MATHEMATICS OF COMPUTATION
Volume 73, Number 247, Pages 1153–1165
S 0025-5718(03)01580-1
Article electronically published on August 12, 2003

ALL FIRST-ORDER AVERAGING TECHNIQUES FOR A POSTERIORI FINITE ELEMENT ERROR CONTROL ON UNSTRUCTURED GRIDS ARE EFFICIENT AND RELIABLE

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ABSTRACT. All first-order averaging or gradient-recovery operators for lowest-order finite element methods are shown to allow for an efficient a posteriori error estimation in an isotropic, elliptic model problem in a bounded Lipschitz domain Ω in $\mathbb{R}^d$. Given a piecewise constant discrete flux $p_h \in P_h$ (that is the gradient of a discrete displacement) as an approximation to the unknown exact flux p (that is the gradient of the exact displacement), recent results verify efficiency and reliability of

$$\eta_M := \min\{\|p_h - q_h\|_{L^2(\Omega)} : q_h \in \mathcal{Q}_h\}$$

in the sense that η_M is a lower and upper bound of the flux error $\|p-p_h\|_{L^2(\Omega)}$ up to multiplicative constants and higher-order terms. The averaging space $\mathcal{Q}_h$ consists of piecewise polynomial and globally continuous finite element functions in d components with carefully designed boundary conditions. The minimal value η_M is frequently replaced by some averaging operator $A : P_h \to \mathcal{Q}_h$ applied within a simple post-processing to p_h. The result $q_h := Ap_h \in \mathcal{Q}_h$ provides a reliable error bound with $\eta_M \le \eta_A := \|p_h - Ap_h\|_{L^2(\Omega)}$.

This paper establishes $\eta_A \le C_{\text{eff}}\, \eta_M$ and so equivalence of η_M and η_A. This implies efficiency of η_A for a large class of patchwise averaging techniques which includes the ZZ-gradient-recovery technique. The bound $C_{\text{eff}} \le 3.88$ established for tetrahedral P_1 finite elements appears striking in that the shape of the elements does *not* enter: The equivalence $\eta_A \approx \eta_M$ is robust with respect to anisotropic meshes. The main arguments in the proof are Ascoli's lemma, a strengthened Cauchy inequality, and elementary calculations with mass matrices.

1. Introduction

Suppose p_h is the discrete flux obtained from a conforming, nonconforming, or mixed low-order finite element method (FEM) based on a regular triangulation $\mathcal{T}$ of the domain Ω. That is, p_h is the piecewise polynomial but globally discontinuous elementwise gradient of the finite element displacement approximations u_h or a discrete flux variable (for a mixed FEM) that approximates the unknown exact flux p. It is the aim of a posteriori error control to bound the error $\|p - p_h\|_{L^2(\Omega)}$ from above and below by computable estimators [AO, BS, V]. It has recently been proven for several examples [CB, BC1, CF3, CF4] that the error $\|p - p_h\|_{L^2(\Omega)}$ in

Received by the editor July 26, 2002 and, in revised form, January 1, 2003.
2000 *Mathematics Subject Classification.* Primary 65N30; Secondary 65N15.
Key words and phrases. A posteriori error estimate, efficiency, finite element method, gradient recovery, averaging operator, mixed finite element method, nonconforming finite element method.

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second-order elliptic boundary value problems is bounded by $\|p_h - q_h\|_{L^2(\Omega)}$ for any continuous and piecewise polynomial q_h in the sense that

$$\|p - p_h\|_{L^2(\Omega)} \le C_{\mathrm{rel}} \|p_h - q_h\|_{L^2(\Omega)} + \text{h.o.t.}$$

The boundary values are included in the set $\mathcal{Q}_h$ of possible averages q_h. The surprising aspect is that *all* averaging techniques which, given p_h, compute $q_h \in \mathcal{Q}_h$ are *reliable* in the sense that

$$\|p - p_h\|_{L^2(\Omega)} \le C_{\mathrm{rel}} \eta_M + \text{h.o.t.} \quad \text{for } \eta_M := \min_{q_h \in \mathcal{Q}_h} \|p_h - q_h\|_{L^2(\Omega)}.$$

The minimum η_M is frequently replaced by an upper bound η_A,

$$\eta_M \le \eta_A := \|p_h - Ap_h\|_{L^2(\Omega)},$$

where $Ap_h \in \mathcal{Q}_h$ is computed with some local averaging operator A. One striking feature of η_M is its immediate efficiency,

$$\begin{aligned} \eta_M &= \min_{q \in \mathcal{Q}_h} \|p_h - p + p - q_h\|_{L^2(\Omega)} \\ &\le \|p - p_h\|_{L^2(\Omega)} + \min_{q_h \in \mathcal{Q}_h} \|p - q_h\|_{L^2(\Omega)} \\ &= \|p - p_h\|_{L^2(\Omega)} + \text{h.o.t.} \end{aligned}$$

This follows from a simple triangle inequality plus some considerations of the minimal $\|p - q_h\|_{L^2(\Omega)}$. The latter argument requires smoothness of p and the correct treatment of boundary conditions that restrict the set $\mathcal{Q}_h$. Note that the multiplicative constant in the efficiency estimate

$$\eta_M \le \|p - p_h\| + \text{h.o.t.} \tag{1.1}$$

is one; i.e. η_M is a lower bound up to higher-order terms. This is, in general, untrue for its upper bound η_A. The possible overestimation of the error $\|p - p_h\|_{L^2(\Omega)}$ by $C_{\mathrm{rel}}\eta_A$ might be very large. In [CB, BC1] a local (edge-oriented) averaging is suggested and shown to be equivalent to η_M (cf. Theorem 3.2 in [CB]). In this paper we analyse a different and more popular averaging operator defined by

$$(Ap_h)(z) = A_z(p_h|_{\omega_z}) \in \mathbb{R}^d \quad \text{for each node } z$$

and its patch ω_z (cf. Section 2 for notation). Here, $A_z := \pi_z \circ M_z$ for some continuous averaging M_z that is exact for constants and the orthogonal projection π_z onto an affine subspace $\mathcal{A}_z \subset \mathbb{R}^d$ that carries proper boundary conditions. The main result, Theorem 4.1, reads

$$\eta_M \le \eta_A \le C_{\mathrm{eff}}\, \eta_M. \tag{1.2}$$

It is remarkable that the constant C_{eff} depends only on the norm of A_z and so it holds for any unstructured grid as well as for a quite large class of averaging and finite element schemes. For the popular choice of integral means

$$M_z(p_h) := \int_{\omega_z} p_h \, dx / |\omega_z| \tag{1.3}$$

for any node z with patch ω_z of area or volume $|\omega_z|$ we establish in Corollary 5.3 for P_1 finite elements the estimates

$$1 \le C_{\mathrm{eff}} \le \sqrt{10} \text{ for } d = 2 \quad \text{and} \quad 1 \le C_{\mathrm{eff}} \le \sqrt{15} \text{ for } d = 3. \tag{1.4}$$

This is surprisingly sharp and does not depend on any detail of the regular triangulation with (possibly) degenerating triangles or tetrahedra.

