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An a priori error estimate for finite element discretizations in nonlinear elasticity for polyconvex materials under small loads

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Summary. We prove optimal a priori error estimates in $W^{1,p}$ for finite element minimizers of polyconvex energy functionals with small applied loads. The proof relies on a quantitative version of Zhang's stability estimate (K. Zhang, Arch. Rat. Mech. Anal. 114 (1991), 95-117).

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1 Introduction

Compared to the large number of contributions on computational finite elasticity in the engineering literature, very little is based on a rigorous mathematical foundation. We refer to the recent survey [T] in the handbook of numerical analysis and emphasize that it does not include any error *estimates*. The mathematical existence theory is more developed and offers a local approach via the implicit function theorem and a global approach via minimization of a polyconvex energy functional in the sense of Ball [C2,V,B2].

The numerical analysis in the context of the implicit function theorem then involves a linearisation about a smooth solution but does not address

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the question of whether the local finite element approximation is a global or local minimizer of a discrete energy. This difficulty is recast into a path-following ansatz where, e.g., the load serves as a control parameter (see, e.g., [C2, D]). In this paper we focus on a discrete global minimization problem and employ energy estimates. For several mathematical difficulties discussed in Section 5 we succeeded only for small loads where the two approaches coincide as shown by Zhang [Z].

2 Variational methods and implicit function theorems in finite elasticity

The variational approach to the existence of equilibrium solutions in nonlinear elasticity is based on the minimization of an energy functional of the form

$$I(u) = \int_\Omega \big(W(Du) - f \cdot u\big)\,\mathrm{d}x \tag{2.1}$$

in a suitable class $\mathcal{A}$ of admissible deformations. For conceptual reasons, one cannot assume local conditions such as convexity for W, see [B1]. However, for a large class of constitutive relations, including the Ogden materials, the free energy density W is polyconvex and the existence of minimizers follows from Ball's seminal paper [B2]. Throughout the paper we assume that the subsequent hypotheses are satisfied.

(H1) Ω is a bounded C^3-domain in $\mathbb{R}^2$;

(H2) $G \in C^3(\mathbb{M}^{2\times 2} \times (0,\infty); \mathbb{R})$ is convex and satisfies the growth condition

$$G(F,\delta) \ge \alpha + \beta |F|^p \qquad \forall F \in \mathbb{M}^{2\times 2},\ \delta > 0,$$

with $p > 2$, $\alpha \in \mathbb{R}$, and $\beta > 0$;

(H3) $W : \mathbb{M}^{2\times 2}_+ \to \mathbb{R}$ is defined by

$$W(F) = \gamma |F|^p + G(F, \det F)$$

with $\gamma > 0$; here $\mathbb{M}^{2\times 2}_+ = \{F \in \mathbb{M}^{2\times 2} :\ \det F > 0\}$.

(H4) $g \in C^3(\Omega; \mathbb{R}^2)$, $f \in L^r(\Omega; \mathbb{R}^2)$ with $r > 2$, and the class $\mathcal{A}$ of admissible functions, defined by

$$\mathcal{A} = \{u \in W^{1,p}(\Omega; \mathbb{R}^2) : u = id + g \text{ on } \partial\Omega,\ \det Du > 0 \text{ a.e. in } \Omega\}$$

is not empty.

(H5) we have for all sequences $\{F_j\}_{j\in\mathbb{N}}$ in $\mathbb{M}^{2\times 2}$ and $\{\delta_j\}_{j\in\mathbb{N}}$ in $(0,\infty)$ with $\lim_{j\to\infty} F_j = F$ and $\lim_{j\to\infty} \delta_j = 0$ that

$$\lim_{j\to\infty} G(F_j, \delta_j) = +\infty.$$

The existence of a minimizer for (2.1) follows from the general theory for polyconvex functionals even for $\gamma = 0$ and under weaker assumptions on the smoothness of W. In our situation, Zhang's results [Z] imply uniqueness of the solution if the loads are sufficiently small.

Theorem 2.1 ([B2]) *Suppose (H1)-(H5) and that the total energy*

$$I(u) = \int_\Omega \big(W(Du) - f \cdot u\big)\, \mathrm{d}x$$

is finite for at least one $u \in \mathcal{A}$. Then there exists a minimizer of I in $\mathcal{A}$. □

A different technique for proving the existence of an equilibrium solution in finite elasticity is based on the implicit function theorem (see, e.g., [C2,V, Z]). We use the following result.

