lemma 1 in (2.9) we deduce the assertion. □

The following lemmas are needed to obtain other bounds for $(\varrho_h; e - w)$.

Lemma 7. *Let $z \in \mathcal{N}$ be either an interior point of Ω or suppose that each open half-space with boundary point z has a non-void intersection with Ω. Suppose $T \in \mathcal{T}$ is such that $z \in \overline{\omega}_T$ and set $\tilde{\Omega}_z := \Omega_z \cup \omega_T$. Let $w_h \in \mathcal{S}^1(\mathcal{T})$ satisfy $w_h(z) = 0$ and $0 \le w_h$ on $\tilde{\Omega}_z$. Then,*

$$\|w_h\|_{2,\tilde{\Omega}_z} \lesssim h_z \min_{q_z \in \mathcal{S}^1(\mathcal{T}|_{\tilde{\Omega}_z})^d} \|\nabla w_h - q_z\|_{2,\tilde{\Omega}_z}. \tag{3.1}$$

Proof. The left- and right-hand side of (3.1) define semi-norms $\|\cdot\|_l$ and $\|\cdot\|_r$, respectively, on $\mathcal{S}^1(\mathcal{T}|_{\tilde{\Omega}_z})$. We claim that $\|w_h\|_r = 0$ implies $\|w_h\|_l = 0$ for all $w_h \in \mathcal{S}^1(\mathcal{T})$ with $w_h(z) = 0 \le w_h|_{\tilde{\Omega}_z}$. Indeed, if ∇w_h equals some $q_z \in \mathcal{S}^1(\mathcal{T}|_{\tilde{\Omega}_z})^d$ it is $(\mathcal{T}|_{\tilde{\Omega}_z})$-piecewise constant and continuous, whence w_h is affine on $\tilde{\Omega}_z$. Since $w_h(z) = 0$ we obtain that $w_h(x)$ equals $\alpha\, n \cdot (x - z)$ for all $x \in \tilde{\Omega}_z$ and some $n \in \mathbb{R}^d$, $|n| = 1$, and some $\alpha \in \mathbb{R}$. Let $H := \{y \in \mathbb{R}^d : m \cdot (y - z) < 0\}$ intersect with $\tilde{\Omega}_z$ ($H \cap \tilde{\Omega}_z \neq \emptyset$ is obvious for $z \in \Omega$ and assumed for $z \in \mathcal{N}\backslash\Omega$). For $x \in H \cap \tilde{\Omega}_z$,

$$0 \le w_h(x) = \alpha\, n \cdot (x - z) \text{ and } m \cdot (x - z) < 0. \tag{3.2}$$

For $m = +n$, (3.2) implies $\alpha \le 0$ and for $m = -n$, (3.2) yields $0 \le \alpha$. Together, $\alpha = 0$, i.e., $w_h = 0$ and so $\|w_h\|_l = 0$. Since $\|\cdot\|_l$ and $\|\cdot\|_r$ are norms on the finite-dimensional affine space $\{w_h \in \mathcal{S}^1(\mathcal{T}|_{\tilde{\Omega}_z}) : w_h(z) = 0, 0 \le w_h|_{\tilde{\Omega}_z}\}$, they are equivalent. The constant $C > 0$ in $\|\cdot\|_l \le C\,\|\cdot\|_r$ depends on $\mathcal{T}|_{\tilde{\Omega}_z}$. A scaling argument concludes the proof. □

Remark 4. If $z \in \mathcal{N}$ is a boundary point of Ω and $\{x \in \Omega : |x - z| < \varepsilon\}$ is convex for some $\varepsilon > 0$ then z does not satisfy the condition of Lemma 7. Convex corners may yield difficulties for positive second order approximation operators [NW].

The next result shows that ϱ_h can be controlled by averaging terms.

Lemma 8. *We have, for $T \in \mathcal{T}$,*

$$\begin{aligned} h_T\|\varrho_h\|_{2,T} &\lesssim h_T\|f\|_{2,\omega_T} \\ &\quad + \min_{q_T \in \mathcal{S}^1(\mathcal{T}|_{\omega_T})^d} \big(\|\nabla u_h - q_T\|_{2,\omega_T} + h_T^{1/2}\|(g - q_T \cdot n)\|_{2,\Gamma_N \cap \partial\omega_T}\big), \\ h_T^2|\varrho_h|_{1,2,T} &\lesssim h_T^2|f|_{1,2,\omega_T} \\ &\quad + \min_{q_T \in \mathcal{S}^1(\mathcal{T}|_{\omega_T})^d} \big(\|\nabla u_h - q_T\|_{2,\omega_T} + h_T^{1/2}\|(g - q_T \cdot n)\|_{2,\Gamma_N \cap \partial\omega_T}\big). \end{aligned}$$

Proof. Set $J^*f := \sum_{z \in \mathcal{K}} (f;\varphi_z)/(1;\varphi_z)\ \psi_z$ and note that J^*f is the first summand in the definition of ϱ_h in (2.8). We have $\|J^*f\|_{2,T} \lesssim \|f\|_{2,\omega_T}$ and $|J^*f|_{1,2,T} \lesssim |f|_{1,2,\omega_T}$ for $T \in \mathcal{T}$, cf. [Ci,CV,CB]. Note also that $h_T^d \lesssim$

$(1, \varphi_z) \lesssim h_T^d$ and $|\psi_z|_{1,2,T} \lesssim h_T^{d/2-1}$, $\|\psi_z\|_{2,T} \lesssim h_T^{d/2}$ for all $T \in \mathcal{T}$ with $T \subseteq \overline{\Omega}_z$. Using this in (2.8) we deduce

$$\|\varrho_h\|_{2,T} \lesssim \|f\|_{2,\omega_T} + \sum_{z\in\mathcal{K}, T\subseteq\overline{\Omega}_z} h_T^{-d/2} |(\nabla u_h; \nabla\varphi_z) - \int_{\Gamma_N} g\varphi_z \, ds|,$$

$$|\varrho_h|_{1,2,T} \lesssim |f|_{1,2,\omega_T} + \sum_{z\in\mathcal{K}, T\subseteq\overline{\Omega}_z} h_T^{-d/2-1} |(\nabla u_h; \nabla\varphi_z) - \int_{\Gamma_N} g\varphi_z \, ds|. \tag{3.3}$$