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Remark 1.1. The averaging technique (1.3) is our interpretation of the ZZ-estimator [ZZ, V] for which reliability and efficiency have been observed before [R1, R2, N, BR] (without treatment of mixed boundary conditions).

Remark 1.2. The averaging estimator η_A can be shown to be equivalent to the edge contributions

$$\eta_{\mathcal{E}} := (\sum_{E\in\mathcal{E}} h_E \|[p_h]|_E\|^2_{L^2(E)})^{1/2},$$

where $[p_h]|_E$ denotes the jump of p_h across the edge $E \in \mathcal{E}$ (with proper modifications on the boundary). Thus our qualitative results (partly) follow from reliability and efficiency of $\eta_{\mathcal{E}}$ as well [C, CV, R1, V].

Remark 1.3. The above estimates on C_{eff} yield lower bounds $C_{\text{eff}}^{-1} \le C_{\text{rel}}$ on the reliability constant (up to higher-order terms). Upper bounds on C_{rel} for related estimators with a best value around 1 can be found in [CF1, CF2].

Remark 1.4. As important corollaries of $\eta_M \approx \eta_A$ and (1.1) we obtain efficiency

$$\eta_A \le C_{\text{eff}}\, \|p - p_h\| + \text{h.o.t.} \tag{1.5}$$

of the reliable error estimation by η_A in [CA, CB, BC1, CF3, CF4].

The remaining part of the paper is organised as follows. Section 2 presents the necessary technical notation. The preliminaries of Section 3 include Ascoli's lemma, the strengthened Cauchy inequality, and eigenvalues of mass matrices. The main result (1.2) is stated as Theorem 4.1 in Section 4 with a proof. An analysis of C_{eff} in a model situation of Section 5 leads to (1.4) shown in Corollary 5.3.

2. Assumptions

2.1. **Regular triangulation.** The bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, with piecewise affine boundary Γ is exactly covered by a triangulation $\mathcal{T}$, $\bigcup \mathcal{T} = \overline{\Omega}$. Each element $T \in \mathcal{T}$ is a compact interval $T = \text{conv}\{a, b\}$ if $d = 1$, a triangle $T = \text{conv}\{a, b, c\}$ if $d = 2$, or a tetrahedron $T = \text{conv}\{a, b, c, d\}$ if $d = 3$. The element's vertices $a, \ldots, d$ are called nodes; $\mathcal{N}$ denotes the set of all nodes. Each flat boundary E of an element $T \in \mathcal{T}$ is either a point $E = \{a\}$, an edge $E = \text{conv}\{a, b\}$, or a face $E = \text{conv}\{a, b, c\}$; $\mathcal{E}$ denotes the set of all such E; $\mathcal{E}_\Omega$ denotes the interior edges or faces and $\mathcal{E}_\Gamma := \{E \in \mathcal{E} : E \subset \Gamma\} = \mathcal{E}\backslash\mathcal{E}_\Omega$ denotes the boundary edges. Analogous notation apply to parallelograms ($d = 2$) or parallelepipeds ($d = 3$) which are possible elements in $\mathcal{T}$ as well. Intersecting distinct elements share either one vertex, an edge, or a common face. Hanging nodes are excluded for simplicity. For each node $z \in \mathcal{N}$ let $\mathcal{E}_z := \{E \in \mathcal{E} : z \in E \cap \mathcal{N}\}$ and the patch $\omega_z := \text{int}(\bigcup \mathcal{T}_z)$, $\mathcal{T}_z := \{T \in \mathcal{T} : z \in T \cap \mathcal{N}\}$. Each edge or face E is associated to a unit normal vector ν_E with fixed orientation; if $E \subseteq \partial\Omega$, set $\nu_E = \nu$, the outer unit normal along $\partial\Omega$. The length and area of $E \in \mathcal{E}$ is denoted by $h_E = \text{diam}(E)$ and $|E| = \mathcal{L}^{d-1}(E)$, respectively; $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure along any affine subspace of $\mathbb{R}^d$. Similarly the length and volume of $T \in \mathcal{T}$ is denoted by $h_T = \text{diam}(T)$ and $|T| = \mathcal{L}^d(T)$, respectively.

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2.2. **Boundary data.** The boundary $\Gamma = \bigcup \mathcal{E}_\Gamma$ is split into a relatively closed part Γ_D and a remaining part $\Gamma_N := \Gamma \backslash \Gamma_D$ such that any edge $E \in \mathcal{E}_\Gamma$ belongs either to Γ_D or to $\overline{\Gamma}_N$. Two disjoint subsets $\mathcal{E}_D$ and $\mathcal{E}_N$ of $\mathcal{E}_\Gamma$ are supposed to satisfy

$$\mathcal{E}_D = \emptyset \quad \text{or} \quad \mathcal{E}_D = \{E \in \mathcal{E}_\Gamma : \, E \subset \Gamma_D\},$$
$$\mathcal{E}_N = \emptyset \quad \text{or} \quad \mathcal{E}_N = \{E \in \mathcal{E}_\Gamma : \, E \subset \overline{\Gamma_N}\}.$$

Given $\mathcal{E}_D$ and $\mathcal{E}_N$, the boundary data $g \in L^2(\Gamma_N)$ and $u_D \in H^{1/2}(\Gamma_D) \cap C(\Gamma_D)$ (i.e. u_D is continuous on Γ_D and can be extended to a function in $H^1(\Omega)$) satisfy $g \in C(\mathcal{E}_D)$ and $u_D \in C^1(\mathcal{E}_N)$; i.e.