Theorem 2.2 *Suppose (H1)-(H5). Then there exists an $\varepsilon_0 > 0$ and a function $\eta : [\,0, \varepsilon_0\,] \to [\,0, \eta_0\,]$ such that the following holds: For all $\varepsilon \in (0, \varepsilon_0)$, for all $g \in C^3(\overline{\Omega}; \mathbb{R}^2)$, and for all $f \in L^r(\Omega; \mathbb{R}^2)$ with*

$$\|g\|_{3,\infty} < \varepsilon \tag{2.2}$$

and

$$\|f + \operatorname{div} \frac{\partial W}{\partial F}(I + Dg)\|_r < \varepsilon, \tag{2.3}$$

there exists a unique weak solution $u = U(f, g) \in W^{2,r}(\Omega; \mathbb{R}^2)$ of the equilibrium equations

$$\operatorname{div} \frac{\partial W}{\partial F}(Du) + f = 0 \tag{2.4}$$

that satisfies the boundary conditions

$$u = id + g \text{ on } \partial\Omega \tag{2.5}$$

and the estimate

$$\|u - id\|_{2,r} < \eta(\varepsilon). \tag{2.6}$$

Moreover, η is continuous at zero with $\eta(0) = 0$.

Proof. This result is essentially a two-dimensional version of Theorem 2.10 in [Z] in the three-dimensional setting. The continuity of η at $\varepsilon = 0$ is central for our arguments below. We include a sketch of the proof of the theorem using the notation in [V] (which is also used in [Z]). The idea is to use the implicit function theorem in suitable function spaces. Since polyconvexity implies quasiconvexity and hence global stability of affine mappings, $u = id$

is a solution of (2.4)-(2.5) with $f = 0$ and $g = 0$. This suggests to consider the mapping

$$\begin{aligned} F : \big(W_0^{1,r}(\Omega; \mathbb{R}^2) \cap W^{2,r}(\Omega; \mathbb{R}^2)\big) \times L^r(\Omega; \mathbb{R}^2) \\ \times C^3(\overline{\Omega}; \mathbb{R}^2) \to L^r(\Omega; \mathbb{R}^2), \\ (v, f, g) \mapsto -f - \operatorname{div} \frac{\partial W}{\partial F}(I + Dv + Dg) \end{aligned}$$

in a neighbourhood of $(0, 0, 0)$. An orientation-preserving C^1 diffeomorphism u of Ω onto Ω is called an admissible deformation if $u = id$ on $\partial\Omega$. We denote by $\mathcal{A}_{m+2,r}$ the set of those admissible deformations that lie in $W^{m+2,r}(\Omega; \mathbb{R}^2)$. Finally $\mathcal{S}_{m+2,r}$ is the subset of all admissible deformations in $\mathcal{A}_{m+2,r}$ for which

$$\int_\Omega \frac{\partial^2 W}{\partial F_{ij} \partial F_{k\ell}}(Du) \frac{\partial \psi^i}{\partial x_j} \frac{\partial \psi^k}{\partial x_\ell} \,\mathrm{d}x \geq c(u) \|\psi\|_{1,2}^2 \qquad \text{for all } \psi \in \mathcal{D}(\Omega; \mathbb{R}^2),$$

where the constant $c(u) > 0$ does not depend on ψ. The arguments on page 101 in [Z] show that $id \in \mathcal{S}_{2,r}$. We can then follow the discussion on page 95 in [V] to see that the differential of F with respect to v at $f = 0$ and $g = 0$ is a linear bijection of $W_0^{1,r}(\Omega; \mathbb{R}^2) \cap W^{2,r}(\Omega; \mathbb{R}^2)$ onto $L^r(\Omega; \mathbb{R}^2)$. Then the implicit function theorem implies that there exists a local representation $V(f, g)$ of the zeros of F, i.e. $F(V(f, g), f, g) = 0$ for f and g close to zero in $L^r(\Omega; \mathbb{R}^2)$ and $C^3(\overline{\Omega}; \mathbb{R}^2)$. Moreover, the function $V(f, g)$ satisfies $V(0, 0) = 0$ and depends continuously on f and g in the corresponding norms. Finally the function $U = id + V$ has the properties stated in the theorem. □

Under the same hypotheses, Zhang [Z] proved that any minimizer of I in Theorem 2.1 solves (2.4)-(2.6) if the assumptions (2.2)-(2.3) hold.

Theorem 2.3 ([Z]) *Suppose (H1)-(H5). Then there exists an $\varepsilon_1 \in (0, \varepsilon_0)$ such that for all $\varepsilon \in (0, \varepsilon_1)$ the following holds: If $g \in C^3(\overline{\Omega}; \mathbb{R}^2)$ and $f \in L^r(\Omega; \mathbb{R}^2)$ satisfy (2.2) and (2.3), respectively, then any minimizer u of I in $\mathcal{A}$ belongs to $W^{2,r}(\Omega; \mathbb{R}^2)$ and satisfies (2.4)–(2.6).* □

Remark. In this paper, we restrict our attention to the two-dimensional situation for notational simplicity. The extension to the three-dimensional setting follows along the same lines, but the algebra is more involved. Theorem 2.3 is the two-dimensional version of Theorem 3.4 in [Z]. We sketch the proof following Zhang's arguments in the appendix. Incompressible materials can be treated by the same techniques, see [B2,C2,V,Z].

