Numer. Math. (2016) 133:557–597
DOI 10.1007/s00211-015-0755-0

Numerische Mathematik

Error analysis of nonconforming and mixed FEMs for second-order linear non-selfadjoint and indefinite elliptic problems

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Received: 2 August 2013 / Revised: 19 May 2015 / Published online: 18 July 2015

Abstract The state-of-the art proof of a global inf-sup condition on mixed finite element schemes does not allow for an analysis of truly indefinite, second-order linear elliptic PDEs. This paper, therefore, first analyses a nonconforming finite element discretization which converges owing to some *a priori* L^2 error estimates even for reduced regularity on non-convex polygonal domains. An equivalence result of that nonconforming finite element scheme to the mixed finite element method (MFEM) leads to the well-posedness of the discrete solution and to *a priori* error estimates for the MFEM. The explicit residual-based *a posteriori* error analysis allows some reliable and efficient error control and motivates some adaptive discretization which improves the empirical convergence rates in three computational benchmarks.

Mathematics Subject Classification 65N30 · 65N50

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1 Introduction

The general second-order linear elliptic PDE on a simply-connected bounded polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$ reads for given right-hand side $f \in L^2(\Omega)$ as

$$\mathcal{L}u := -\nabla \cdot (\mathbf{A}\nabla u + u\mathbf{b}) + \gamma\, u = f \text{ in } \Omega \qquad u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

The coefficients are all essentially bounded functions and the eigenvalues of the symmetric matrix $\mathbf{A}$ are all positive and uniformly bounded away from zero. The point is that the convective term $\mathbf{b}$ and the reaction term γ may be arbitrary as long as the boundary value problem (1.1) is well-posed in the sense that zero is not an eigenvalue. In other words, $\mathcal{L} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is supposed to be injective, where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega) := \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. Since $\mathcal{L}$ is a bounded linear operator between Hilbert spaces, this is equivalent to assume that $\mathcal{L}$ is an isomorphism.

It is known since [21] for conforming finite element discretization and it will be proved in this paper for nonconforming and for mixed finite element methods that sufficiently fine triangulations allow for unique discrete solution. One key argument in the proof is some representation formula for the lowest-order Raviart–Thomas solution to (1.1) in terms of the Crouzeix–Raviart solution. This circumvents the extra conditions on the coefficients from [12] to deduce the solvability of the mixed finite element scheme and, thereby, allows a numerical analysis of the general linear indefinite problem at hand. The *a priori* error analysis shows a quasi-optimal error estimate by best-approximation errors.

The robust *a posteriori* error control is feasible for sufficiently fine (although unstructured but shape-regular) meshes on the basis of some *a priori* L^2 control for the nonconforming FEM by duality. This allows for reliable and efficient error estimates in terms of the explicit residual-based error estimators up to generic constants and data approximation errors.

This paper is devoted to another approach to generalized saddle-point problems via an explicit equivalence to nonconforming finite element schemes for general second-order linear indefinite and non-symmetric elliptic PDEs. The standard generalization of the Brezzi splitting lemma [5] to more general possibly non-symmetric bilinear forms in [12] formulates various conditions on several boundedness and inf-sup constants. Those are essentially sufficient conditions and not equivalent to well-posedness. Observe that all conditions in [12] hold as well for some bilinear form which involves a homotopy parameter λ which takes away the non-symmetry or indefiniteness for $\lambda = 0$ and equals the bilinear form considered in [12] for $\lambda = 1$. For such a homotopy and certain critical values of $0 < \lambda < 1$, the underlying PDE may have a zero eigenvalue, while the sufficient condition of [12] is convex in λ and so holds for that critical value as well. This illustrates that we may encounter some general second-order linear PDE, where the conditions in [12] do not guarantee any well-posedness of the continuous or the discrete situation, while the continuous problem is well-posed, and hence, some novel mathematical ideas are required to ensure the solvability of the discrete solution in MFEM and their uniform boundedness *a priori* for small meshes.