$$g|_E \in C(E) \text{ for all } E \in \mathcal{E}_N \quad \text{and} \quad u_D|_E \in C^1(E) \text{ for all } E \in \mathcal{E}_D.$$

On each $E \in \mathcal{E}_D$, let $\tau_E^{(j)}$ denote a tangential unit vector for $j = 1, \ldots, d-1$ such that $(\nu_E, \tau_E^{(1)}, \ldots, \tau_E^{(d-1)})$ is a Cartesian basis of $\mathbb{R}^d$. Then, $\nabla_E u_D$ denotes the tangential derivative and, given $a \in \mathbb{R}^d$, $(a)_E$ denotes the vector of all components of a in $(\tau_E^{(j)})_{j=1}^{d-1}$, e.g. $(a)_E = (\tau_E^{(1)} \cdot a, \tau_E^{(2)} \cdot a)$ for $d = 3$; $\nabla_E u_D = (\nabla u_D)_E = \partial u_D / \partial s$ for $d = 2$.

The Dirichlet and Neumann boundary conditions on the gradient $p = \nabla u$ are asserted at each boundary node $z \in \mathcal{N}$ by $p(z) \in \mathcal{A}_z$ for the affine subspace

$$\begin{aligned} (2.1) \quad \mathcal{A}_z := \{a \in \mathbb{R}^d : \forall E \in \mathcal{E}_z \cap \mathcal{E}_N, \, g(z) = a \cdot \nu_E \\ \text{and } \forall E \in \mathcal{E}_z \cap \mathcal{E}_D, \nabla_E u_D(z) = (a)_E\} \end{aligned}$$

of $\mathbb{R}^d$. Set $\mathcal{A}_z = \mathbb{R}^d$ for $z \in \mathcal{N} \cap \Omega$ and suppose $\mathcal{A}_z \neq \emptyset$ for all $z \in \mathcal{N}$. Finally, let $\pi_z : \mathbb{R}^d \to \mathbb{R}^d$ denote the orthogonal projection onto $\mathcal{A}_z$,

$$\mathcal{A}_z = \pi_z(0) + \mathcal{V}_z,$$

where $\mathcal{V}_z$ is a linear subspace of $\mathbb{R}^d$. The (nonlinear) orthogonal projection π_z is Lipschitz continuous with $\mathrm{Lip}(\pi_z) \leq 1$ and, for each $a \in \mathbb{R}^d$, $a - \pi_z(a) \perp \mathcal{V}_z$.

Remark 2.1. As an intersection of hyperplanes, $\mathcal{A}_z$ is an affine subspace of $\mathbb{R}^d$. The condition $\mathcal{A}_z \neq \emptyset$ is essentially a consistency condition on the boundary data: If $u \in C^1(\overline{\omega}_z)$ satisfies $u = u_D$ on $\Gamma_D = \bigcup \mathcal{E}_D$ and $\partial u / \partial \nu = g$ on $\overline{\Gamma}_N = \bigcup \mathcal{E}_N$, then $\nabla u(z) \in \mathcal{A}_z$.

Remark 2.2. The condition $(a)_E = \nabla_E u_D(z)$ in (2.1) is equivalent to

$$a \cdot \tau_E = \partial u_D(z) / \partial \tau_E \quad \text{for all vectors } \tau_E \in \mathbb{R}^d \text{ with } \tau_E \perp \nu_E.$$

This is a Dirichlet boundary condition $u = u_D$ on E in terms of $a = p(z) = \nabla u(z)$ at z.

Remark 2.3. In case $\mathcal{E}_D \cap \mathcal{E}_z = \emptyset$, the condition $p \in \mathcal{A}_z$ asserts Neumann boundary conditions at the node z with respect to all normals on neighbouring $\mathcal{E}_z \cap \mathcal{E}_N$. (Here, p is assumed to be a flux and not necessarily a gradient.)

Remark 2.4. The condition $p(z) \in \mathcal{A}_z$ with simultaneous Dirichlet and Neumann conditions, i.e. with $\mathcal{E}_z \cap \mathcal{E}_N \neq \emptyset \neq \mathcal{E}_z \cap \mathcal{E}_D$, is based on the interpretation of p as both a flux and a gradient. Hence, the model example is the Laplace equation with mixed boundary conditions. Nonconforming finite element methods require the case $\mathcal{E}_D \neq \emptyset$ [CB, CBJ].

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Remark 2.5. It is by no means obvious that averaging concerns the fluxes and the gradients simultaneously. The positive examples in [CBJ, CF3, CF4, BC2, CA] may be seen as exceptions. In general, the flux and the gradient approximations may be averaged separately. In the latter case we encounter $\mathcal{E}_N = \emptyset$ or $\mathcal{E}_D = \emptyset$.

2.3. **Discrete spaces.** On each element there exists a set of shape functions, namely, $P_{(k)}(T) := P_k(T)$ if T is triangular and $P_{(k)}(T) := Q_k(T)$ if T is rectangular; $P_k(T)$ and $Q_k(T)$ denote algebraic polynomials on $T \subseteq \mathbb{R}^d$ of total and partial degree $\leq k$, respectively. Furthermore, for each $T \in \mathcal{T}$ let $P(T)$ satisfy $P_{(0)}(T) \subset P(T) \subset P_{(1)}(T)$. Then, set

$$\begin{aligned}
\mathcal{L}^k(\mathcal{T}) &:= \{v_h \in L^\infty(\Omega) : \forall T \in \mathcal{T},\, v_h|_T \in P_{(k)}(T)\} \text{ for } k = 0, 1,\\
\mathcal{S}^1(\mathcal{T}) &:= \mathcal{L}^1(\mathcal{T}) \cap C(\Omega) = \operatorname{span}\{\varphi_z : z \in \mathcal{N}\},\\
P_h &:= P(\mathcal{T}) := \{p_h \in L^\infty(\Omega)^d : \forall T \in \mathcal{T},\, p_h|_T \in P(T)\} \subseteq \mathcal{L}^1(\mathcal{T})^d,\\
\mathcal{Q}_h &:= \{q_h \in \mathcal{S}^1(\mathcal{T})^d : \forall z \in \mathcal{N} \cap \Gamma,\, q_h(z) \in \mathcal{A}_z\}.
\end{aligned}$$