Notation. We denote with $\|\cdot\|_p$ and $\|\cdot\|_{k,p}$ the norms in $L^p(\Omega)$ and $W^{k,p}(\Omega)$, respectively. The $W^{k,p}(\Omega)$-seminorm is defined by $|u|_{k,p} = \|D^k u\|_p$. If $A \in \mathbb{M}^{k\times k}$, then $|A|$ is the Frobenius norm of A, i.e., the Euclidean norm in $\mathbb{R}^{k^2}$, induced by the scalar product $A : B = \sum_{i,j=1}^{k} A_{ij} B_{ij}$. For $k = 1, 2$ we denote the matrix of all partial derivatives of G of order k with $D^k G$ and we let

$$c_k = \max\left\{|D^k G(F, \delta)| : F \in \mathbb{M}^{2\times 2}, |I - F| \le \frac{1}{4}, |\delta - 1| \le \frac{1}{32} + \frac{\sqrt{2}}{4}\right\}.$$

3 Stability of global minimizers

In this section we modify Zhang's arguments to prove a stability estimate for the global minimizer $u \in \mathcal{A}$ and an approximation $w \in \mathcal{A} \cap W^{1,\infty}(\Omega; \mathbb{R}^2)$ with $\|u - id\|_{1,\infty}$ and $\|w - id\|_{1,\infty}$ small enough. It is important to note that Theorem 2.3 implies regularity for the minimizer $u \in \mathcal{A}$ of the variational problem. In particular, $u \in W^{2,r}(\Omega; \mathbb{R}^2) \hookrightarrow W^{1,\infty}(\Omega; \mathbb{R}^2)$. More precisely, we prove that for *any* $v \in \mathcal{A}$ with $I(u) \le I(v) \le I(w)$ the estimate

$$\|D(u - v)\|_2 \le c_3 \|D(u - w)\|_2$$

holds. As an application, we prove error estimates for finite element approximations in Section 4.

Theorem 3.1 *Suppose (H1)-(H5). Then there exist $\varepsilon > 0$, $\delta_0 > 0$ small enough, and $c_3 > 0$ such that for all $g \in C^3(\overline{\Omega}; \mathbb{R}^2)$ and for all $f \in L^r(\Omega; \mathbb{R}^2)$ satisfying (2.2)–(2.3) the following implication holds: If $u \in \mathcal{A}$ is the minimizer of I,*

$$I(u) = \min\{I(\tilde{u}) : \tilde{u} \in \mathcal{A}\},$$

with $|u - id|_{1,\infty} < \delta_0$, and if $v \in \mathcal{A}$, and $w \in \mathcal{A} \cap W^{1,\infty}(\Omega; \mathbb{R}^2)$ are such that $|w - id|_{1,\infty} < \delta_0$ and

$$I(u) \le I(v) \le I(w)$$

then

$$\|D(u - v)\|_p^p + \|D(u - v)\|_2^2 \le c_3 \|D(u - w)\|_2^2.$$

Remark. The point is that we do *not* assume in Theorem 3.1 that $|v - id|_{1,\infty} < \delta_0$.

The proof of the theorem requires two elementary estimates which we state in the following two lemmas. The first is a consequence of Lemma 8.2 in [E]. We include the proof of the second for the convenience of the reader.

Lemma 3.2 ([E]) *For $2 \le p < \infty$ there exists a constant $c_4 > 0$ which only depends on k and p such that for all $A, B \in \mathbb{R}^k$ the following estimate holds:*

$$c_4\left(|A|^{p-2}|B|^2 + |B|^p\right) \le |A + B|^p - |A|^p - p|A|^{p-2} A \cdot B. \qquad \square$$

Lemma 3.3 *For $2 \le p < \infty$ and $A, B \in \mathbb{R}^k$ we have*

$$\big||A|^{p-2}A - |B|^{p-2}B\big| \le (p-1)\,|B-A|\,\max\big\{|A|^{p-2},\ |B|^{p-2}\big\}.$$

Proof. For $\lambda \in [\,0,1\,]$ we define $F_\lambda = A + \lambda(B-A)$. Then

$$\begin{aligned}
&\Big|\frac{\partial}{\partial\lambda}\big(|F_\lambda|^{p-2}F_\lambda\big)\Big| \\
&\quad \le (p-2)\big||F_\lambda|^{p-4}\big(F_\lambda : (B-A)\big)\,F_\lambda\big| + |F_\lambda|^{p-2}|B-A| \\
&\quad \le (p-1)|F_\lambda|^{p-2}|B-A| \le (p-1)|B-A|\max\{|A|^{p-2}, |B|^{p-2}\}
\end{aligned}$$

and the assertion follows from

$$|B|^{p-2}B - |A|^{p-2}A = \int_0^1 \frac{\partial}{\partial\lambda}(|F_\lambda|^{p-2}F_\lambda)d\lambda. \qquad \square$$