This paper assumes that the parameters in the general second-order linear elliptic PDE are such that the associated boundary value problem is well-posed on the continuous level and shows with arguments like those in [21] for the conforming case that there exists discrete solutions for a first-order nonconforming finite element method provided the mesh is sufficiently fine. Based on general conforming companions as part of the novel medius analysis, which utilizes mathematical arguments between *a priori* and *a posteriori* analysis, this paper proves L^2 error and piecewise H^1 error estimates.

The remaining parts of the paper are organized as follows. Section 2 introduces the weak and mixed weak formulations and equivalence of primal and mixed methods. Section 3 presents the Crouzeix–Raviart nonconforming finite element methods (NCFEM) and discusses the solvability of the discrete problem and the related *a priori* and *a posteriori* error estimates. Section 4 focuses on Raviart–Thomas mixed finite element methods (RTFEM), the representation of RTFEM solution via NCFEM, and *a priori* error estimates for RTFEM. Section 5 establishes *a posteriori* error estimates for the discrete mixed formulation and its efficiency. Numerical experiments in Sect. 6 concern to sensitivity of the *a priori* and *a postriori* error bounds and study the performance of the related adaptive algorithms.

This section concludes with some notation used through out this paper. An inequality $A \lesssim B$ abbreviates $A \leq CB$, where $C > 0$ is a mesh-size independent constant that depends only on the domain and the shape of finite elements; $A \approx B$ means $A \lesssim B \lesssim A$. Standard notation applies to Lebesgue and Sobolev spaces and $\|\cdot\|$ abbreviates $\|\cdot\|_{L^2(\Omega)}$ with L^2 scalar product denoted as $(\cdot,\cdot)_{L^2(\Omega)}$ or $(\cdot,\cdot)$ (whenever there is no chance of confusion). Let $H^m(\Omega)$ denote the Sobolev spaces of order m with norm given by $\|\cdot\|_m$. The space of $\mathbb{R}^2$-valued L^2 and H^1 functions defined over the domain Ω is denoted by $L^2(\Omega;\mathbb{R}^2)$ and $H^1(\Omega;\mathbb{R}^2)$ respectively. Let $H(\operatorname{div},\Omega) = \{\mathbf{q} \in L^2(\Omega;\mathbb{R}^2) : \operatorname{div}\mathbf{q} \in L^2(\Omega)\}$ with the norm $\|\cdot\|_{H(\operatorname{div},\Omega)}$ and its dual space $H(\operatorname{div},\Omega)^*$.

2 On weak and mixed formulations

This section introduces the minimal assumptions, the weak formulation with a reference to solvability, and the mixed formulation for the problem (1.1) and their equivalence. Define the bilinear form $a(\cdot,\cdot)$ for $u, v \in H_0^1(\Omega)$ by

$$a(u,v) = (\mathbf{A}\nabla u + u\mathbf{b}, \nabla v)_{L^2(\Omega)} + (\gamma\, u, v)_{L^2(\Omega)}.$$

The weak formulation of (1.1) reads: Given $f \in L^2(\Omega)$, seek a function $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v)_{L^2(\Omega)} \quad \text{for all} \quad v \in H_0^1(\Omega). \tag{2.1}$$

Throughout this paper, the following assumptions **(A1)–(A2)** are posed on the coefficients and solution to the problem (1.1).

(A1) The coefficient matrix $\mathbf{A} \in L^\infty(\Omega; \mathbb{R}^{2\times 2}_{sym})$ is positive definite; that is, there exist positive numbers α_0 and Λ such that $\alpha_0|\boldsymbol{\xi}|^2 \le \mathbf{A}(x)\boldsymbol{\xi}\cdot\boldsymbol{\xi} \le \Lambda|\boldsymbol{\xi}|^2$ for a.e. $x \in \Omega$ and for all $\boldsymbol{\xi} \in \mathbb{R}^2$. Further, the coefficient matrix $\mathbf{A}$, vector $\mathbf{b}$ and γ are Lipschitz continuous.