Let q_T be an element of $\mathcal{S}^1(\mathcal{T}|_{\omega_T})^d$. An elementwise inverse estimate shows

$$h_z \| \operatorname{div}_{\mathcal{T}}(q_T - \nabla u_h)\|_{2,\omega_T} \lesssim \|\nabla u_h - q_T\|_{2,\omega_T}.$$

This, an integration by parts, $\operatorname{div}_{\mathcal{T}}\nabla u_h = 0$, $|\varphi_z|_{1,2} \lesssim h_z^{d/2-1}$, $\|\varphi_z\| \lesssim h_z^{d/2}$, and noting that for $z \in T \cap \mathcal{K}$ there holds $\omega_z \subseteq \omega_T$ lead to

$$\begin{aligned}(\nabla u_h; \nabla\varphi_z) - \int_{\Gamma_N} \varphi_z q_T \cdot n \, ds &= (\nabla u_h - q_T; \nabla\varphi_z) \\ -(\operatorname{div}_{\mathcal{T}} q_T; \varphi_z) &\lesssim h_T^{d/2-1} \|\nabla u_h - q_T\|_{2,\omega_T}.\end{aligned}$$

This, the fact that each element T belongs to a finite number of patches $\overline{\Omega}_z$ only, (3.3), and $\int_{\partial\omega_T\cap\Gamma_N} \varphi_z(g - q_T \cdot n) \, ds \lesssim h_z^{d/2-1} \|h_z^{1/2}(g - q_T \cdot n)\|_{2,\Gamma_N\cap\partial\omega_T}$ conclude the proof of the lemma. □

Definition 3. *With each $T \in \mathcal{T}_i$,*

$$\begin{aligned}\mathcal{T}_i := \{T \in \mathcal{T} : T \cap \Gamma_D = \emptyset, \exists x, y \in \mathcal{K} \cap \overline{\omega}_T,\\ \chi_h(x) = u_h(x), \chi_h(y) < u_h(y)\},\end{aligned}$$

we associate some $z_T \in \mathcal{K} \cap \overline{\omega}_T$ such that $\chi_h(z_T) = u_h(z_T)$ and set $\tilde{\Omega}_{z_T} := \Omega_{z_T} \cup \omega_T$,

$$\mathcal{T}_s := \{K \in \mathcal{T} : \exists T \in \mathcal{T}_i, K \subseteq \overline{\tilde{\Omega}}_{z_T}\}, \qquad \textit{and} \qquad \Omega_s := \bigcup_{T\in\mathcal{T}_s} T.$$

Remark 5. For each $T \in \mathcal{T}_i$ we preferably choose $z_T \in \Omega$ (i.e., $z_T \notin \partial\Omega$ if possible). This allows us to impose the condition of Lemma 7 in as few nodes as possible on the boundary.

Remark 6. The region Ω_s may be regarded as a layer between the discrete contact zone and the discrete non-contact zone.

Lemma 9. *Assume that for each $T \in \mathcal{T}_i$ for which $z_T \in \mathcal{K} \cap \Gamma_N$ the intersection of Ω with any open half-space with boundary point z_T is non-void. For all $w \in H^1(\Omega)$ satisfying $w|_{\Gamma_D} = u_D - u_{D,h}$ and $\chi - u_h \leq w$, we have*

$$\begin{aligned}
(\varrho_h; e - w) \lesssim & \sum_{T \in \mathcal{T} \setminus \mathcal{T}_D} \|h_T^{-1}(\chi - \chi_h - w)_-\|_{2,T}^2 \\
& + \sum_{T \in \mathcal{T}_i} \min_{q_z \in \mathcal{S}^1(\mathcal{T}|_{\tilde{\Omega}_{z_T}})^d} \|\nabla(\chi_h - u_h) - q_z\|_{2,\tilde{\Omega}_{z_T}}^2 + \|h_{\mathcal{T}}^2 \nabla \varrho_h\|^2 \\
& + \sum_{T \in \mathcal{T}_D} h_T \|\varrho_h\|_{2,T} |e - w|_{1,2,\omega_T} \\
& + \sum_{\mathcal{T} \in \mathcal{T}_c} h_T \|\varrho_h\|_{2,T} \|h_T^{-1}(\chi - \chi_h - w)_-\|_{2,T}.
\end{aligned}$$

Proof. Since $(\varrho_h; e - w) = \sum_{T \in \mathcal{T}} \int_T \varrho_h(e - w)\,dx$ we may estimate the contribution from each $T \in \mathcal{T}$ separately. In the first case we assume $T \in \mathcal{T}_D$. Since $e - w = 0$ on Γ_D, Friedrichs' inequality implies

$$\int_T \varrho_h(e - w)\,dx \lesssim h_T \|\varrho_h\|_{2,T} |e - w|_{1,2,\omega_T}. \tag{3.4}$$

In the second case we assume $T \in \mathcal{T}$, $T \cap \Gamma_D = \emptyset$, and $\chi_h|_{\omega_T} < u_h|_{\omega_T}$. Lemma 6 guarantees $\varrho_h|_T = 0$ and so

$$\int_T \varrho_h(e - w)\,dx = 0. \tag{3.5}$$

In the third case we assume $T \in \mathcal{T}_i$ and so there exist $y, z_T \in \mathcal{K} \cap \overline{\omega}_T$ such that $\chi_h(z_T) = u_h(z_T)$ and $\chi_h(y) < u_h(y)$. The conditions of Lemma 7 are satisfied for z_T by assumption. Since $\chi_h \leq u_h$, $\chi \leq u$,

$$(\chi - \chi_h - w)_- + \chi_h - u_h \leq (\chi - u_h - w)_- \leq e - w.$$

Because of $\varrho_h \leq 0$, this leads to

$$\int_T \varrho_h(e - w)\,dx \leq \int_T \varrho_h(\chi - \chi_h - w)_-\,dx + \int_T \varrho_h(\chi_h - u_h)\,dx. \tag{3.6}$$