Proof. Let $\partial G/\partial F$ and $\partial G/\partial\delta$ denote the partial derivatives of G with respect to the (four components of the) first and the last argument, respectively. The convexity of G implies that

$$\begin{aligned}
0 &\le I(w) - I(v) \\
&\le \gamma\big(\|Dw\|_p^p - \|Dv\|_p^p\big) \\
&\quad - \int_\Omega f\cdot(w-v)\,\mathrm{d}x \\
&\quad + \int_\Omega \frac{\partial G}{\partial F}(Dw, \det Dw) : D(w-v)\,\mathrm{d}x \\
&\quad + \int_\Omega \frac{\partial G}{\partial\delta}(Dw, \det Dw)(\det Dw - \det Dv)\,\mathrm{d}x.
\end{aligned} \tag{3.1}$$

The integral that involves f can be substituted by the weak formulation of the Euler-Lagrange equations (2.4) for the minimizer u,

$$\begin{aligned}
\int_\Omega f\cdot\zeta\,\mathrm{d}x &= \int_\Omega \frac{\partial G}{\partial F}(Du, \det Du) : D\zeta\,\mathrm{d}x + \gamma p\int_\Omega |Du|^{p-2}Du : D\zeta\,\mathrm{d}x \\
&\quad + \int_\Omega \frac{\partial G}{\partial\delta}(Du, \det Du)\,\mathrm{cof}\,Du : D\zeta\,\mathrm{d}x
\end{aligned} \tag{3.2}$$

for all $\zeta \in W_0^{1,p}(\Omega;\mathbb{R}^2)$. Lemma 3.2 yields with $A = Dw$ and $B = D(v-w)$

$$\begin{aligned}
&\gamma\big(\|Dw\|_p^p - \|Dv\|_p^p\big) \\
&\quad \le \gamma p\int_\Omega |Dw|^{p-2}Dw : D(w-v)\,\mathrm{d}x \\
&\quad\quad -\gamma c_4\int_\Omega \big(|Dw|^{p-2}|D(v-w)|^2 + |D(v-w)|^p\big)\,\mathrm{d}x.
\end{aligned} \tag{3.3}$$

We infer from (3.1)–(3.3) that

$$
\begin{aligned}
&\gamma c_4 \int_\Omega \big(|Dw|^{p-2}|D(v-w)|^2 + |D(v-w)|^p\big)\,\mathrm{d}x \\
&\quad\le \int_\Omega \big(\frac{\partial G}{\partial F}(Dw, \det Dw) - \frac{\partial G}{\partial F}(Du, \det Du)\big) : D(w-v)\,\mathrm{d}x \\
&\qquad + \int_\Omega \frac{\partial G}{\partial \delta}(Dw, \det Dw)\big(\det Dw - \det Dv - \operatorname{cof} Dw : D(w-v)\big)\,\mathrm{d}x \\
&\qquad + \int_\Omega \big(\frac{\partial G}{\partial \delta}(Dw, \det Dw) \operatorname{cof} Dw \\
&\qquad - \frac{\partial G}{\partial \delta}(Du, \det Du) \operatorname{cof} Du\big) : D(w-v)\,\mathrm{d}x \\
&\qquad + \gamma p \int_\Omega \big(|Dw|^{p-2}Dw - |Du|^{p-2}Du\big) : D(w-v)\,\mathrm{d}x.
\end{aligned}
$$

The estimate $|\det A| \le \frac{1}{2}|A|^2$ and the expansion

$$\det A = \det(A-B) + \det B + \operatorname{cof} A : (A-B)$$

show that for all $A, B \in \mathbb{M}^{2\times 2}$ with $|A-I|, |B-I| \le \delta_0 < 1/4$ we have

$$
\begin{aligned}
|\det A - \det B| &\le |\det(A-B)| + |\operatorname{cof} B : (A-B)| \\
&\le |A-B|(\sqrt{2} + 2\delta_0) \le 2|A-B|.
\end{aligned}
$$

Moreover, $|tA + (1-t)B - I| \le \delta_0$ for $t \in [0,1]$ and

$$
\begin{aligned}
|\det A - 1| &= \big|(a_{11}-1)(a_{22}-1) + a_{11} + a_{22} - a_{12}a_{21} - 2\big| \\
&\le \frac{1}{2}|A-I|^2 + \sqrt{2}|A-I| \le \frac{1}{32} + \frac{\sqrt{2}}{4}
\end{aligned}
$$

for $|A-I| < 1/4$. We therefore obtain for $X = F$ and $X = \delta$,

$$
\begin{aligned}
&\big|\frac{\partial G}{\partial X}(A, \det A) - \frac{\partial G}{\partial X}(B, \det B)\big| \\
&\quad = \Big|\int_0^1 \frac{\partial}{\partial t}\frac{\partial G}{\partial X}\big(tA + (1-t)B, t\det A + (1-t)\det B\big)\,\mathrm{d}t\Big| \\
&\quad \le c_2\big(|A-B| + |\det A - \det B|\big) \\
&\quad \le 3\,c_2\,|A-B|.
\end{aligned}
$$