(A2) Given any $f \in L^2(\Omega)$, the problem (1.1) has a unique weak solution $u \in H^1_0(\Omega)$. The dual problem reads: Given $g \in L^2(\Omega)$, seek a solution $\Phi \in H^1_0(\Omega)$ such that

$$a(v, \Phi) = (g, v)_{L^2(\Omega)} \qquad \text{for all } v \in H^1_0(\Omega). \tag{2.2}$$

The unique solvability of (2.2) follows by duality from the well-posedness of $\mathcal{L}$, in **(A2)** and, as a consequence, $\|\Phi\|_1 \le C\|g\|$.

(A3) Suppose that there exist some constants $0 < \delta < 1$ and $C(\delta) < \infty$ such that the unique solution $\Phi = \mathcal{L}^{-1}g$ of (2.2) satisfies $\Phi \in H^{1+\delta}(\Omega) \cap H^1_0(\Omega)$ and

$$\|\Phi\|_{1+\delta} \le C(\delta)\|g\|. \tag{2.3}$$

Since 0 is not part of the spectrum of $\mathcal{L}$, the Fredholm alternative [16, Theorem 5, pp 305–306] proves that the problem (1.1) has a unique weak solution for each $f \in L^2(\Omega)$. For more detailed information on existence and uniqueness result of the weak solution to (1.1) or to (2.2), see [16, Theorem 4, pp 303–305] or [17, Theorem 8.3, pp 181–182]. For (2.3), refer to [15, cf. § 5.e and § 14.A].

Introduce new variables $\mathbf{p} = -(\mathbf{A}\nabla u + u\mathbf{b})$ and $\mathbf{b}^* = \mathbf{A}^{-1}\mathbf{b}$ and rewrite (1.1) as a first-order system

$$\mathbf{A}^{-1}\mathbf{p} + u\mathbf{b}^* + \nabla u = 0 \ \text{ and } \ \operatorname{div}\mathbf{p} + \gamma\, u = f \ \text{ in } \Omega. \tag{2.4}$$

The mixed formulation seeks $(\mathbf{p}, u) \in H(\operatorname{div}, \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\mathbf{A}^{-1}\mathbf{p} + u\mathbf{b}^*, \mathbf{q})_{L^2(\Omega)} - (\operatorname{div}\mathbf{q}, u)_{L^2(\Omega)} &= 0 && \text{for all } \mathbf{q} \in H(\operatorname{div}, \Omega),\\ (\operatorname{div}\mathbf{p}, v)_{L^2(\Omega)} + (\gamma\, u, v)_{L^2(\Omega)} &= (f, v)_{L^2(\Omega)} && \text{for all } v \in L^2(\Omega). \end{aligned} \tag{2.5}$$

Theorem 2.1 (Equivalence of primal and mixed formulation) *The pair* $(\mathbf{p}, u) \in H(\operatorname{div}, \Omega) \times L^2(\Omega)$ *solves (2.5) if and only if* $u \in H^1_0(\Omega)$ *solves (1.1) and* $\mathbf{p} = -(\boldsymbol{A}\nabla u + u\mathbf{b})$.

Proof Let $(\mathbf{p}, u) \in H(\operatorname{div}, \Omega) \times L^2(\Omega)$ solve (2.5) and let $\Phi \in \mathcal{D}(\Omega)$. Since $\mathbf{q} :=$ Curl $\Phi := (-\partial\Phi/\partial x_2, \partial\Phi/\partial x_1)$ is divergence-free and an admissible test function in the first equation of (2.5), a formal integration by parts with curl defined for any smooth vector field $\mathbf{r} = (r_1, r_2)$ by curl $\mathbf{r} := \partial \mathrm{r}_1/\partial \mathrm{x}_2 - \partial \mathrm{r}_2/\partial \mathrm{x}_1$ proves

$$\operatorname{curl}\,(\mathbf{A}^{-1}\mathbf{p} + u\mathbf{b}^*) = 0 \text{ in } \mathcal{D}'(\Omega).$$

The Helmholtz decomposition shows for the simply-connected domain Ω that $A^{-1}\mathbf{p} + u\mathbf{b}^*$ is the gradient of some $v \in H^1_0(\Omega)$; namely,

$$\mathbf{A}^{-1}\mathbf{p} + u\mathbf{b}^* = \nabla v.$$

The substitution of this in the first equation of (2.5) followed by an integration by parts shows

$$(\operatorname{div} \mathbf{q}, v + u)_{L^2(\Omega)} = 0 \quad \text{for all} \quad \mathbf{q} \in H(\operatorname{div}, \Omega).$$

