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New adaptive mixed finite element method (AMFEM)

Carsten Carstensen[1,2,3]* und **Hella Rabus**[1,2] **

[1] Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin

[2] Supported by the DFG Research Centre Matheon "Mathematics for key technologies" in Berlin, Germany

[3] Supported by the Hausdorff Institute of Mathematics in Bonn, Germany

The need to develop reliable and efficient adaptive algorithms using mixed finite element methods arises from various applications in fluid dynamics and computational continuum mechanics. In order to save degrees of freedom, not all but just some selected set of finite element domains are refined and hence the fundamental question of convergence requires a new mathematical argument as well as the question of optimality.

We will present a new adaptive algorithm for mixed finite element methods to solve the model Poisson problem, for which optimal convergence can be proved. The a posteriori error control of mixed finite element methods dates back to Alonso (1996) *Error estimators for a mixed method.* and Carstensen (1997) *A posteriori error estimate for the mixed finite element method.* The error reduction and convergence for adaptive mixed finite element methods has already been proven by Carstensen and Hoppe (2006) *Error Reduction and Convergence for an Adaptive Mixed Finite Element Method*, *Convergence analysis of an adaptive nonconforming finite element methods* .

Recently, Chen, Holst and Xu (2008) *Convergence and Optimality of Adaptive Mixed Finite Element Methods.* presented convergence and optimality for adaptive mixed finite element methods following arguments of Rob Stevenson for the conforming finite element method. Their algorithm reduces oscillations, before applying and a standard adaptive algorithm based on usual error estimation. The proposed algorithm does this in a natural way, by switching between the reduction of either the estimated error or oscillations.

1 Introduction

The need to develop optimal adaptive algorithms to approximate solutions of PDEs arises from several application. Complex problems need to be solved in a reasonable amount of time and computational costs. Especially for mixed finite element methods the theory of adaptive algorithms is still under development, in particular the design of algorithms, for which optimality can be proven is a challenging task.

A posteriori error control for mixed finite element methods (MFEM) dates back to Alonso (1996) [1] and Carstensen (1997) [2]. The proof of error reduction and convergence by Carstensen and Hoppe [3, 4] in 2006 was a mile stone into that direction.

In contrast to the algorithm proposed by [5] our approach is to reduce the error and oscillations simultaneously, rather than to to reduce oscillations first up to a given tolerance and then approximate the solution by some standard adaptive algorithm. Chen, Holst and Xu proved convergence and optimality for their method.

This talk proposes an optimal adaptive mixed finite element algorithm for the Poisson model problem

$$p + \nabla u = 0,\ \operatorname{div} p = f \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega \tag{1.1}$$

with $f \in L^2(\Omega)$. The discrete mixed variational formulation reads: Seek $(p_\ell, u_\ell) \in RT_0(\mathcal{T}_\ell) \times P_0(\mathcal{T}_\ell)$ such that $\forall (q_\ell, v_\ell) \in RT_0(\mathcal{T}_\ell) \times P_0(\mathcal{T}_\ell)$ the following equations hold

$$(p_\ell, q_\ell)_{L^2(\Omega)} = (\operatorname{div} q_\ell, u_\ell)_{L^2(\Omega)}\,, \quad (\operatorname{div} p_\ell, v_\ell)_{L^2(\Omega)} = (f, v_\ell)_{L^2(\Omega)}\,,$$

on lowest order Raviart Thomas FEM spaces

$$RT_0(\mathcal{T}_\ell) = \{q \in \mathrm{H}(\mathrm{div}, \Omega) \mid \forall T \in \mathcal{T} \quad \exists a \in \mathbb{R}^2, b \in \mathbb{R} \quad \forall x \in T \quad q(x) = a + bx\}.$$

It is well known, that the Ladyschenskaja-Babuška-Brezzi (LBB) condition holds and insures existence and uniqueness of the discrete solution.

The essential steps in the algorithm are the steps MARK & REFINE. Having calculated the edge-based error estimator η_ℓ and the oscillations osc_ℓ in step ESTIMATE the mesh is refined with respect to the chosen refinement indicator, depending on whether the error estimator dominates oscillations or $\kappa\eta_\ell^2 < \mathrm{osc}_\ell^2$ for given $\kappa > 0$.

* Carsten Carstensen E-mail: cc@math.hu-berlin.de, Phone: +00 49 30 20935844, Fax: +00 49 30 20935444

** Hella Rabus E-mail: rabus@math.hu-berlin.de, Phone: +00 49 30 20932360, Fax: +00 49 30 20932234

2 Adaptive Mixed Finite Element Method (AMFEM)

The standard loop of adaptive FEM reads

Solve → Estimate → Mark → Refine

and has to be modified as follows to ensure optimal convergence rate for the lowest order mixed finite element method.