The nodal basis functions $(\varphi_z : z \in \mathcal{N})$ are defined by $\varphi_z \in \mathcal{S}^1(\mathcal{T})$ with $\varphi_z(z) = 1$ and $\varphi_z(x) = 0$ for all $z, x \in \mathcal{N}$ with $x \neq z$. Without further explicit notice, we shall make frequent use of

$$0 \leq \varphi_z \leq 1, \quad \operatorname{supp} \varphi_z = \overline{\omega}_z, \quad \text{and} \quad \textstyle\sum_{z \in \mathcal{N}} \varphi_z = 1.$$

2.4. **Averaging operators.** Given $p_h \in P_h$ (not necessarily globally continuous), the operator $A : P_h \to \mathcal{Q}_h$ is supposed to average p_h on each patch ω_z and adopt to boundary conditions. Therefore,

$$Ap_h := \sum_{z \in \mathcal{N}} A_z(p_h|_{\omega_z})\, \varphi_z \quad \text{and} \quad A_z := \pi_z \circ M_z : \mathcal{L}^1(\mathcal{T}_z)^d \to \mathbb{R}^d.$$

Recall that $\mathcal{L}^1(\mathcal{T}_z)$ denotes the $\mathcal{T}_z$ piecewise polynomials of degree ≤ 1 and that $p_h|_{\omega_z}$ belongs to $\mathcal{L}^1(\mathcal{T}_z)$. The local operator A_z is the composition of an averaging process $M_z : \mathcal{L}^1(\mathcal{T}_z)^d \to \mathbb{R}^d$ and the orthogonal projection $\pi_z : \mathbb{R}^d \to \mathbb{R}^d$ onto the affine subspace $\mathcal{A}_z \subset \mathbb{R}^d$.

The operator M_z is supposed to be linear and exact on continuous functions in $\mathcal{P}(\mathcal{T}_z) := \{p_h \in L^\infty(\omega_z)^d : \forall T \in \mathcal{T}_z,\, p_h|_T \in P(T)\}$; i.e.

$$M_z(f) = f(z) \quad \text{for all } f \in \mathcal{P}(\mathcal{T}_z) \cap C(\omega_z)^d \text{ and } z \in \mathcal{N}. \tag{2.2}$$

The master example for M_z reads

$$M_z(f) := \sum_{T \in \mathcal{T}_z} \lambda_{z,T}(f|_T)(z) \quad \text{for all } f \in \mathcal{P}(\mathcal{T}_z),\, z \in \mathcal{N}. \tag{2.3}$$

A necessary condition for (2.2) on the real coefficients $(\lambda_{z,T} : T \in \mathcal{T}_z)$ in (2.3) reads

$$\textstyle\sum_{T \in \mathcal{T}_z} \lambda_{z,T} = 1.$$

For a practical realization of A_z and for numerical examples we refer to [CB, CF3, CF4].

2.5. **Estimators.** Given the spaces P_h and $\mathcal{Q}_h$ of subsection 2.3 and the averaging operator $A : P_h \to \mathcal{Q}_h$ of subsection 2.4, we define, for any fixed $p_h \in P_h$, the averaging estimators

$$\eta_M := \min_{r_h \in \mathcal{Q}_h} \|p_h - r_h\|_{L^2(\Omega)} \leq \eta_A := \|p_h - Ap_h\|_{L^2(\Omega)}.$$

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3. Preliminaries

This section establishes some tools in an abstract frame to clarify the arguments below. Attention is on the arising constants: In contrast to earlier work based on a compactness arguments which led to unknown constants, we aim to quantify C_{eff}.

3.1. **Ascoli's lemma.** Given a linear and bounded map $L : H \to \mathbb{R}^n$ in a (real) Hilbert space H with norm $\|\cdot\|$, there holds, for $f \in H$,

$$|L(f)| \le \|L\| \operatorname{dist}(f; \ker L). \tag{3.1}$$

Here, $\operatorname{dist}(f; \ker L) := \min\{\|f-g\| : g \in \ker L\}$ is the distance to the (closed) kernel $\ker L$ of L and

$$\|L\| := \sup_{g\in X\setminus\{0\}} |L(g)|/\|g\| \tag{3.2}$$

is the operator norm of L; $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. The proof of (3.1) is by definition of $\|L\|$,

$$|L(f)| = |L(f-g)| \le \|L\|\,\|f-g\| \quad \text{for all } g \in \ker L.$$

In case $n = 1$, i.e. $L \in H^*$, there even holds equality in (3.1), which is known as Ascoli's lemma. A simple proof for the converse inequality of (3.1) follows for $g \in H$ with $\|g\| = 1$, $L(g) = \|L\|$ and so with $f - g\,L(f)/\|L\| \in \ker L$ from

$$\operatorname{dist}(f; \ker L) \le \|f - \Big(f - g\,L(f)/\|L\|\Big)\| = |L(f)|/\|L\|.$$

Suppose $n \ge 1$ again, let e_j be the jth canonical unit vector in $\mathbb{R}^n$, and set $L_j := e_j \cdot L$. Then there holds

$$|L_j(f)| = \|L_j\| \operatorname{dist}(f; \ker L_j).$$

The sum over all $j = 1, \dots, n$ squared components shows

$$|L(f)|^2 = \textstyle\sum_{j=1}^n \|L_j\|^2 \operatorname{dist}(f; \ker L_j)^2 \quad \text{for all } f \in H. \tag{3.3}$$

Compared with (3.1), the operator norm $\|L_j\|$ in (3.3) is smaller than $\|L\|$ while the kernel of L_j is larger than $\ker L \subseteq \ker(L_j)$.

3.2. **Strengthened Cauchy inequality.** Let H be a (real) Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let V and W be closed subspaces of H. Owing to the Cauchy inequality, the constant $\gamma_{V,W}$,

$$\gamma_{V,W} := \sup_{v\in V\setminus\{0\}} \sup_{w\in W\setminus\{0\}} \langle v, w\rangle / (\|v\|\,\|w\|) \le 1, \tag{3.4}$$

defines the angle $\angle(v, w)$ between v and w by $0 \le \cos(\angle(v, w)) = \gamma_{V,W} \le 1$. The spaces V and W satisfy a strengthened Cauchy inequality if $\gamma_{V,W} < 1$, that is, if $\angle(V, W)$ is positive.