It is known that the divergence operator $\operatorname{div} : H(\operatorname{div}, \Omega) \to L^2(\Omega)$ is surjective and so the preceding identity proves $u + v = 0$. (A direct proof follows with the test function $\mathbf{q} = \nabla\psi$ for the solution $\psi \in H_0^1(\Omega)$ of the Poisson problem $-\Delta\psi = u + v$ in Ω.) This implies $u \in H_0^1(\Omega)$ and

$$\mathbf{A}^{-1}\mathbf{p} + u\mathbf{b}^* = -\nabla u. \tag{2.6}$$

This identity is recast into $\mathbf{p} = -(\mathbf{A}\nabla u + u\mathbf{b})$ so that the second equation of (2.5) leads to (1.1).

Conversely, let u be a solution to (1.1) and define $\mathbf{p} := -(\mathbf{A}\nabla u + u\mathbf{b}) \in L^2(\Omega; \mathbb{R}^2)$. Then (1.1) reads

$$\operatorname{div} \mathbf{p} + \gamma\, u = f \quad \text{in } \mathcal{D}'(\Omega).$$

Since $f - \gamma\, u \in L^2(\Omega)$, this implies $\mathbf{p} \in H(\operatorname{div}, \Omega)$ and the previous identity leads to

$$\operatorname{div} \mathbf{p} + \gamma\, u = f \quad \text{a.e. in } \Omega.$$

Now, an immediate consequence is the second identity in (2.5).

The definition of $\mathbf{p}$ is equivalent to (2.6). The multiplication of (2.6) with any $\mathbf{q} \in H(\operatorname{div}, \Omega)$ followed by an integration over the domain Ω leads on the right-hand side to the $L^2(\Omega)$ product of $-\nabla u$ and $\mathbf{q}$. That term allows for an integration by parts and so leads to the first identity in (2.5). This concludes the proof. □

The well-posedness of (1.1) states that $\mathcal{L} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is bounded and has a bounded inverse. This is an assumption on the coefficients which excludes zero eigenvalues in the Fredholm alternative, see [17, Sect. 8.2]. The system (1.1) is equivalent to (2.5) which implies that the operator

$$\mathcal{M} : \begin{cases} H(\operatorname{div}, \Omega) \times L^2(\Omega) \to H(\operatorname{div}, \Omega)^* \times L^2(\Omega), \\ (\mathbf{q}, v) \mapsto (\mathbf{A}^{-1}\mathbf{q} + v\mathbf{b}^* + \nabla v,\ \operatorname{div} \mathbf{q} + \gamma v) \end{cases} \tag{2.7}$$

has a range which includes $\{0\} \times L^2(\Omega)$; that is, for any $f \in L^2(\Omega)$ there exists $\mathcal{M}^{-1}(0, f)$, which solves (2.5) with the zero right-hand side in the first equation of (2.5). The *a posteriori* error analysis relies on the well-posedness of the operator $\mathcal{M}$ even with a general right-hand side $\mathbf{g} \in H(\operatorname{div}, \Omega)^*$ in the first equation of (2.5).

Theorem 2.2 (Well-posedness of mixed formulation) *The linear operator $\mathcal{M}$ from (2.7) is bounded and has a bounded inverse.*

Proof The injectivity follows from that of $\mathcal{L}$ and the equivalence of (1.1) and (2.5) in Theorem 2.1 for $\mathbf{g} = 0$. The more delicate surjectivity follows in several steps. The step one is that for $\mathbf{g} = 0$ and any $f \in L^2(\Omega)$, there exists some unique $\mathcal{M}^{-1}(0, f)$ in (2.7), because of the equivalence of (1.1) and (2.5).