Owing to Lemma 6 we have $\varrho_h(y) = 0$ and so $\|\varrho_h\|_{2,T} \lesssim h_T |\varrho_h|_{1,2,\omega_T}$ by a discrete Friedrichs' inequality. This, (3.6), and Lemma 7 show

$$\begin{aligned}
\int_T \varrho_h(e - w)\,dx \lesssim\ & h_T^2 |\varrho_h|_{1,2,\omega_T} \big(\|h_T^{-1}(\chi - \chi_h - w)_-\|_{2,T} \\
& + \min_{q_z \in \mathcal{S}^1(\mathcal{T}|_{\tilde{\Omega}_{z_T}})^d} \|\nabla(\chi_h - u_h) - q_z\|_{2,\tilde{\Omega}_{z_T}}\big).
\end{aligned} \tag{3.7}$$

In the remaining fourth case we assume $T \in \mathcal{T}_c$ and obtain with (3.6)

$$\int_T \varrho_h (e-w)\,dx \lesssim h_T \|\varrho_h\|_{2,T} \|h_T^{-1}(\chi - \chi_h - w)_-\|_{2,T}. \tag{3.8}$$

The summation of (3.4), (3.5), (3.7), and (3.8) verifies the assertion. □

The combination of the next result with Lemma 4 provides the proof of (1.7) and so leads to the reliability of all averaging techniques.

Theorem 3 (A posteriori estimate II). *Assume that for each $T \in \mathcal{T}_i$ such that $z_T \in \mathcal{K} \cap \Gamma_N$ the intersection of Ω with any open half-space with boundary point z_T is non-void. For all $w \in H^1(\Omega)$ satisfying $w|_{\Gamma_D} = u_D - u_{D,h}$ and $\chi - u_h \le w$, we have*

$$\begin{aligned}
|e-w|_{1,2} + |e|_{1,2} \lesssim{} & \min_{p_h \in \mathcal{S}^1(\mathcal{T})^d} (\|\nabla u_h - p_h\| + \|h_{\mathcal{E}}^{1/2}(g - p_h\cdot n)\|_{2,\Gamma_N}) \\
&+ \min_{q_h \in \mathcal{S}^1(\mathcal{T}_s)^d} \|\nabla(\chi_h - u_h) - q_h\|_{2,\Omega_s} \\
&+ \Big(\sum_{T\in\mathcal{T}_c} \|f\|_{2,\omega_T} \|(\chi - \chi_h - w)_-\|_{2,T}\Big)^{1/2} + |w|_{1,2} + \|h_{\mathcal{T}}^2 \nabla f\| \\
&+ \Big(\sum_{T\in\mathcal{T}\setminus\mathcal{T}_D} \|h_T^{-1}(\chi - \chi_h - w)_-\|_{2,T}^2\Big)^{1/2} + \Big(\sum_{T\in\mathcal{T}_D} h_T^2 \|f\|_{2,\omega_T}^2\Big)^{1/2}.
\end{aligned}$$

Proof. As in the proof of Theorem 2 we have

$$\begin{aligned}
|e-w|_{1,2} + |e|_{1,2} \lesssim{} & \min_{p_h \in \mathcal{S}^1(\mathcal{T})^d} (\|\nabla u_h - p_h\| + \|h_{\mathcal{E}}^{1/2}(g - p_h\cdot n)\|_{2,\Gamma_N}) \\
& + \|h_{\mathcal{T}}^2 \nabla f\| + |w|_{1,2} + (\varrho_h; e-w).
\end{aligned}$$

Employing Lemma 9 and absorbing $|e-w|_{1,2}$ we have

$$\begin{aligned}
|e-w|_{1,2}^2 + |e|_{1,2}^2 \lesssim{} & \min_{p_h \in \mathcal{S}^1(\mathcal{T})^d} (\|\nabla u_h - p_h\| \\
&+ \|h_{\mathcal{E}}^{1/2}(g - p_h\cdot n)\|_{2,\Gamma_N})^2 + \|h_{\mathcal{T}}^2 \nabla f\|^2 + |w|_{1,2}^2 \\
&+ \sum_{T\in\mathcal{T}\setminus\mathcal{T}_D} \|h_T^{-1}(\chi - \chi_h - w)_-\|_{2,T}^2 + \|h_{\mathcal{T}}^2 \nabla \varrho_h\|^2 \\
&+ \sum_{T\in\mathcal{T}_D} h_T^2 \|\varrho_h\|_{2,T}^2 + \sum_{T\in\mathcal{T}_i} \min_{q_z \in \mathcal{S}^1(\mathcal{T}|_{\tilde\Omega_{z_T}})^d} \|\nabla(\chi_h - u_h) - q_z\|_{2,\tilde\Omega_{z_T}}^2 \\
&+ \sum_{T\in\mathcal{T}_c} h_T \|\varrho_h\|_{2,T} \|h_T^{-1}(\chi - \chi_h - w)_-\|_{2,T}.
\end{aligned} \tag{3.9}$$

Lemma 8 shows

$$
\begin{aligned}
&\|h_{\mathcal{T}}^2\nabla\varrho_h\|^2+\sum_{T\in\mathcal{T}_D}h_T^2\|\varrho_h\|_{2,T}^2+\sum_{\mathcal{T}\in\mathcal{T}_c}h_T\|\varrho_h\|_{2,T}\|h_T^{-1}(\chi-\chi_h-w)_-\|_{2,T}\\
&\quad\lesssim\|h_{\mathcal{T}}^2\nabla f\|^2+\sum_{T\in\mathcal{T}\setminus\mathcal{T}_D}\|h_T^{-1}(\chi-\chi_h-w)_-\|_{2,T}^2\\
&\qquad+\sum_{T\in\mathcal{T}}\min_{q_T\in\mathcal{S}^1(\mathcal{T}|_{\omega_T})^d}\big(\|\nabla u_h-q_T\|_{2,\omega_T}+h_T^{1/2}\|(g-q_T\cdot n)\|_{2,\Gamma_N\cap\partial\omega_T}\big)^2\\
&\qquad+\sum_{T\in\mathcal{T}_c}\|h_Tf\|_{2,\omega_T}\|h_T^{-1}(\chi-\chi_h-w)_-\|+\sum_{T\in\mathcal{T}_D}h_T^2\|f\|_{2,\omega_T}^2.
\end{aligned}
\tag{3.10}
$$

Let $\tilde{p}_h\in\mathcal{S}^1(\mathcal{T})^d$ denote the minimiser of

$$\min_{p_h\in\mathcal{S}^1(\mathcal{T})^d}(\|\nabla u_h-p_h\|+\|h_{\mathcal{E}}^{1/2}(g-p_h\cdot n)\|_{2,\Gamma_N}).$$