Lemma 3.1 ([B])**.** For a constant c with $0 < c < 1$ and $\gamma_{V,W}$ from (3.4), the following assertions (a), (b), and (c) are pairwise equivalent.

(a) $\gamma_{V,W} \le c$;
(b) $\forall v \in V$, $\sqrt{1-c^2}\|v\| \le \operatorname{dist}(v; W)$;
(c) $\forall v \in V\,\forall w \in W$, $\sqrt{(1-c^2)/2}(\|v\| + \|w\|) \le \|v + w\|$. □

We are particularly interested in (a)⇔(b) also considered in [AO].

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Lemma 3.2. Let X and Y be closed linear subspaces of a Hilbert space H with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Set

$$V := \{x \in X : \forall a \in X \cap Y, \langle x, a\rangle = 0\} = X \cap (X \cap Y)^{\perp} \tag{3.5}$$

and suppose that V and Y are nontrivial and that V has positive finite dimension. Set

$$\gamma_{V,Y} := \sup_{v\in V\setminus\{0\}} \sup_{y\in Y\setminus\{0\}} \langle v, y\rangle /(\|v\|\,\|y\|). \tag{3.6}$$

Then $0 \le \gamma_{V,Y} < 1$ and $\gamma_{V,Y} = \langle v, y\rangle$ for some $v \in V$ and $y \in Y$ with $\|v\| = 1 = \|y\|$. Moreover,

$$\operatorname{dist}(x; X \cap Y) \le (1 - \gamma_{V,Y}^2)^{-1/2} \operatorname{dist}(x; Y) \quad \text{for all } x \in X \tag{3.7}$$

and the factor $(1-\gamma_{V,Y}^2)^{-1/2}$ is optimal in the sense that (3.7) fails to hold for any smaller constant.

Proof. Owing to the definition in (3.6) there exist sequences (x_j) and (y_j) in V and Y, respectively, with $\|x_j\| = 1 = \|y_j\|$ and

$$\lim_{j\to\infty} \langle x_j, y_j\rangle = \gamma_{V,Y}.$$

Since (x_j) and (y_j) are bounded in a Hilbert space, there exists a subsequence (not relabeled) with $(x_j) \to x$ and $(y_j) \rightharpoonup y$ in H. The strong convergence of (x_j) follows from the finite dimension of V. Hence, $\|x\| = 1 \ge \|y\|$ and $\lim_{j\to\infty} \langle x_j, y_j\rangle = \langle x, y\rangle$. If $y \neq 0$, we have

$$\gamma_{V,Y} = \langle x, y\rangle \le \langle x, y\rangle /(\|x\|\,\|y\|) \le \gamma_{V,Y}.$$

(The last inequality follows from (3.6) and $x \in V$, $y \in Y$.) Hence we have $\gamma_{V,Y} = \langle x/\|x\|, y/\|y\|\rangle$ for some $x/\|x\| \in V$ and $y/\|y\| \in Y$ with norm 1 if $0 < \gamma_{V,Y} < \infty$.

If $y = 0$, $\gamma_{V,Y} = 0$ and each $v \in V$ is perpendicular to Y.

In both cases, $\gamma_{V,Y} = \langle v, y\rangle$ for some $v \in V$ and $y \in Y$ with $\|v\| = \|y\| = 1$. This proves the attainment result.

A Cauchy inequality shows $\gamma_{V,Y} \le 1$. If $\gamma_{V,Y} = 1 = \langle v, y\rangle$ for $v \in V$ and $y \in Y$ with $\|v\| = 1 = \|y\|$, we have equality in the Cauchy inequality and hence $v = y$. Thus, $v \in V \cap Y \subseteq X \cap Y$ and so $\|v\|^2 = 0$ owing to (3.5). This contradicts $\|v\| = 1$ and proves $\gamma_{V,Y} \neq 1$.

It remains to apply Lemma 3.1 for V and $W := Y$. Then $\gamma_{V,Y}$ in (3.4) and (3.6) coincide and the equivalence (a)$\Leftrightarrow$(b) of Lemma 3.1 proves, first,

$$\|v\| \le (1 - \gamma_{V,Y}^2)^{-1/2} \operatorname{dist}(v; Y) \quad \text{for all } v \in V \tag{3.8}$$

and, second, that the constant factor $(1-\gamma_{V,Y}^2)^{-1/2}$ in (3.8) cannot be smaller.

Given $x \in X$ and the closed subspace $X \cap Y$ of X, there exists an orthogonal decomposition

$$x = v + w \quad \text{with } v \in V \text{ and } w \in X \cap Y.$$

Moreover, $\operatorname{dist}(x; X \cap Y) = \|v\|$ and $\operatorname{dist}(v; Y) = \operatorname{dist}(v + w; w + Y) = \operatorname{dist}(x; Y)$. This and (3.8) conclude the proof. $\square$

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TABLE 1. Mass matrices $M(T)$ (scaled with $|T|^{-1}$) for some elements T and their eigenvalues $\lambda_1 \ldots, \lambda_m$ and $\lambda(T)$ of Lemma 3.3

pT	$M(T)$	Eigenvalues	$\lambda(T)$
interval	$1/6 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	1, 3	1
triangle	$1/12 \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$	1/12, 1/12, 1/4	12
parallelogram	$1/36 \begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$	1/36, 1/12, 1/12, 1/4	36
tetrahedron	$1/20 \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$	1/20, 1/20, 1/20, 1/5	20

3.3. **Eigenvalues of mass matrices.** This subsection summarises a few inequalities and the constants therein. For each element $T \in \mathcal{T}$ with volume $|T| = \mathcal{L}^d(T)$ we associate $m := \operatorname{card}(\mathcal{N} \cap T)$ nodal basis functions $\varphi_1, \ldots, \varphi_m$ called shape functions with

$$\int_T \varphi_j \, dx = |T|/m \quad \text{for } j = 1, \ldots, m \tag{3.9}$$

(as $\sum_{j=1}^m \varphi_j = 1$ and the forms of $\varphi_1, \ldots, \varphi_m$ are identical). Scaled with $|T|^{-1}$, the mass matrix reads

$$M(T) := \Big(\int_T \varphi_j \varphi_k \, dx/|T| : \, j, k = 1, \ldots, m\Big). \tag{3.10}$$

Table 1 displays some mass matrices and their eigenvalues $\lambda_1, \ldots, \lambda_m$.