Anmerkung 2.1 The main challenge in designing optimal adaptive algorithms for mixed finite element methods is to overcome the lack of Galerkin-orthogonality. For that purpose quasi-orthogonality is needed [4, 5]. Beyond orthogonality, the major objective is to embed both Cases (A) and (B) in an optimal refinement strategy, which allows the return to standard arguments for adaptive schemes.

The proposed algorithm reduces estimated error and oscillations simultaneously, in contrast to the one introduced by Chen, Holst and Xu [5] where an initial triangulation approximating the data up to a given tolerance, is generated, before the solution is approximated by an adaptive algorithm for mixed methods.

The basic scheme of AMFEM reads as follows:

Algorithm 2.2 (AMFEM) For given positive κ and $0 < \theta < 1$, do until termination

SOLVE: Solve the discrete problem (1.1) on $\mathcal{T}_\ell$.

ESTIMATE: Compute data oscillations and edge-based error estimator.
Choose refinement indicator: osc_ℓ or η_ℓ depending on whether $\mathrm{osc}_\ell^2 \le \kappa\eta_\ell^2$.

MARK: Mark elements or edges with respect to the refinement indicator.

REFINE: Generate $\mathcal{T}_{\ell+1}$ by refining $\mathcal{T}_\ell$ with respect to the marked set by applying Newest Vertex Bisection.
Possible refinments of a triangle $T \in \mathcal{T}_\ell$ depending on which edges are marked for refinement are depicted in Figure 2.1.

Whereas the procedure of the first two steps follows standard adaptive algorithms, the design of the two steps MARK and REFINE is crucial for the proof of contraction and optimal convergence and needs to be modified properly. Each step of AMFEM is described in detail in the remaining part of this section.

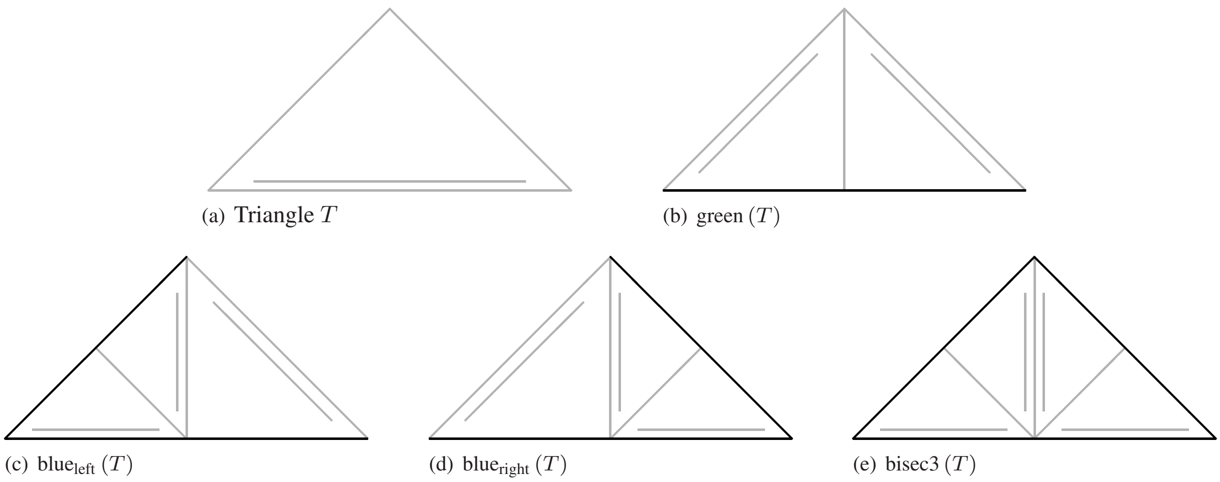


Abb. 2.1 Possible refinement of a triangle T. The reference edge of each (sub-)triangle is identified through an additional parallel line, while themarked edges of T are drawn in black.

Steps SOLVE & ESTIMATE

On each level ℓ, let $\mathcal{T}_\ell$ denotes the partitioning of Ω into triangles, $\mathcal{E}_\ell$ the set of interior edges.

In steps SOLVE & ESTIMATE, solve the discrete problem (1.1) on $\mathcal{T}_\ell$ and compute edge-oriented error estimator

$$\eta_\ell^2 := \eta_\ell^2(\mathcal{E}_\ell) = \sum_{E \in \mathcal{E}_\ell} \eta_\ell^2(E), \text{ with}$$

$$\eta_\ell(E) := |E|^{1/2} \, \|[p_\ell]_E\|_{L^2(E)} \text{ for all } E \in \mathcal{E}_\ell$$

and $[q]_E := q_{T_+} - q_{T_-}$ denoting the jump of q across an edge E and element-wise oscillations

$$\operatorname{osc}_\ell := \operatorname{osc}(f,\mathcal{T}_\ell) := \left(\sum_{T\in\mathcal{T}_\ell} \operatorname{osc}^2(f,T)\right)^{1/2}, \text{ where}$$

$$\operatorname{osc}(f,T) := |T|^{1/2}\, \|f - f_T\|_{L^2(T)}, \text{ and}$$

$$f_T := \fint_T f\,\mathrm{dx} := |\omega|^{-1} \int_\omega f(x)\,\mathrm{dx}\,.$$

Implementational details are given in [6].