Finally, since the determinant is a null-Lagrangian, i.e., $\int_\Omega \det D\zeta\ \mathrm{d}x = 0$ for all $\zeta \in W_0^{1,p}(\Omega;\mathbb{R}^2)$, and $w - v \in W_0^{1,p}(\Omega;\mathbb{R}^2)$ it follows that

$$\int_\Omega \frac{\partial G}{\partial \delta}(I,1)\ \det D(w-v)\,\mathrm{d}x = 0. \tag{3.4}$$

We may therefore estimate

$$
\begin{aligned}
&\gamma c_4 \int_\Omega \big(|Dw|^{p-2}|D(v-w)|^2 + |D(v-w)|^p\big)\,\mathrm{d}x \\
&\quad\le 3\,c_2\,|u-w|_{1,2}|w-v|_{1,2} \\
&\qquad - \int_\Omega \big(\frac{\partial G}{\partial \delta}(Dw, \det Dw) - \frac{\partial G}{\partial \delta}(I,1)\big)\det D(w-v)\,\mathrm{d}x \\
&\qquad + \int_\Omega \big(\frac{\partial G}{\partial \delta}(Dw, \det Dw) - \frac{\partial G}{\partial \delta}(Du, \det Du)\big)\operatorname{cof} Dw : D(w-v)\,\mathrm{d}x \\
&\qquad + \int_\Omega \frac{\partial G}{\partial \delta}(Du, \det Du)\operatorname{cof} D(w-u) : D(w-v)\,\mathrm{d}x \\
&\qquad + \gamma p \big\| |Du|^{p-2}Du - |Dw|^{p-2}Dw \big\|_2 |w-v|_{1,2}\,.
\end{aligned}
$$

Since $|Dw| \le |Dw - I| + |I| \le \delta_0 + \sqrt{2} \le 2$ and analogously $|Du| \le 2$ we obtain in view of Lemma 3.3 as an upper bound of the foregoing inequality

$$
\begin{aligned}
&3c_2|u-w|_{1,2}|v-w|_{1,2} + 3c_2\delta_0\frac{1}{2}|w-v|_{1,2}^2 + 6c_2|u-w|_{1,2}|v-w|_{1,2} \\
&\quad + c_1|u-w|_{1,2}|v-w|_{1,2} + \gamma p(p-1)2^{p-2}|u-w|_{1,2}|v-w|_{1,2} \\
&\le \big(c_1 + 9c_2 + \gamma p(p-1)2^{p-2}\big)|u-w|_{1,2}|v-w|_{1,2} + \frac{3}{2}c_2\delta_0|w-v|_{1,2}^2.
\end{aligned}
$$

We apply Young's inequality to the first term on the right-hand side and use $|Dw| \ge |I| - |Dw - I| \ge 1$. This implies that

$$
\begin{aligned}
&\gamma c_4 \int_\Omega \big(|D(v-w)|^2 + |D(v-w)|^p\big)\,\mathrm{d}x \\
&\quad \le \frac{c_5}{c_6}\int_\Omega |Du - Dw|^2\,\mathrm{d}x + \big(c_6 + \frac{3}{2}\,c_2\,\delta_0\big)\int_\Omega |Dw - Dv|^2\,\mathrm{d}x.
\end{aligned}
$$

We absorb the resulting quadratic term in $|w-v|_{1,2}$ on the left-hand side for δ_0 and $c_6 > 0$ small enough and obtain the assertion of the theorem. □

4 Finite element discretizations

In this section we apply the stability result in Theorem 3.1 to prove a priori error estimates for finite element approximations. The idea is to use the $W^{2,r}$-regularity of the minimizer u to show that $|\Pi u - id|_{1,\infty} < \delta_0$ for an interpolation Πu of u in a suitable finite element space $\mathcal{A}_h$. Since the minimizer u_h in $\mathcal{A}_h$ satisfies $I(u) \le I(u_h) \le I(\Pi u)$ we can apply Theorem 3.1 and the a priori estimates follow easily.

The definition of $\mathcal{A}_h$ requires some care since the existence results assume the boundary $\partial\Omega$ to be smooth. This excludes corners in the domain and enforces isoparametric finite elements. In order to focus on the idea and keep the proofs short we derive instead a convergence estimate for the simplest case of Courant elements.

Suppose that $\mathcal{T}$ is a regular triangulation in the sense of Ciarlet [C1] that consists of closed triangles with diameters $h \le h_0$ and angles in the interval $(\omega, \pi - \omega)$ with $\omega > 0$. Assume, moreover, that $\Omega_h = \cup\mathcal{T} \subseteq \overline{\Omega}$ is an approximation of Ω in the sense that $|\Omega \setminus \Omega_h| \le \kappa h^2$ and $\operatorname{dist}(z, \partial\Omega) \le \kappa h^2$ for all nodes z on $\partial\Omega_h$. The constants in our estimates depend on the positive parameters κ and ω and require a sufficiently small h_0, but are independent of h_0. Define

$$\mathcal{S}_0^1(\mathcal{T}) = \Big\{u \in C^0(\Omega; \mathbb{R}^2) : u \equiv 0 \text{ on } \Omega \setminus \Omega_h, \\ u \text{ affine on all triangles } T \in \mathcal{T}\Big\}$$

and the finite-dimensional space of admissible deformations

$$\mathcal{A}_h = \Big\{u_h \in W^{1,p}(\Omega; \mathbb{R}^2) : u_h \in id + g + \mathcal{S}_0^1(\mathcal{T}),\ \det Du_h > 0 \text{ a.e. in } \Omega\Big\}.$$

The existence of a minimizer in $\mathcal{A}_h$ is a consequence of general existence theorems for polyconvex functionals, see, e.g., [B1] for an overview and [B2] for details.