Since $\tilde{p}_h|_{\omega_T}\in\mathcal{S}^1(\mathcal{T}|_{\omega_T})^d$ for all $T\in\mathcal{T}$ and since the patches ω_T have a finite overlap we have

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}}\min_{q_T\in\mathcal{S}^1(\mathcal{T}|_{\omega_T})^d}\big(\|\nabla u_h-q_T\|_{2,\omega_T}+h_T^{1/2}\|(g-q_T\cdot n)\|_{2,\Gamma_N\cap\partial\omega_T}\big)^2\\
&\quad\lesssim\min_{p_h\in\mathcal{S}^1(\mathcal{T})^d}\big(\|\nabla u_h-p_h\|+\|h_{\mathcal{E}}^{1/2}(g-p_h\cdot n)\|_{2,\Gamma_N}\big)^2.
\end{aligned}
\tag{3.11}
$$

A similar argument and the definition of $\mathcal{T}_s$ show

$$
\begin{aligned}
&\sum_{\mathcal{T}\in\mathcal{T}_i}\min_{q_z\in\mathcal{S}^1(\mathcal{T}|_{\tilde{\Omega}_{z_T}})^d}\|\nabla(\chi_h-u_h)-q_z\|_{2,\tilde{\Omega}_{z_T}}^2\\
&\quad\lesssim\min_{q_h\in\mathcal{S}^1(\mathcal{T}_s)^d}\|\nabla(\chi_h-u_h)-q_h\|_{2,\Omega_s}^2.
\end{aligned}
\tag{3.12}
$$

The combination of (3.9)–(3.12) proves the theorem. □

Remark 7. Two applications of the triangle inequality indicate efficiency of the error estimate of Theorem 3 in case $\chi_h=\chi$,

$$
\begin{aligned}
&\min_{p_h\in\mathcal{S}^1(\mathcal{T})^d}(\|\nabla u_h-p_h\|+\|h_{\mathcal{E}}^{1/2}(g-p_h\cdot n)\|_{2,\Gamma_N})\\
&\quad+\min_{q_h\in\mathcal{S}^1(\mathcal{T}_s)^d}\|\nabla(u_h-\chi_h)-q_h\|_{2,\Omega_s}\le|e|_{1,2}\\
&\quad+|e|_{1,2,\Omega_s}+|\chi-\chi_h|_{1,2,\Omega_s}\\
&\quad+2\min_{p_h\in\mathcal{S}^1(\mathcal{T})^d}(\|\nabla u-p_h\|+\|h_{\mathcal{E}}^{1/2}(g-p_h\cdot n)\|_{2,\Gamma_N})\\
&\quad+\min_{q_h\in\mathcal{S}^1(\mathcal{T}_s)^d}\|\nabla\chi-q_h\|_{2,\Omega_s}.\quad\square
\end{aligned}
$$

4 Numerical Realisation

In the examples presented in the subsequent section we have $\chi_h = \chi, \Gamma_N = \emptyset$, $\chi|_{\Gamma_D} \leq u_{D,h}$, and $f \in H^2(\Omega)$. Then the error estimate of Theorem 3 reduces to

$$|e|_{1,2} \lesssim \min_{p_h \in \mathcal{S}^1(\mathcal{T})^d} \|\nabla u_h - p_h\| + \min_{q_h \in \mathcal{S}^1(\mathcal{T}_s)^d} \|\nabla(u_h - \chi_h) - q_h\|_{2,\Omega_s} + \text{h.o.t.}, \tag{4.1}$$

where h.o.t. denotes higher order contributions which only depend on given right-hand sides. In the numerical experiments we do not compute the minima in (4.1) but calculate an upper bound for the first two terms by applying the operator $\mathcal{A} : L^2(\Omega)^2 \to \mathcal{S}^1_N(\mathcal{T}, g)$ and $\mathcal{B} : L^2(\Omega)^2 \to \mathcal{S}^1(\mathcal{T})^2$ of [CB], defined for $p \in L^2(\Omega)^2$ and $\Gamma_N = \emptyset$ by

$$\mathcal{A}p = \mathcal{B}p := \sum_{z \in \mathcal{N}} p_z \varphi_z \tag{4.2}$$

with $p_z := \frac{1}{|\omega_z|} \int_{\omega_z} p\, dx \in \mathbb{R}^2$ for $z \in \mathcal{N}$; i.e., we calculate

$$\|\nabla u_h - \mathcal{A}\nabla u_h\| + \|\nabla(u_h - \chi_h) - \mathcal{B}\nabla(u_h - \chi_h)\|_{2,\Omega_s}.$$

We refer to [CB] for a definition of $\mathcal{A}$ in case $\Gamma_N \neq \emptyset$ where $g - (\mathcal{A}\nabla u_h) \cdot n$ vanishes at all nodes $z \in \mathcal{N} \cap \overline{\Gamma}_N$.

Our numerical approximations are obtained with the following projected SOR algorithm proposed in [E].

Algorithm (A^{pSOR}) Input: $0 < \omega := 3/2 < 2$, $\delta = 10^{-3} > 0$, and $u_h^{(0)} \in \mathcal{S}^1(\mathcal{T})$ with $u_h^{(0)}|_{\Gamma_D} = u_{D,h}$.

(a) Set $u_h^{(0)} := \max\{u_h^{(0)}, \chi_h\}$ and $k := 1$.
(b) Set $\tilde{\mathcal{K}} := \mathcal{K}$ and $u_h^{(k)} := u_h^{(k-1)}$.

(i) Choose $z \in \tilde{\mathcal{K}}$.
(ii) Compute the minimising t^* for

$$\frac{1}{2}||\nabla(u_h^{(k)} + t\varphi_z)||^2 - \big(f, u_h^{(k)} + t\varphi_z\big) - \int_{\Gamma_N} g\,(u_h^{(k)} + t\varphi_z)\, ds$$

among all $t \in \mathbb{R}$ such that $u_h^{(k)} + t\varphi_z \in K_h$.
(iii) Set $v_z := u_h^{(k)}(z) + t^*$. If

$$\big((1-\omega)u_h^{(k)}(z) + \omega v_z - \chi_h(z)\big)\big(u_h^{(k)}(z) - \chi_h(z)\big) > 0$$

set $u_h^{(k)} := (1-\omega)u_h^{(k)} + \omega t^* \varphi_z$ and otherwise $u_h^{(k)} := u_h^{(k)} + t^*\varphi_z$.