Lemma 3.3. Suppose $T \in \mathcal{T}$ and $f \in P_{(1)}(T)^d$. Then

$$|T| \sum_{z \in \mathcal{N} \cap T} |f(z)|^2 \leq \lambda(T) \|f\|^2_{L^2(T)} \tag{3.11}$$

where $\lambda(T) = 1/\lambda_1$ for the minimal eigenvalue λ_1 of the matrix (3.10) displayed in Table 1.

Proof. Let $f_j := e_j \cdot f$ be the jth component of f and let $\{z_1, \ldots, z_m\} = \mathcal{N} \cap T$ denote the vertices of T. With the m components $\xi_k := f_j(z_k)$ of $\xi \in \mathbb{R}^m$ and a standard estimation of the Rayleigh quotient there holds

$$\lambda_1 \sum_{z \in \mathcal{N} \cap T} f_j(z)^2 = \lambda_1 |\xi|^2 \leq \xi \cdot M(T) \xi = |T|^{-1} \|f_j\|^2_{L^2(T)}.$$

The sum over all components $j = 1, \ldots, d$ verifies assertion (3.11). $\square$

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4. Equivalence of η_M and η_A

This section is devoted to the proof of the equivalence of η_M and η_A under the present assumptions. A discussion of the constant C_{eff} follows in Section 5. Theorem 4.1 covers efficiency (1.5) for conforming, nonconforming, and mixed finite element methods [CB].

Theorem 4.1. *There exists a mesh-size independent positive constant C_{eff} with*

$$\eta_M \le \eta_A \le C_{\text{eff}}\,\eta_M.$$

Proof. The first inequality is obvious and the proof concerns the second. Throughout the first step and main part of the proof let T denote a fixed element. Set

$$p_h|_T = \sum_{z\in\mathcal{N}\cap T} p_z\varphi_z|_T \quad \text{and} \quad q_h := Ap_h = \sum_{z\in\mathcal{N}} q_z\varphi_z \text{ for } q_z := A_z(p_h|_{\omega_z}).$$

(Notice that the representation of p_h is local on the fixed element T; p_h may be discontinuous on Ω and so has different coefficients on different elements.) A Cauchy inequality in $\mathbb{R}^m$, $m = \operatorname{card}(\mathcal{N}\cap T)$, shows, pointwise on T,

$$\begin{aligned} |p_h - q_h|^2 &= |\sum_{z\in\mathcal{N}\cap T} \varphi_z(p_z - q_z)|^2 \\ &\le (\sum_{z\in\mathcal{N}\cap T}\varphi_z)(\sum_{z\in\mathcal{N}\cap T}\varphi_z|p_z-q_z|^2) \\ &= \sum_{z\in\mathcal{N}\cap T}\varphi_z|p_z-q_z|^2. \end{aligned} \tag{4.1}$$

Since $q_z = \pi_z(m_z)$ for $m_z := M_z(p_h|_{\omega_z})$ and $p_z - \pi_z(p_z)\perp\mathcal{V}_z$ in $\mathbb{R}^d$, there holds

$$|p_z - q_z|^2 = |p_z - \pi_z(p_z)|^2 + |\pi_z(p_z) - \pi_z(m_z)|^2.$$

With any $r_z \in \mathcal{A}_z = \pi_z(0) + \mathcal{V}_z$ and $\operatorname{Lip}(\pi_z) \le 1$, this yields

$$|p_z - q_z|^2 \le |p_z - r_z|^2 + |p_z - m_z|^2. \tag{4.2}$$

The combination of (4.1)-(4.2) is integrated over the fixed T and shows

$$\|p_h - q_h\|^2_{L^2(T)} \le \sum_{z\in\mathcal{N}\cap T}|p_z - r_z|^2\int_T\varphi_z\,dx + \sum_{z\in\mathcal{N}\cap T}|p_z-m_z|^2\int_T\varphi_z\,dx.$$

With (3.9) and Lemma 3.3 this gives, for $r_h := \sum_{z\in\mathcal{N}} r_z\varphi_z \in \mathcal{Q}_h$,

$$\|p_h - q_h\|^2_{L^2(T)} \le \frac{\lambda(T)}{m}\|p_h - r_h\|^2_{L^2(T)} + \frac{|T|}{m}\sum_{z\in\mathcal{N}\cap T}|p_z - m_z|^2. \tag{4.3}$$

The second step focuses on the estimation of $p_z - m_z$ and introduces the finite-dimensional Hilbert space $X := P(\mathcal{T}_z) \subseteq \mathcal{L}^1(\mathcal{T}_z)^d$ with the inner product $\langle\cdot,\cdot\rangle$,

$$\langle f, g\rangle := \int_{\omega_z}\varphi_z\, f\cdot g\,dx/|T| \quad \text{for } f,g\in L^2(\omega_z)^d =: H. \tag{4.4}$$

Define $\delta_{T,z}(f) := (f|_T)(z)$ for all $f \in X$ and consider

$$L_{T,z} := \delta_{T,z} - M_z : X \to \mathbb{R}^d \quad \text{linear} \tag{4.5}$$

and continuous with the bound

$$\|L_{T,z}\| := \sup_{f\in P(\mathcal{T}_z)\setminus\{0\}}\Big|f|_T(z) - M_z(f)\Big|/(|T|^{-1/2}\|\varphi_z^{1/2}f\|_{L^2(\omega_z)}). \tag{4.6}$$

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A scaling argument shows that $\|L_{T,z}\|$ does not depend on the diameter of ω_z because of the factor $|T|^{-1/2}$. (Details on the constant $\|L_{T,z}\|$ from (4.6) follow for specific examples after the proof.) Since M_z is exact on $\mathcal{P}(\mathcal{T}_z) \cap C(\omega_z)^d$,

$$\mathbb{R}^d \subseteq \mathcal{P}(\mathcal{T}_z) \cap C(\omega_z)^d \subseteq \ker L =: Z \subseteq X \quad \text{and} \quad Y := \mathcal{S}^1(\mathcal{T}_z)^d.$$

Ascoli's lemma (formula (3.1)) shows

$$|p_z - m_z| = |L_{T,z}(p_h)| \leq \|L_{T,z}\| \operatorname{dist}(p_h|_{\omega_z}; Z). \tag{4.7}$$

Lemma 3.2 and $X \cap Y \subseteq Z$ prove $0 \leq \gamma < 1$ for the constant γ of (3.6) and

$$\operatorname{dist}(p_h|_{\omega_z}; Z) \leq \operatorname{dist}(p_h|_{\omega_z}; X \cap Y) \leq (1-\gamma^2)^{-1/2} \operatorname{dist}(p_h|_{\omega_z}; Y). \tag{4.8}$$

(The constant (3.6) will be discussed at the end of this section for specific examples.)