In step two, let $\mathbf{g} = \nabla v$ be the gradient of some Sobolev function $v \in H_0^1(\Omega)$, i.e.,

$$\begin{aligned}\langle \mathbf{g}, \mathbf{q}\rangle_{H(\mathrm{div},\Omega)^*\times H(\mathrm{div},\Omega)} &= \int_\Omega \nabla v \cdot \mathbf{q}\, dx\\ &= -\int_\Omega v \,\mathrm{div}\, \mathbf{q}\, dx \quad \text{for all} \quad \mathbf{q} \in H(\mathrm{div}, \Omega).\end{aligned}$$

Then, $\mathcal{M}(\mathbf{p}, u) = (\mathbf{g}, f)$ is equivalent to

$$\mathbf{p} = \mathbf{A}\nabla(v - u) - u\mathbf{b} \quad \text{and} \quad \mathrm{div}\, \mathbf{p} + \gamma u = f.$$

The substitution of $\mathbf{p}$ in the second equation shows

$$-\mathrm{div}(\mathbf{A}\nabla u + u\mathbf{b}) + \gamma u = f - \mathrm{div}\,(\mathbf{A}\nabla v) \in H^{-1}(\Omega).$$

Since Eq. (1.1) has a unique weak solution for a given right-hand side in $H^{-1}(\Omega)$ (from (**A2**) and the Fredholm alternative), the previous equation has unique solution

$$u = \mathcal{L}^{-1}(f - \mathrm{div}\,(\mathbf{A}\nabla v)\,) \in H_0^1(\Omega).$$

Since

$$\mathbf{p} := \mathbf{A}\nabla(v - u) - u\mathbf{b} \in L^2(\Omega;\ \mathbb{R}^2)$$

satisfies $\mathrm{div}\, \mathbf{p} = f - \gamma u \in L^2(\Omega)$, it follows $\mathbf{p} \in H(\mathrm{div}, \Omega)$. Altogether,

$$\mathcal{M}(\mathbf{p}, u) = (\nabla v, f).$$

In step three, let $\mathbf{g} \in L^2(\Omega; \mathbb{R}^2) \subseteq H(\mathrm{div}, \Omega)^*$ and consider the Helmholtz decomposition of $\mathbf{g}$ in the format

$$\mathbf{A}\mathbf{g} = \mathbf{A}\nabla\alpha + \mathrm{Curl}\,\beta$$

for $\alpha \in H_0^1(\Omega)$ and $\beta \in H^1(\Omega)/\mathbb{R}$. This decomposition follows from the solution α of $-\mathrm{div}(\mathbf{A}\nabla\alpha) = -\mathrm{div}(\mathbf{A}\mathbf{g})$ and the fact that the divergence free function $\mathbf{A}(\mathbf{g} - \nabla\alpha)$ equals a rotation in the simply-connected domain Ω.

Since $\mathbf{g} = \nabla\alpha + \mathbf{A}^{-1}\mathrm{Curl}\,\beta$ and from step two, the superposition principle shows that it remains to verify that

$$\mathcal{M}(\mathbf{p}, u) = (\mathbf{A}^{-1}\mathrm{Curl}\,\beta, 0)$$

has a unique solution. Since div (Curl β) = 0, this is equivalent to

$$\mathcal{M}(\mathbf{p} - \mathrm{Curl}\,\beta, u) = 0$$

with the obvious solution $\mathbf{p} = \mathrm{Curl}\,\beta \in H(\mathrm{div}, \Omega)$ and $u = 0$.

In step four, let $\mathbf{g} = \nabla v$ for some $v \in L^2(\Omega)$ such that

$$\langle \mathbf{g}, \mathbf{q} \rangle_{H(\mathrm{div},\Omega)^* \times H(\mathrm{div},\Omega)} = -\int_\Omega v \,\mathrm{div}\,\mathbf{q}\,dx \quad \text{for all } \mathbf{q} \in H(\mathrm{div}, \Omega).$$

This generalizes the step two in the sense that $v \in L^2(\Omega)$. The equation $\mathcal{M}(\mathbf{p}, u) = (\nabla v, 0)$ is equivalent to

$$\mathcal{M}(\mathbf{p}, u - v) = (-v\mathbf{b}^*, -\gamma v).$$

This has a unique solution $(\mathbf{p}, u - v)$ in $H(\mathrm{div}, \Omega) \times L^2(\Omega)$, because of step three (owing to $(\mathbf{g}, f) \in L^2(\Omega; \mathbb{R}^2 \times \mathbb{R})$).