Lemma 4.1 *Assume that there exists a $w_h \in \mathcal{A}_h$ with $I(w_h) < \infty$. Then there exists a minimizer u_h of I in $\mathcal{A}_h$.*

We need some additional notation for the construction of the interpolation operator. Assume that z_α, $\alpha = 1, \dots, M$, are the nodes in the triangulation and that the nodes contained in the boundary $\partial\Omega_h$ are labeled $z_1, \dots, z_N$. Let $\{\varphi_\alpha\}$ be the nodal basis in $\mathcal{S}_0^1(\mathcal{T})$ with $\varphi_\alpha(z_\beta) = \delta_{\alpha\beta}$. We define a modification of the standard interpolation operator Π_1 onto $\mathcal{S}_0^1$ with zero boundary values on $\partial\Omega_h$ by

$$\hat{\Pi}_1 w = \sum_{\alpha=N+1}^{M} w(z_\alpha)\varphi_\alpha.$$

This operator is well-defined if w is continuous. In particular, if u is the minimizer constructed in Theorem 2.2, then $u \in W^{2,r}(\Omega; \mathbb{R}^2) \hookrightarrow W^{1,\infty}(\Omega; \mathbb{R}^2)$ and $\hat{\Pi}_1 u$ exists. Finally we let

$$\Pi : W^{2,r}(\Omega; \mathbb{R}^2) \to \mathcal{A}_h, \quad \Pi u = id + g + \hat{\Pi}_1(u - id - g),$$

where $\hat{\Pi}_1$ acts on the two components of its argument.

The next lemma provides the crucial estimates for the interpolation Πu of u. Similar interpolation estimates in L^2 will be important in the proof of Theorem 4.3.

Lemma 4.2 *Assume that $\varepsilon > 0$, $\delta_0 > 0$, that f and g are as in Theorem 3.1, and that u is the unique minimizer of I in $\mathcal{A}$. Then there exists a constant $c_7 > 0$, which depends only on ω, such that*

$$|\Pi u - id|_{1,\infty} \leq c_7\big(h^{1-2/r} \parallel u - id - g \parallel_{2,r} + h + \varepsilon + \eta(\varepsilon)\big).$$

In particular, $\Pi u \in \mathcal{A}_h$ for sufficiently small ε and h.

Proof. On each element $T \in \mathcal{T}$ without nodes in $\partial\Omega_h$ the operator $\hat{\Pi}_1$ coincides with the usual nodal interpolation operator Π_1 onto piecewise affine functions. Therefore we may use the standard interpolation estimate (see, e.g., [C1], Theorem 3.1.6) to conclude

$$\begin{aligned} |u - \Pi u|_{W^{1,\infty}(T)} &= |u - id - g - \Pi_1(u - id - g)|_{W^{1,\infty}(T)} \\ &\leq c_8\, h^{1-2/r}|u - id - g|_{W^{2,r}(T)}. \end{aligned} \tag{4.1}$$

This implies together with the triangle inequality, the Sobolev embedding $W^{2,r}(\Omega;\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\Omega;\mathbb{R}^2)$, and (2.6) that

$$\begin{aligned} |\Pi u - id|_{W^{1,\infty}(T)} &\leq |\Pi u - u|_{W^{1,\infty}(T)} + |u - id|_{W^{1,\infty}(T)} \\ &\leq c_8\, h^{1-2/r}|u - id - g|_{W^{2,r}(T)} + c_9\, \eta(\varepsilon). \end{aligned} \tag{4.2}$$

The estimate (4.1) cannot be applied to a triangle T with one or more nodes z on $\partial\Omega_h$ since $\hat{\Pi}_1(u - id - g)$ has been set to zero in the boundary nodes. However, since u is the minimizer, we obtain from Theorem 2.2

$$|u - id - g|_{1,\infty} \leq c_9\|u - id\|_{W^{2,r}(T)} + \|g\|_{3,\infty} \leq c_9(\eta(\varepsilon) + \varepsilon)$$

(see (2.1) and (2.5)). By our assumptions on $\mathcal{T}$ we have $\operatorname{dist}(z, \partial\Omega) = \mathcal{O}(h^2)$ for all nodes $z \in \partial\Omega_h$ and thus $|(u - id - g)(z)| = \mathcal{O}(h^2)$. This allows us to estimate

$$\begin{aligned} |u - \Pi u|_{W^{1,\infty}(T)} \leq\ & |u - id - g - \Pi_1(u - id - g)|_{W^{1,\infty}(T)} \\ & + \sum_{z_\alpha \in \overline{T} \cap \partial\Omega_h} |(u - id - g)(z_\alpha)\varphi_\alpha|_{W^{1,\infty}(T)}. \end{aligned}$$