(iv) Set $\tilde{\mathcal{K}} := \tilde{\mathcal{K}} \setminus \{z\}$ and go to (i) if $\tilde{\mathcal{K}} \neq \emptyset$.
(c) Set $u_h := u_h^{(k)}$ and stop if $|u_h^{(k)} - u_h^{(k-1)}|_{1,2} \leq \delta$.
(d) Set $k := k + 1$ and go to (b).
Output: approximation u_h of the discrete solution.

The implementation was performed in Matlab in the spirit of [ACF]. The following adaptive algorithm generates the sequences of meshes $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ in this paper which are uniform for $\Theta = 0$ or adapted for $\Theta = 1/2$ in (4.3). For details on the red-blue-green-refinements in the algorithm we refer to [V].

Algorithm (A_Θ) (a) Start with a coarse mesh $\mathcal{T}_0$, $\ell = 0$.
(b) Run Algorithm (A^{pSOR}) to compute u_h on the actual mesh $\mathcal{T}_\ell$.
(c) Define

$$\begin{aligned} \mathcal{M} &:= \{z \in \mathcal{K} : u_h(z) = \chi_h(z), \\ &\qquad \exists T \in \mathcal{T}_k \, \exists y \in \mathcal{N} \cap T, \; z \in T, \; \chi_h(y) < u_h(y)\}, \\ \tilde{\mathcal{T}}_s &:= \{T \in \mathcal{T}_k : \exists z \in \mathcal{M}, T \subseteq \overline{\omega}_z\}. \end{aligned}$$

For $T \in \mathcal{T}_k$ compute the refinement indicator

$$\eta_T := \begin{cases} \|\nabla u_h - \mathcal{A}\nabla u_h\|_{2,T} & \text{if } T \notin \tilde{\mathcal{T}}_s, \\ \frac{1}{2}\big(\|\nabla u_h - \mathcal{A}\nabla u_h\|_{2,T} & \\ \quad + \|\nabla(u_h - \chi_h) - \mathcal{B}\nabla(u_h - \chi_h)\|_{2,T}\big) & \text{if } T \in \tilde{\mathcal{T}}_s, \end{cases}$$

and the estimator $\eta_N := \big(\sum_{T\in\mathcal{T}} \eta_T^2\big)^{1/2}$ for the energy error $e_N := \|\nabla(u - u_h)\|_{2,\Omega}$.
(d) Mark the element T for *red*-refinement provided

$$\eta_T \geq \Theta \max_{T' \in \mathcal{T}_\ell} \eta_{T'}. \tag{4.3}$$

(e) Mark further elements (*red–blue–green*-refinement) to avoid hanging nodes and generate a new triangulation $\mathcal{T}_{\ell+1}$. Update ℓ and go to (b).

Remark 8. (i) For simplicity, we computed $\tilde{\mathcal{T}}_s$ to approximate $\mathcal{T}_s$.
(ii) The choice of the factors in the definition of $\eta_{A,T}$ is motivated by the efficiency estimate of Remark 7.
(iii) The computed estimator η_N is in fact efficient according to [C2].

5 Numerical Experiments

The theoretical results of this paper are supported by numerical experiments. In this section, we report on three examples of problem (1.1) on uniform and adapted meshes.

5.1 Example with smooth rotational symmetric solution [LLT]

Let $f := -2$ on $\Omega := (-3/2, 3/2)^2$ and $u_D(x, y) := r^2/2 - \ln(r) - 1/2$ where $r := (x^2 + y^2)^{1/2}$ on the Dirichlet boundary $\Gamma_D := \partial\Omega$. For $\chi_h = \chi := 0$ the exact solution of problem (1.1) reads

$$u(x, y) = \begin{cases} r^2/2 - \ln(r) - 1/2 & \text{if } r \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $u \in H^2(\Omega)$. In our numerical experiments the coarse triangulation $\mathcal{T}_0$ of Fig. 1 consists of 16 squares halved along a diagonal.

The top plot in Fig. 1 shows a sequence of triangulations generated by Algorithm $(A_{1/2})$. The algorithm refines the mesh in the complement of the contact zone $\{(x, y) \in \Omega : x^2 + y^2 \leq 1\}$ in which the solution vanishes. The approximate discrete contact zone $\{T \in \mathcal{T}_k : u_h(x_T) = \chi_h(x_T)\}$, where x_T denotes the center of a triangle T, is plotted in white while its complement is shaded (we chose this color since in most of the examples the complement of the contact zone is refined and appears darker). The bottom plot of Fig. 1 displays the solution u_h on the adaptively generated mesh $\mathcal{T}_8$ with 881 degrees of freedom. In Fig. 2 we plotted the error and its estimator versus the degrees of freedom for uniform and adaptive mesh refinement. A logarithmic scaling used for both axes allows a slope $-\alpha$ to be interpreted as an experimental convergence rate 2α (owing to $h \propto N^{-2}$ in two dimensions). We obtain experimental convergence rates 1 for both refinement strategies. The error on the adaptively refined meshes is however smaller than the error on uniform meshes at comparable numbers of degrees of freedoms. The plot shows that η_N serves as a very accurate approximate of the error e_N: The entries (N, e_N) and (N, η_N) almost coincide.