Step three combines (4.3) and (4.7)-(4.8) with

$$\operatorname{dist}(p_h|_{\omega_z}; \mathcal{S}^1(\mathcal{T}_z)^d) \leq |T|^{-1/2} \|\varphi_z^{1/2}(p_h - r_h)\|_{L^2(\omega_z)}$$

and (writing γ_z for γ) results in

$$\|p_h - q_h\|^2_{L^2(T)} \leq \lambda(T)/m \, \|p_h - r_h\|^2_{L^2(T)} + \sum_{z \in \mathcal{N} \cap T} \|L_{T,z}\|^2 \, (1-\gamma_z^2)^{-1}/m \, \|\varphi_z^{1/2}(p_h - r_h)\|^2_{L^2(\omega_z)}.$$

In step four, the sum over all elements $T \in \mathcal{T}$ and the fact

$$\sum_{z \in \mathcal{N}} \|\varphi_z^{1/2}(p_h - r_h)\|^2_{L^2(\omega_z)} = \|p_h - r_h\|^2_{L^2(\Omega)}$$

show the assertion

$$\eta_A = \|p_h - q_h\|_{L^2(\Omega)} \leq C_{\text{eff}} \, \|p_h - r_h\|_{L^2(\Omega)} \quad \text{for all } r_h \in \mathcal{Q}_h. \tag{4.9}$$

The constant C_{eff} depends on $m = m_T$, $\lambda(T)$, and $\|L_{T,z}\|^2/(1-\gamma_z^2)$ as

$$C_{\text{eff}}^2 = \max_{T \in \mathcal{T}} \big(\lambda(T) + \max_{z \in \mathcal{N} \cap T} \|L_{T,z}\|^2/(1-\gamma_z^2)\big)/m_T. \tag{4.10}$$

This concludes the proof of $\eta_A \leq C_{\text{eff}} \, \eta_M$. □

5. Example

The constant C_{eff} and its possible dependence on mesh will be studied for the P_1 FEM with piecewise constant discrete fluxes. Recall that

$$X := \mathcal{P}(\mathcal{T}_z) \subseteq \mathcal{L}^1(\mathcal{T}_z)^d \subset H := L^2(\omega_z)^d$$

and $Y = \mathcal{S}^1(\mathcal{T}_z)^d$ with the scalar product (4.4) on H.

The following lemma provides coarse but uniform estimates of eigenvalues which could be computed as a function of $\operatorname{card}(\mathcal{T}_z)$.

Lemma 5.1. Suppose that $\mathcal{P}(\mathcal{T}_z) = \mathcal{L}^0(\mathcal{T}_z)^d$ and that $\mathcal{T}_z$ consists of simplices in $\mathbb{R}^d$. Then the constant $\gamma = \gamma_z \geq 0$ from (3.5)-(3.6) satisfies

$$\gamma^2 \leq 5/6 \text{ for } d = 2 \quad \text{and} \quad \gamma^2 \leq 9/10 \text{ for } d = 3.$$

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Proof. Given any $v_h \in \mathcal{L}^0(\mathcal{T}_z)^d$ and $y_h \in \mathcal{S}^1(\mathcal{T}_z)^d$, set $v_T := v_h|_T \in \mathbb{R}^d$ and $y_T = \int_T \varphi_z\, y_h\, dx$ for $T \in \mathcal{T}_z$. Then, (3.6) reads

$$\gamma^2 = \max_{v_h, y_h} \left(\sum_{T\in\mathcal{T}_z} v_T \cdot y_T \right)^2 \Big/ \left(\left(\sum_{T\in\mathcal{T}_z} |T||v_T|^2/m \right) \left(\int_{\omega_z} \varphi_z |y_h|^2\, dx \right) \right)$$

where, by definition of V, $v_h \in V$ satisfies $\sum_{T\in\mathcal{T}_z} |T|\, v_T = 0$. Consequently, the sum

$$\textstyle\sum_{T\in\mathcal{T}_z} v_T \cdot y_T$$

does not depend on an additive constant in y_h which, therefore, is determined to minimise $\int_{\omega_z} \varphi_z\, |y_h|^2\, dx$. This results in the condition $\int_{\omega_z} \varphi_z\, y_h\, dx = 0$; i.e.

$$\textstyle\sum_{T\in\mathcal{T}_z} y_T = 0. \tag{5.1}$$

A Cauchy inequality yields

$$\gamma^2 = \max_{y_h} \left(m \sum_{T\in\mathcal{T}} |y_T|^2/|T| \right) \Big/ \int_{\omega_z} \varphi_z\, |y_h|^2\, dx \tag{5.2}$$

and equality is indeed attained for $v_T = y_T/|T|$ (compatible with $v_h \in V$ and (5.1)). Given y_h in $\mathcal{S}^1(\mathcal{T}_z)^2$ with (5.1) and nodal values $y_0 = y_h(z)$, $y_{a,T} = y_h(a)$, $y_{b,T} = y_h(b)$ on $T = \operatorname{conv}\{z, a, b\} \in \mathcal{T}_z$ for $d = 2$, a straightforward calculation shows

$$y_T = |T|(2y_0 + y_{a,T} + y_{b,T})/12 \quad \text{for } T \in \mathcal{T}_z.$$