Anmerkung 2.3 For the chosen edge-error estimator η_ℓ efficiency and reliability hold

$$C_{\text{eff}}\eta_\ell^2 \le \|p - p_\ell\|^2_{L^2(\Omega)} \le C_{\text{rel}}\left(\eta_\ell^2 + \operatorname{osc}_\ell^2\right).$$

Steps MARK & REFINE

The design of these two steps is essential for embedding the two Cases (A) and (B) into an optimal mesh. Depending on $\operatorname{osc}_\ell^2 \le \kappa\eta_\ell^2$ on each level either *error estimator reduction* or *oscillation reduction* is performed.

In **Case (A):** $\operatorname{osc}_\ell^2 \le \kappa\eta_\ell^2$ and the edge-based error estimator is to be reduced.

MARK: Compute minimal set of marked edges, which fulfil bulk criterion $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell$ such that

$$\theta\eta_\ell^2 \le \sum_{E\in\mathcal{M}_\ell} \eta_E^2.$$

REFINE: Refine all edges in $\mathcal{C}\ell(\mathcal{M}_\ell)$ using Newest Vertex Bisection.

In **Case (B):** $\operatorname{osc}_\ell^2 > \kappa\eta_\ell^2$ the oscillations dominate the error estimator, therefore osc_ℓ is to be reduced in an optimal way.

MARK: Get triangulation $\mathcal{T}_{\epsilon_\ell}$ by *Thresholding Second Algorithm* [7] for $\operatorname{Tol}(\ell) := \rho_2^{1/2}\operatorname{osc}_\ell$ with $0 < \rho_2 < 1$, such that

$$\operatorname{osc}^2(f,\mathcal{T}_{\epsilon_\ell}) < \operatorname{Tol}(\ell)^2 \text{ and } |\mathcal{T}_{\epsilon_\ell}| - |\mathcal{T}_0| \lesssim \operatorname{Tol}^{-1/s}(\ell).$$

REFINE: Compute overlay triangulation $\mathcal{T}_{\ell+1} := \mathcal{T}_\ell \oplus \mathcal{T}_{\epsilon_\ell}$.

3 Contraction & Optimality

The convergence of adaptive algorithms is not obvious, since the intention of adaptivity is to let refine some regions of the mesh, whereas in other regions, $\operatorname{diam}(T)$ might stay well away from zero.

Satz 3.1 (Contraction [8]) *For a special choice of positive reals $\alpha,\beta,\kappa,\theta$, for each level ℓ with*

$$\xi_\ell^2 := \eta_\ell^2 + \alpha\varepsilon_\ell^2 + \beta\operatorname{osc}_\ell^2$$

there exists $0 < \rho < 1$, such that the following contraction property holds

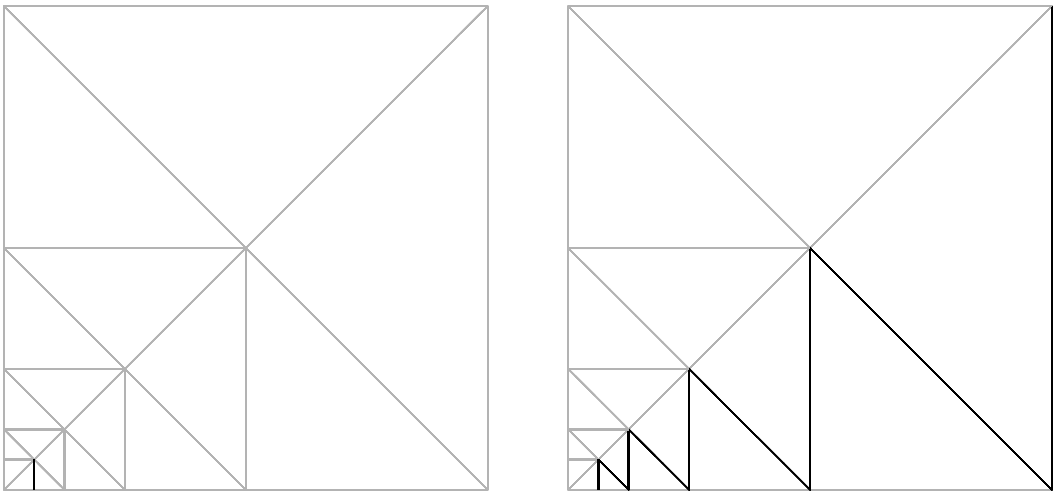
$$\xi_{\ell+1} \le \rho\xi_\ell.$$

Satz 3.2 (Overhead of CLOSURE [7,9]) *Let $\mathcal{T}_\ell$ be some triangulation refined from $\mathcal{T}_0$ refining with respect to some sequence of marked edges $\mathcal{M}_k \subseteq \mathcal{T}_k$, $0 \le k < \ell$. Then, there exists some constant $C_0 > 0$, depending solely on the shape regularity of $\mathcal{T}_0$ such that*