5.2 Example with corner singularity

Using polar coordinates (r, φ) on the L-shaped domain $\Omega := (-2, 2)^2 \setminus [0, 2] \times [-2, 0]$, $u_D := 0$ on $\Gamma_D := \partial\Omega$, $\chi_h = \chi := 0$, let

$$\begin{aligned} f(r, \varphi) := &-r^{2/3} \sin(2\varphi/3)\big(\gamma_1'(r)/r + \gamma_1''(r)\big) \\ &-\frac{4}{3} r^{-1/3} \gamma_1'(r) \sin(2\varphi/3) - \gamma_2(r) \end{aligned}$$

where,

$$\gamma_1(r) := \begin{cases} 1 & \text{if } \overline{r} < 0, \\ -6\overline{r}^5 + 15\overline{r}^4 - 10\overline{r}^3 + 1 & \text{if } 0 \leq \overline{r} < 1, \\ 0 & \text{if } 1 \leq \overline{r}, \end{cases}$$

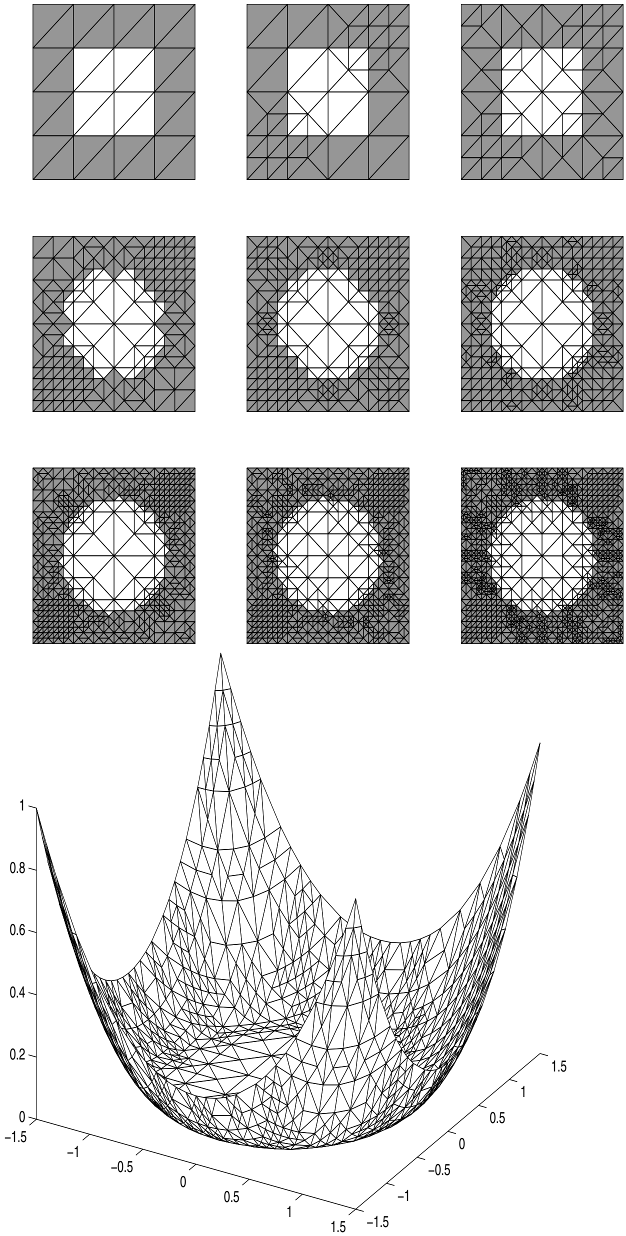

Fig. 1. Adaptively refined meshes $\mathcal{T}_0$ (left upper) to $\mathcal{T}_8$ (right lower) (top) with approximate discrete contact zone shown in white and solution u_h on $\mathcal{T}_8$ with 881 free nodes (bottom) in Example 5.1.

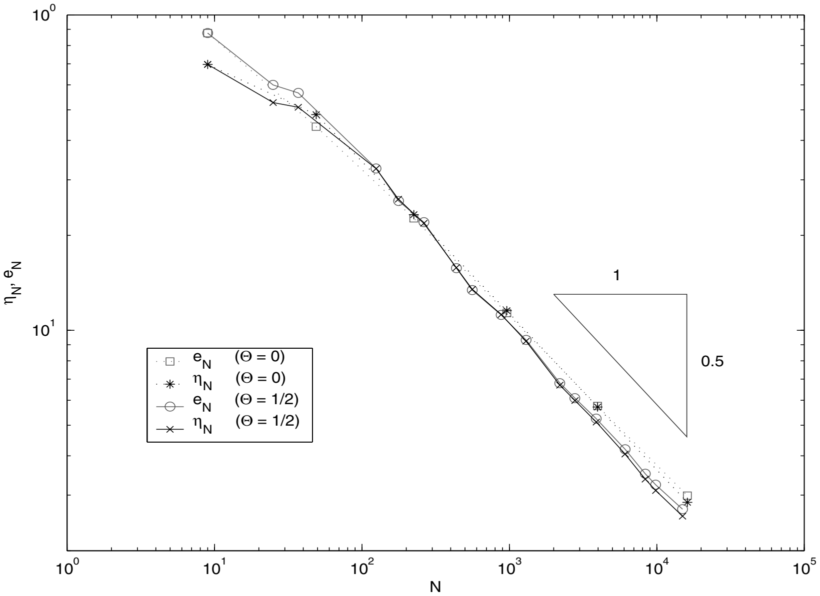

Fig. 2. Error and error estimator for uniform ($\Theta = 0$) and adaptive ($\Theta = 1/2$) mesh-refinement in Example 5.1.

for $\overline{r} := 2(r - 1/4)$, and $\gamma_2(r) := 0$ if $r \leq 5/4$ and $\gamma_2(r) := 1$ otherwise. The exact solution of (1.1) is then given by $u(r, \varphi) := r^{2/3}\,\gamma_1(r)\,\sin(2\varphi/3)$ and has a typical corner singularity at the origin. The coarsest triangulation $\mathcal{T}_0$ of Fig. 3 consists of 48 halved squares.

The sequence of triangulations generated by Algorithm $(A_{1/2})$ in Example 5.2 and displayed in the top plot of Fig. 3 shows a refinement towards the origin where the solution has a singularity in the gradient and a refinement in the region $\{(x, y) \in \Omega : 1/4 \leq (x^2 + y^2)^{1/2} \leq 3/4\}$ where the solution has large gradients. This behavior can also be seen in the bottom plot of Fig. 3 where we plotted the numerical solution u_h on triangulation $\mathcal{T}_8$ with 572 degrees of freedom. Fig. 4 shows that the adaptive Algorithm $(A_{1/2})$ improves the experimental convergence rate about 3/4 for uniform mesh-refinement to the optimal value 1. Note that for uniform mesh-refinement we expect an asymptotic convergence rate 2/3 due to the corner singularity. The error in the region where u has a large gradient seems to dominate in this preasymptotic range with $N \leq 10^5$. Again, the entries for η_N and e_N almost coincide and this behavior improves for increasing numbers of degrees of freedom.