In step five, let $G \in H(\mathrm{div}, \Omega)^*$ with its Riesz representation $\mathbf{g} \in H(\mathrm{div}, \Omega)$ in the Hilbert space $H(\mathrm{div}, \Omega)$, i.e.,

$$\forall \mathbf{q} \in H(\mathrm{div}, \Omega) \qquad G(\mathbf{q}) = \int_\Omega (\mathbf{g} \cdot \mathbf{q} + \mathrm{div}\,\mathbf{g}\,\mathrm{div}\,\mathbf{q})\,dx.$$

Then, $\mathcal{M}(\mathbf{p}_1, u_1) = (\mathbf{g}, f)$ has a unique solution $(\mathbf{p}_1, u_1)$ from step three and $\mathcal{M}(\mathbf{p}_2, u_2) = (-\nabla \mathrm{div}\,\mathbf{g}, 0)$ has a unique solution $(\mathbf{p}_2, u_2)$ from step four with $v = \mathrm{div}\,\mathbf{g} \in L^2(\Omega)$. In conclusion, $(\mathbf{p}, u) := (\mathbf{p}_1 + \mathbf{p}_2, u_1 + u_2) = \mathcal{M}^{-1}(G, f)$. This concludes the proof. □

3 Non-conforming finite element methods

This section describes the Crouzeix–Raviart non-conforming finite element methods (NCFEM) for the problem (2.1) and discusses *a priori* error estimates.

3.1 Regular triangulation

Let $\mathcal{T}$ be a regular triangulation of the bounded simply-connected polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$ into triangles such that $\cup_{T\in\mathcal{T}} T = \overline{\Omega}$. Let $\mathcal{E}$ denote the set of all edges in $\mathcal{T}$, $\mathcal{E}(\partial\Omega)$ denote the set of all boundary edges in $\mathcal{T}$ and let $\mathcal{N}$ denote the set of vertices in $\mathcal{T}$. Let $\mathrm{mid}(E)$ denote the midpoint of the edge E and $\mathrm{mid}(T)$ denote the centroid of the triangle T. The set of edges of the element T is denoted by $\mathcal{E}(T)$. Let h_T denote the diameter of the element $T \in \mathcal{T}$ and $h_\mathcal{T} \in P_0(\mathcal{T})$ the piecewise constant mesh-size, $h_\mathcal{T}|_T := h_T$ for all $T \in \mathcal{T}$ with $h := \max_{T\in\mathcal{T}} h_T$. Let $|E|$ be the length of the edge $E \in \mathcal{E}$ with unit outward normal ν_E.

Let Π_0 be the L^2 projection onto $P_0(\mathcal{T})$ and define $osc(f, \mathcal{T}) := \|h_{\mathcal{T}}(1 - \Pi_0)f\|$, where

$$P_r(\mathcal{T}) = \{v \in L^2(\Omega) : \forall T \in \mathcal{T}, v|_T \in P_r(T)\}.$$

Here and throughout this paper, $P_r(T)$, denotes the algebraic polynomials of total degree at most $r \in \mathcal{N}$ as functions on the triangle $T \in \mathcal{T}$. The P_1 conforming finite element space reads

$$V(\mathcal{T}) := P_1(\mathcal{T}) \cap V, \quad \text{where} \quad V := H_0^1(\Omega).$$

The jump of $\mathbf{q}$ across E is denoted by $[\mathbf{q}]_E$; that is, for two neighbouring triangles T_+ and T_-,

$$[\mathbf{q}]_E(x) := (\mathbf{q}|_{T_+}(x) - \mathbf{q}|_{T_-}(x)) \quad \text{for} \quad x \in E = \partial T_+ \cap \partial T_-.$$

The sign of $[\mathbf{q}]_E$ is defined by the convention that there is a fixed orientation of ν_E pointing outside of T_+. Let $H^m(\mathcal{T})$ be the broken Sobolev space of order m with broken Sobolev norm

$$\|\cdot\|_{H^m(\mathcal{T})} := \left(\sum_{T \in \mathcal{T}} \|\cdot\|^2_{H^m(T)} \right)^{1/2}.$$