This and (5.1) plus a Cauchy inequality yield

$$\begin{aligned} 12 \sum_{T\in\mathcal{T}_z} |y_T|^2/|T| &= \sum_{T\in\mathcal{T}_z} y_T \cdot (2y_0 + y_{a,T} + y_{b,T}) \\ &= \sum_{T\in\mathcal{T}_z} y_T \cdot (y_{a,T} + y_{b,T}) \\ &\le \left(\sum_{T\in\mathcal{T}_z} |y_T|^2/|T| \right)^{1/2} \left(\sum_{T\in\mathcal{T}_z} |T|\, |y_{a,T} + y_{b,T}|^2 \right)^{1/2} \end{aligned}$$

and so (divide by $\left(\sum_{T\in\mathcal{T}_z} |y_T|^2/|T|\right)^{1/2}$ and square) leads to

$$144 \sum_{T\in\mathcal{T}_z} |y_T|^2/|T| \le \sum_{T\in\mathcal{T}_z} |T||y_{a,T} + y_{b,T}|^2. \tag{5.3}$$

The summand on the right-hand side is

$$\begin{aligned} &|T|\, |y_{a,T} + y_{b,T}|^2 \\ &\qquad \le 2|T| \left(2|y_0|^2 + |y_{a,T}|^2 + |y_{b,T}|^2\right) \\ &\qquad = 120 \int_T \varphi_z\, |y_h|^2\, dx - 288|y_T|^2/|T|. \end{aligned} \tag{5.4}$$

The latter equality follows with lengthy but straightforward calculation with the well-known formula $\int_T \lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma\, dx = 2|T|\, \alpha!\beta!\gamma!/(2+\alpha+\beta+\gamma)!$ for the barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$ on the triangle T. The combination of (5.3)-(5.4) verifies

$$\sum_{T\in\mathcal{T}_z} |y_T|^2/|T| \le 5/18 \int_{\omega_z} \varphi_z\, |y_h|^2\, dx.$$

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Using this in (5.2) shows $\gamma^2 \leq 5/6$. The proof for $d = 3$ follows with the same arguments modified for $y_T = |T|\,(2y_0 + y_{a,T} + y_{b,T} + y_{c,T})/20$, etc.; the details are omitted. $\square$

To study $\|L_{z,T}\|$, let M_z be given by (2.3); i.e.

$$M_z(f) = \textstyle\sum_{T\in\mathcal{T}_z} \lambda_{z,T}\, f_T \quad \text{for } f|_T = f_T \in \mathbb{R}^d,\, T \in \mathcal{T}_z, \text{ and } f \in \mathcal{L}^0(\mathcal{T}_z)^d.$$

The real coefficients $\lambda_{z,T}$ sum up to $1 = \sum_{T\in\mathcal{T}_z} \lambda_{z,T}$ (some are possibly negative). For comparison, a popular choice for the coefficient $\lambda_{z,T}$ reads

$$\mu_{z,T} := |T|/|\omega_z| \quad \text{for } T \in \mathcal{T}_z. \tag{5.5}$$

Lemma 5.2. Suppose (4.4)-(4.6) for fixed $z \in T \cap \mathcal{N}$ and that $\mathcal{P}(\mathcal{T}_z) = \mathcal{L}^0(\mathcal{T}_z)^d$ and that $\mathcal{T}_z$ consists of simplices. Then $m = d + 1$ and

$$\|L_{z,T}\|^2 = m\left((1-\lambda_{z,T})^2 + \sum_{K\in\mathcal{T}_z\setminus\{T\}} \lambda_{z,K}^2 \mu_{z,T}/\mu_{z,K}\right).$$

Proof. Given any $f \in \mathcal{L}^0(\mathcal{T}_z)^d$ (write $f_K := f_K$ for each $K \in \mathcal{T}$),

$$L_{z,T}(f) = (1-\lambda_{z,T})f_T - \textstyle\sum_{K\in\mathcal{T}_z\setminus\{T\}} \lambda_{z,K} f_K$$

is independent of a global additive constant in f. To minimise $\|\varphi_z^{1/2} f\|$, this constant is such that $\int_{\omega_z} \varphi_z\, f\, dx = 0$. Hence

$$\textstyle\sum_{K\in\mathcal{T}_z} |K|\, f_K = 0 \tag{5.6}$$

and (with an argument f in $\mathcal{L}^0(\mathcal{T}_z)^d\setminus\{0\}$ with (5.6) in the supremum)

$$\|L_{z,T}\|^2 = \sup_f |T|\, |L_{z,T}(f)|^2 / \left(\textstyle\sum_{M\in\mathcal{T}_z} |f_M|^2\, |M|/m\right).$$

A Cauchy inequality shows

$$|L_{z,T}(f)| \leq \left(\textstyle\sum_{M\in\mathcal{T}_z} |f_M|^2 |M|\right)^{1/2} \left((1-\lambda_{z,T})^2/|T| + \textstyle\sum_{K\in\mathcal{T}_z\setminus\{T\}} \lambda_{z,K}^2/|K|\right)^{1/2}.$$

Equality holds for $f_T = (1-\lambda_{z,T})/|T|\, e$ and $f_K = -\lambda_K/|K|\, e$ for any other $K \in \mathcal{T}_z\setminus\{T\}$ and some fixed unit vector $e \in \mathbb{R}^d$. Since this choice of f satisfies (5.6), there holds

$$\|L_{z,M}\|^2 = m|T|\left((1-\lambda_{z,T})^2/|T| + \textstyle\sum_{K\in\mathcal{T}_z\setminus\{T\}} \lambda_{z,K}^2/|K|\right). \quad \square$$

The following consequence gives an estimate for the choice (5.5) and indicates that this choice is optimal.

Corollary 5.3. Under the assumptions of the preceding two lemmas (satisfied for all $z \in \mathcal{N}$) and for $\mu_{z,T} = \lambda_{z,T}$ there holds

$$\eta_M \leq \eta_A \leq \sqrt{10}\, \eta_M \text{ for } d = 2 \quad \text{and} \quad \eta_M \leq \eta_A \leq \sqrt{15}\, \eta_M \text{ for } d = 3.$$

Proof. The estimates follow from Theorem 4.1 and Lemmas 5.1 and 5.2 with $\|L_{z,T}\|^2 = m(1-\mu_{z,T}) \leq m$ and $\lambda(T) = 12$, 20 for $d = 2, 3$. $\square$

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