5.3 Example with unknown exact solution

Let $f := 1$ on $\Omega := (-1, 1)^2$, $u_D := 0$ on $\Gamma_D := \partial\Omega$, $\Gamma_N := \emptyset$, and $\chi(x, y) := \mathrm{dist}((x, y), \partial\Omega)$. In this example the exact solution of (1.1) is not

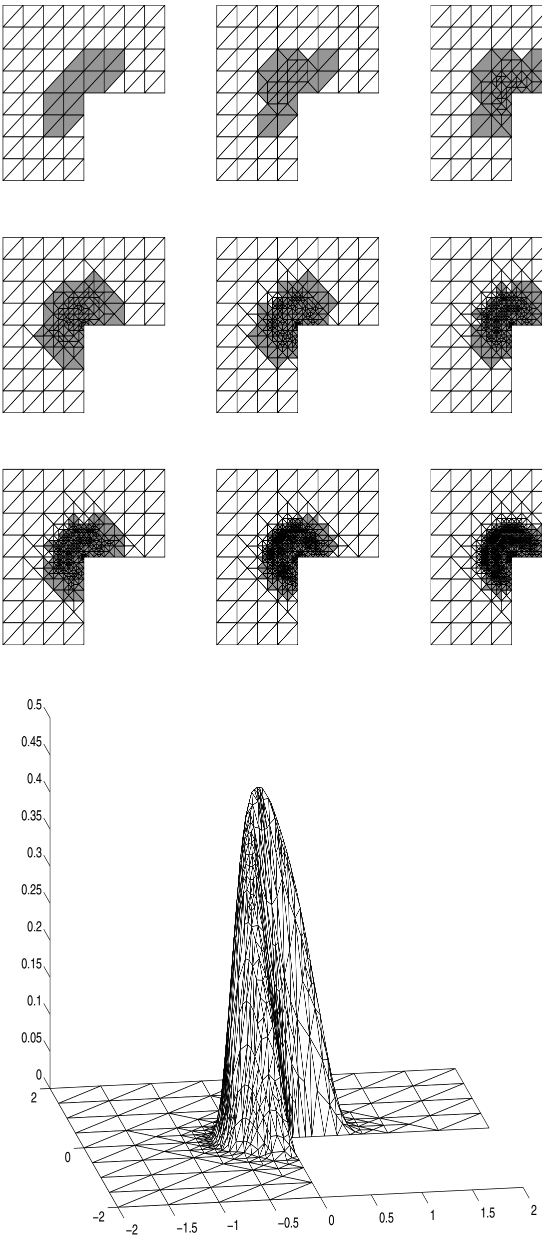

Fig. 3. Adaptively refined meshes $\mathcal{T}_0$ (left upper) to $\mathcal{T}_8$ (right lower) (top) with approximate discrete contact zone shown in white and solution u_h on $\mathcal{T}_8$ with 572 free nodes (bottom) in Example 5.2.

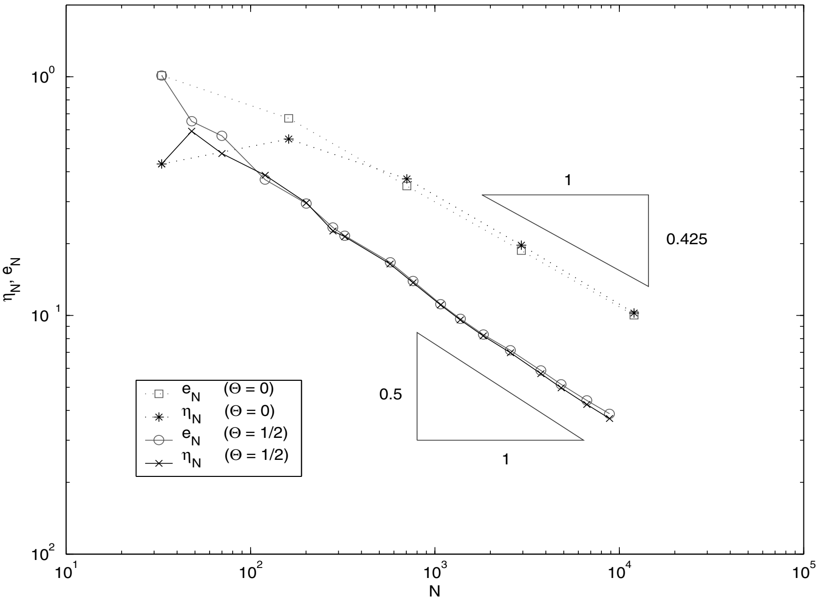

Fig. 4. Error and error estimator for uniform ($\Theta = 0$) and adaptive ($\Theta = 1/2$) mesh–refinement in Example 5.2.

known and cannot be expected to be smooth since $\chi \notin H^2(\Omega)$. The coarsest triangulation $\mathcal{T}_0$ of Fig. 5 consists of 64 elements with $\chi_h = \chi$ on $\mathcal{T}_0$.

This example is different from Examples 5.1 and 5.2 in the sense that the solution and the obstacle are non-smooth along the lines $\mathcal{C} = \{(x, y) \in \Omega : x = y \text{ or } x = 1 - y\}$. Algorithm $(A_{1/2})$ refines the mesh towards these lines as can be seen in the top and bottom plot of Fig. 5. Moreover, the approximate discrete contact zone reduces to these lines. Note that $f + \Delta u = 0$ in $\Omega \setminus \mathcal{C}$. Since the mesh is aligned with the lower dimensional contact zone $\mathcal{C}$ there holds $u - Iu = 0$ on $\mathcal{C}$, where $Iu \in K_h \subseteq K$ denotes the nodal interpolant of the exact solution u. Hence, an integration by parts shows $(f; u - Iu) = (\nabla u; \nabla(u - Iu))$. From this and (1.1)-(1.2) with $v = Iu = v_h$ it follows that

$$|e|_{1,2} \le |u - Iu|_{1,2} \lesssim \|h_{\mathcal{T}} D^2 u\|_{2,\Omega\setminus\mathcal{C}}$$

is linearly convergent and much smaller than the last term in (1.10) which, here, is *not* of higher order. Consequently, the estimator may be expected to be reliable but not efficient as displayed in Fig. 6, where we compared η_A to an approximated error e'_N. In the approximated error $e'_N = |u_k - u'_{k+2}|_{1,2}$ the function $u'_{k+2} \in \mathcal{S}^1(\mathcal{T}'_{k+2})$ is the solution of the discrete problem (1.2) with the triangulation $\mathcal{T}'_{k+2}$ obtained from two red refinements of $\mathcal{T}_k$ if $\Theta = 0$ and two successive adaptive refinements of $\mathcal{T}_k$ based on Algorithm $(A_{1/2})$