The piecewise gradient $\nabla_{NC} : H^1(\mathcal{T}) \longrightarrow L^2(\Omega; \mathbb{R}^2)$ acts as $\nabla_{NC} v|_T = \nabla v|_T$ for all $T \in \mathcal{T}$. The broken Sobolev norm $|||\cdot|||_{NC}$ abbreviates $(\mathbf{A}\nabla_{NC}\cdot, \nabla_{NC}\cdot)^{1/2}_{L^2(\Omega)}$ based on an underlying triangulation $\mathcal{T}$.

3.2 Crouzeix–Raviart non-conforming finite element methods

This subsection defines the non-conforming finite element spaces and discusses the solvability of the discrete problem and the related *a priori* error estimates.

Given $P_1(\mathcal{T})$, the non-conforming Crouzeix–Raviart (CR) finite element space reads

$$CR^1(\mathcal{T}) := \{v \in P_1(\mathcal{T}) : \forall E \in \mathcal{E}, \ v \ \text{is continuous at mid}(E)\},$$
$$CR_0^1(\mathcal{T}) := \{v \in CR^1(\mathcal{T}) : v(\text{mid}(E)) = 0 \ \text{for all } E \in \mathcal{E}(\partial\Omega)\}.$$

Let

$$\begin{aligned} a_{NC}(w_{CR}, v_{CR}) : &= \sum_{T \in \mathcal{T}} \int_T \Big((\mathbf{A}\nabla w_{CR} + w_{CR}\mathbf{b}) \cdot \nabla v_{CR} + \gamma\, w_{CR} v_{CR} \Big)\, dx \\ &= (\mathbf{A}\nabla_{NC} w_{CR} + w_{CR}\mathbf{b}, \nabla_{NC} v_{CR})_{L^2(\Omega)} + (\gamma\, w_{CR}, v_{CR})_{L^2(\Omega)}. \end{aligned} \tag{3.1}$$

The nonconforming finite element method for (2.1) seeks $u_{CR} \in CR_0^1(\mathcal{T})$ such that

$$a_{NC}(u_{CR}, v_{CR}) = (f, v_{CR}) \quad \text{for all } v_{CR} \in CR_0^1(\mathcal{T}). \tag{3.2}$$

Note that, $a_{NC}(v, w) = a(v, w) \quad$ for $v, w \in H^1(\Omega)$. Observe that there are positive constants α_A and M_A such that

$$\alpha_A \|v\|^2_{H^1(\mathcal{T})} \leq |||v|||^2_{NC} \leq M_A \|v\|^2_{H^1(\mathcal{T})} \quad \text{for all } v \in H_0^1(\Omega) + CR_0^1(\mathcal{T}). \tag{3.3}$$

The assumptions (**A1**) implies that, the bilinear form $a_{NC}(\cdot, \cdot)$ satisfies the following properties (i)–(ii).

(i) Boundedness. There exists a positive constant M such that

$$|a_{NC}(v, w)| \leq M\, |||v|||_{NC} \;\; |||w|||_{NC} \quad \text{for all} \quad v, w \in H_0^1(\Omega) + CR_0^1(\mathcal{T}). \tag{3.4}$$

(ii) Gårding-type inequality. There is a positive constant α and a nonnegative constant β such that

$$\alpha\, |||v|||^2_{NC} - \beta \|v\|^2 \leq a_{NC}(v, v) \quad \text{for all} \quad v \in H_0^1(\Omega) + CR_0^1(\mathcal{T}). \tag{3.5}$$

3.3 Existence and uniqueness of the solution to NCFEM

This subsection is devoted to a discussion on the unique solvability of the discrete problem (3.2). The conforming finite element approximation $\Phi_C \in V(\mathcal{T})$ to the problem (2.2) seeks $\Phi_C \in V(\mathcal{T})$ with

$$a(v_C, \Phi_C) = (g, v_C) \;\; \text{for all} \;\; v_C \in V(\mathcal{T}). \tag{3.6}$$