Since $|\varphi_\alpha|_{W^{1,\infty}(T)} = \mathcal{O}(h^{-1})$ (the constant in the estimate depends only on the angles of the triangles $T \in \mathcal{T}$) this implies the estimates

$$|u - \Pi u|_{W^{1,\infty}(T)} \leq c_{10}\big(h^{1-2/r}|u - id - g|_{W^{2,r}(T)} + h\big), \tag{4.3}$$

$$|\Pi u - id|_{W^{1,\infty}(T)} \leq c_{11}\big(h^{1-2/r}|u - id - g|_{W^{2,r}(T)} + h + \eta(\varepsilon)\big). \tag{4.4}$$

In the thin boundary layer $\Omega \setminus \overline{\Omega}_h$ we have $\hat{\Pi}_1(u - id - g) = 0$ and therefore

$$|\Pi u - id|_{W^{1,\infty}(\Omega\setminus\overline{\Omega}_h)} \leq \parallel g \parallel_{W^{1,\infty}(\Omega\setminus\overline{\Omega}_h)} \leq \varepsilon, \tag{4.5}$$

by (2.2). The estimates (4.2), (4.4) and (4.5) now imply the first assertion of the lemma.

It remains to show that $\det D\Pi u > 0$ a.e. Let w be any function with $|w - id|_{W^{1,\infty}} \leq \mu$ for some $\mu > 0$. Then

$$\det Dw \geq 1 - 2\mu(\mu + 1)$$

and this inequality yields the second assertion of the lemma for h and ε small enough. □

We are now in a position to prove the main theorem of this section.

Theorem 4.3 *For any $\omega > 0$ and $\kappa > 0$ there exist $\varepsilon \in (0, \varepsilon_0)$ (ε_0 as in Theorem 2.2) and $h_0 > 0$ with the following properties: For any triangulation $\mathcal{T}$ as in Section 4 with maximal mesh-size $h \leq h_0$ (and shape and boundary approximation according to ω and κ) and for all $f \in L^r(\Omega; \mathbb{R}^2)$ and $g \in C^3(\overline{\Omega}; \mathbb{R}^2)$ with (2.2)–(2.3), the unique minimizer u of I satisfies $\Pi u \in \mathcal{A}_h$ and any minimizer u_h of I in $\mathcal{A}_h$ satisfies*

$$\begin{aligned} \| D(u - u_h) \|_p^p + \| D(u - u_h) \|_2^2 &\leq c_{12} \| D(u - \Pi u) \|_2^2 \\ &\leq c_{13} \big(h^{4(1-1/r)} + h^2\big)(\varepsilon + \eta(\varepsilon)). \end{aligned} \tag{4.6}$$

Remark. Since, in particular, $\| D(u - u_h) \|_2 \leq c_{12}^{1/2} \| D(u - \Pi u) \|_2$, we regard (4.6) as a quasi-optimal a priori error estimate.

Proof. Since the angles of the triangles in $\mathcal{T}$ are contained in a compact interval in $(0, \pi)$ we may choose ε and h_0 sufficiently small to ensure that $\Pi u \in \mathcal{A}_h$ by Lemma 4.2. The existence of a minimizer follows now from Lemma 4.1 and the first inequality in (4.6) is a consequence of Theorem 3.1 since $I(u) \leq I(u_h) \leq I(\Pi u)$ for all minimizers u_h of I in $\mathcal{A}$.

The second inequality in (4.6) uses interpolation estimates in $W^{1,2}$ similar to those in the proof of Lemma 4.2 in $W^{1,\infty}$. For triangles T without boundary nodes we obtain the analogue of (4.1),

$$|u - \Pi u|_{W^{1,2}(T)} \leq c_{14}\, h^{2(1-1/r)} |u - id - g|_{W^{2,r}(T)},$$

while we have for triangles with boundary nodes

$$|u - \Pi u|_{W^{1,2}(T)} \leq c_{15}\big(h^{2(1-1/r)} |u - id - g|_{W^{2,r}(T)} + h^2\big),$$

see (4.3). Finally our assumptions on $\mathcal{T}$ imply that

$$|u - \Pi u|_{W^{1,2}(\Omega \setminus \Omega_h)} \leq \mathcal{O}(h) |u - id - g|_{W^{1,\infty}(\Omega \setminus \Omega_h)}.$$

The proof of the theorem follows easily.

5 Concluding remarks

Several analytical difficulties are listed below which, in our opinion, have to be overcome in order to avoid the assumptions of small right-hand sides in our analysis.