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \le C_0 \sum_{k=1}^{\ell-1} |\mathcal{M}_k|\,.$$

Anmerkung 3.3 (Maintain Shape Regularity and Controlling CLOSURE) To prove optimality of AMFEM the two strategies of refining a mesh have to be combined to estimate the refinement costs in total. In the first case, where oscillations are small compared to the estimated error, the standard estimate by Rob Stevenson is used.

Having marked a set of edges $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell$ in MARK, shape regularity is maintained by application of CLOSURE, which computes $\mathcal{C}\ell(\mathcal{M}_\ell) \supseteq \mathcal{M}_\ell$. Finally the triangulation is refined by bisecting exactly the edges in $\mathcal{C}\ell(\mathcal{M}_\ell)$ using Newest Vertex Bisection, cf. Figure 2.1.



(a) Set of marked edges $\mathcal{M}_\ell$ (b) Set of marked edges after applying CLOSURE - $\mathcal{C}\ell(\mathcal{M}_\ell)$

Abb. 3.1 Noncontrollable overhead of CLOSURE in only one level

Considering only one level ℓ, the control depends on some $C_\ell > 0$, such that for $|\mathcal{M}_\ell| \ll |\mathcal{C}\ell(\mathcal{M}_\ell)|$ there exists $1 \ll C_\ell$ with

$$|\mathcal{T}_{\ell+1}| - |\mathcal{T}_\ell| \leq C_\ell \, |\mathcal{M}_\ell| \, .$$

Clearly, the unboundedness of C_ℓ is not intentionally. One example is depicted in Figure 3.1, obviously the marking of one edge leads to an unbounded number of elements in $\mathcal{C}\ell(\mathcal{M}_\ell)$.

Anmerkung 3.4 (Reducing oscillations) In the second case the number of refinement levels of the resulting triangulation is not bounded. In that case there is need to bound the refinement cost in order to achieve optimality for AMFEM. For more information see [8].

The major result and achievement is the following theorem, ensuring that AMFEM generates a sequence of optimal triangulations in the sense of Stevenson [9]. Having proven contraction of the error, the next step is to consider optimality of the algorithm.

Definition 3.5 (Approximation class) For $s > 0$ and an initial triangulation $\mathcal{T}_0$, set

$$\mathcal{A}_s := \left\{ (p,f) \in \mathrm{H}(\mathrm{div},\Omega) \times L^2(\Omega) \mid \|(p,f)\|_{A_s} < \infty \right\},$$
$$\|(p,f)\|_{\mathcal{A}_s} := \sup_{N \in \mathbb{N}} N^s \inf_{|\mathcal{T}|-|\mathcal{T}_0| \leq N} \left(\mathrm{dist}^2(p, RT_0(\mathcal{T})) + \mathrm{osc}^2(f,\mathcal{T}) \right).$$

A triangulation $\mathcal{T}_\epsilon$ refined from $\mathcal{T}_0$ is called optimal, if for $\epsilon > 0$, $(p,f) \in \mathcal{A}_s$ it holds

$$\eta_\epsilon^2 + \mathrm{osc}^2\left(f, \mathcal{T}_\epsilon\right) \leq \epsilon \text{ and } |\mathcal{T}_\epsilon| - |\mathcal{T}_0| \lesssim \epsilon^{-1/s} \, \|(p,f)\|_{\mathcal{A}_s}$$

Satz 3.6 (Optimal convergence [8]) *For a special choice of* $0 < \theta < 1$ *and positive reals* α, β, κ, ρ *with* $\rho < 1$*; the sequence of triangulations* $\mathcal{T}_\ell$ *with discrete MFEM solutions* $(p_\ell, u_\ell) \in RT_0(\mathcal{T}_\ell) \times P_0(\mathcal{T}_\ell)$ *and the weighted term*

$$\xi_\ell^2 := \eta_\ell^2 + \alpha \varepsilon_\ell^2 + \beta \, \mathrm{osc}_\ell^2$$

generated by algorithm AMFEM is optimal in the sense

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \lesssim \xi_\ell^{-1/s}.$$

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