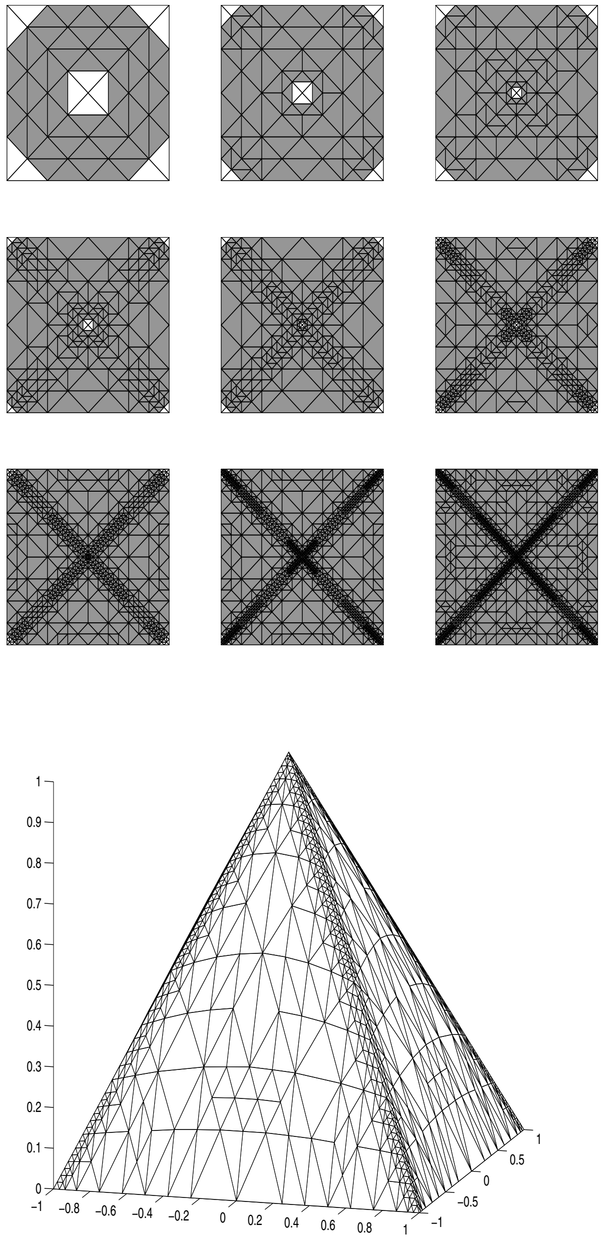

Fig. 5. Adaptively refined meshes $\mathcal{T}_0$ (left upper) to $\mathcal{T}_8$ (right lower) (top) with approximate discrete contact zone shown in white and solution u_h on $\mathcal{T}_8$ with 1149 free nodes (bottom) in Example 5.3.

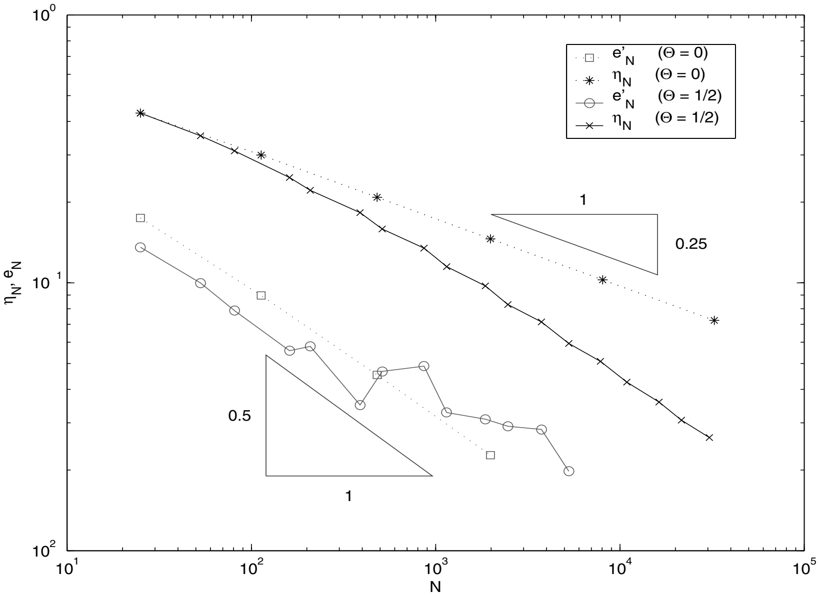

Fig. 6. Error estimator and approximated error for uniform ($\Theta = 0$) and adaptive ($\Theta = 1/2$) mesh–refinement in Example 5.3.

if $\Theta = 1/2$. It remains unclear whether this is a good approximation of the actual error e_N; for uniform mesh refinement the approximated error e'_N converges faster than expected.

5.4 Remarks

(i) The numerical results for Examples 5.1–5.2 show that the adaptive Algorithm $(A_{1/2})$ yields significant error reduction.
(ii) The error estimate performed extremely accurate although its realized version is reliable but not necessarily efficient.
(iii) As an initial function for Algorithm (A^{pSOR}) we defined $u_h^{(0)}(z) = \chi_h(z)$ for all $z \in \mathcal{K}$ and $u_h^{(0)}|_{\Gamma_D} = u_{D,h}$ on $\mathcal{T}_0$ for the first mesh and successively the prolongation to $\mathcal{T}_{k+1}$ of the solution u_h on $\mathcal{T}_k$ for subsequent refinement levels (nested iteration). In the above examples, Algorithm (A^{pSOR}) terminated after at most ten iterations.
(iv) The meshes generated by Algorithm (A_Θ) show local symmetries. A similar error estimator as η_N designed for second order partial differential equations performed well also on randomly perturbed meshes without any symmetry [CB, BC].
(v) The error estimator is reliable and efficient in Examples 5.1 and 5.2. It is reliable (but possibly not efficient) in Example 5.3 owing to non-smoothness of the obstacle.

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