(i) The strong convergence of Πu to u shows the convergence of the energies $I(\Pi u) \to I(u)$ and so the convergence of $I(u_h) \to I(u)$. A typical problem in the computation of energy minimizers is the Lavrientev phenomenon, see, e.g., [BK], where

$$\min_{\mathcal{A}} I < \inf_{\mathcal{A} \cap W^{1,\infty}(\Omega;\mathbb{R}^2)} I \le \inf_{h>0} \inf_{\mathcal{A}_h} I,$$

i.e., there is no convergence of energies.

(ii) Smooth solutions u are required to guarantee $0 < \det(DJu)$ for some finite element interpolant Ju of u.

(iii) The nonlinearity $DW(Du)$ has no structure properties such as monotonicity. Therefore, "a simple energy estimate would not be enough for proving convergence" [D, p. 366]. This led Dobrowolski [D] to study L^∞-estimates for incompressible materials using a homotopy argument.

(iv) A smooth solution u does not allow cavitation, i.e., formation and growth of holes inside the body (see the seminal paper [B3]).

(v) Any stability result such as Theorem 3.1 implies uniqueness of global and separation of local minimizers and excludes bifurcation.

(vi) Zhang's result [Z] is, to the best of our knowledge, the only result that guarantees uniqueness and regularity of minimizers of I. The main drawback is the restriction to small loads and boundary data close to the identity for the pure Dirichlet problem. On the other hand, buckling is expected for large data and therefore uniqueness results require further conditions. It remains an open problem how to obtain a local version of Zhang's result for large data.

(vii) It is unclear how to extend the stability result to mixed boundary conditions. It is not known whether there is a higher regularity result analogous to (2.6). In addition, the proof of Lemma 3.1 uses the fact that the determinant is a null-Lagrangian.

A Proof of Theorem 2.3

The proof of Theorem 2.3 in the two-dimensional setting is considerably easier than the proof in the three-dimensional situation since the terms involving the cofactor matrix are not present. We sketch the argument following [Z] for the convenience of the reader.

Let u be the solution given by the implicit function theorem and let $u + \varphi$ with $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^2)$ be the global minimizer of the polyconvex variational integral. The goal is to prove that $\varphi \equiv 0$. In order to simplify the notation we define for $F \in \mathbb{M}^{2\times 2}$ the differentiable functions

$$A(F) = \frac{\partial G}{\partial F}(F, \det F), \quad C(F) = \frac{\partial G}{\partial \delta}(F, \det F).$$

For $\varepsilon > 0$ given we may choose $\|f\|_r$ and $\|g\|_{3,\infty}$ small enough in order to guarantee that

(A.1)

$$\|I - Du\|_\infty < \varepsilon, \quad \|C(Du) - C(I)\|_\infty < \varepsilon, \quad |Du(x)| \geq 1 \text{ for } x \in \Omega.$$

Since u satisfies the weak form of (2.4), there holds

$$\int_\Omega \big(p\gamma |Du|^{p-2} Du : D\varphi - f\varphi + A(Du) : D\varphi \\ + C(Du) \operatorname{cof} Du : D\varphi\big)\, \mathrm{d}x = 0.$$

By construction, $u+\varphi$ is the global minimizer and thus $I(u+\varphi) - I(u) \leq 0$.
In view of the convexity of G, this implies

$$\begin{aligned} 0 &\geq \int_\Omega \big\{\gamma\big(|Du + D\varphi|^p - |Du|^p\big) - f\varphi\big\}\, \mathrm{d}x \\ &\quad + \int_\Omega \big\{A(Du) : D\varphi + C(Du)\big(\det(Du + D\varphi) - \det Du\big)\big\}\, \mathrm{d}x \\ &\geq \int_\Omega \gamma c_4 \big(|Du|^{p-2}|D\varphi|^2 + |D\varphi|^p\big)\, \mathrm{d}x + \int_\Omega C(Du) \det D\varphi\, \mathrm{d}x + \delta I(u)\varphi \end{aligned}$$

where we used the expansion

$$\det(A + B) = \det A + \operatorname{cof} A : B + \det B \quad \text{for } A,\ B \in \mathbb{M}^{2\times 2}$$

and the estimate in Lemma 3.2. In view of (A.1), the bound $|\det A| \leq \frac{1}{2}|A|^2$ for $A \in \mathbb{M}^{2\times 2}$, and the fact that the determinant is a null-Lagrangian, i.e.,

$$\int_\Omega \det D\varphi\, \mathrm{d}x = 0 \quad \text{for all } \varphi \in W_0^{1,2}(\Omega; \mathbb{R}^2),$$

we deduce

$$0 \geq \int_\Omega \big(\gamma c_4 |D\varphi|^2 - \varepsilon |\det D\varphi|\big)\, \mathrm{d}x \geq \int_\Omega \big(\gamma c_4 - \frac{\varepsilon}{2}\big)|D\varphi|^2\, \mathrm{d}x.$$

For $\varepsilon > 0$ small enough we obtain $D\varphi = 0$ and hence $\varphi = 0$. This establishes the assertion of the theorem.

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