A simple modification of arguments given in [21, Theorem 2] leads to the following error estimate. Given any $\epsilon > 0$, there exists an $h_1 = h_1(\epsilon) > 0$ such that for $0 < h \leq h_1$, if $\Phi \in H_0^1(\Omega)$ is a solution to (2.2) and $\Phi_C \in V(\mathcal{T})$ satisfies (3.6), then it holds

$$\|\Phi - \Phi_C\| \leq \epsilon \|\Phi - \Phi_C\|_1. \tag{3.7}$$

Since $g \in L^2(\Omega)$,

$$\|\Phi - \Phi_C\|_1 \leq \epsilon \|g\|. \tag{3.8}$$

Under the extra regularity assumption that the solutions $u, \Phi \in H^{1+\delta}(\Omega) \cap H_0^1(\Omega)$, it holds [21]

$$\|\Phi - \Phi_C\| \leq C h^{\delta} \|\Phi - \Phi_C\|_1. \tag{3.9}$$

Since $g \in L^2(\Omega)$,

$$\|\Phi - \Phi_C\|_1 \leq Ch^\delta \|g\|. \tag{3.10}$$

Also, the conforming finite element solution to (2.1) $u_C \in V(\mathcal{T})$ satisfies

$$\|u - u_C\| \leq Ch^\delta \|u - u_C\|_1. \tag{3.11}$$

Since $f \in L^2(\Omega)$,

$$\|u - u_C\|_1 \leq Ch^\delta \|f\|. \tag{3.12}$$

The nonconforming finite element method (3.2) is well-posed even for more general right-hand sides.

Theorem 3.1 (Stability) *For sufficiently small maximum mesh size h and for all* $f_0 \in L^2(\Omega)$ *and* $\mathbf{f}_1 \in L^2(\Omega; \mathbb{R}^2)$, *the discrete problem*

$$a_{NC}(u_{CR}, v_{CR}) = (f_0, v_{CR}) + (\mathbf{f}_1, \nabla_{NC} v_{CR}) \quad \text{for all } v_{CR} \in CR_0^1(\mathcal{T}), \tag{3.13}$$

has a unique solution $u_{CR} \in CR_0^1(\mathcal{T})$. *Furthermore, the solution is stable in the sense that*

$$|\!|\!| u_{CR} |\!|\!|_{NC} \lesssim \|f_0\| + \|\mathbf{f}_1\|. \tag{3.14}$$

One of the key arguments in the proof of Theorem 3.1 is the following consistency condition.

Lemma 3.2 (Consistency) *Let* Φ *be the unique solution to (2.2) with the right-hand side* $g \in L^2(\Omega)$. *For* $\epsilon > 0$, *there exists some* $h_2 > 0$ *such that for* $0 < h \leq h_2$ *it holds*

$$\sup_{0 \neq v_{CR} \in CR_0^1(\mathcal{T})} \frac{|a_{NC}(v_{CR}, \Phi) - (g, v_{CR})_{L^2(\Omega)}|}{|\!|\!| v_{CR} |\!|\!|_{NC}} \leq \epsilon \|g\|. \tag{3.15}$$

Proof Given $v_{CR} \in CR_0^1(\mathcal{T})$, define a conforming approximation by the averaging of the possible values (also known as the precise representation)

$$v_1(z) := v_{CR}^*(z) := \lim_{\delta \to 0} \frac{1}{|B(z, \delta)|} \int_{B(z,\delta)} v_{CR} dx$$

of the (possibly) discontinuous v_{CR} at any interior node $z \in \mathcal{N}$, where $B(z, \delta)$ is a ball of radius δ at z. Linear interpolation of those values defines $v_1 \in V(\mathcal{T})$. The second step defines $v_2 \in P_2(\mathcal{T}) \cap C_0(\bar{\Omega}) \subset H_0^1(\Omega)$ which equals v_1 at all nodes $\mathcal{N}$ and satisfies

$$\int_E v_{CR}\, ds = \int_E v_2\, ds \quad \text{for all} \quad E \in \mathcal{E}.$$

The third step adds the cubic bubble-functions to v_2 such that the resulting function $v_3 \in P_3(\mathcal{T}) \cap C_0(\bar{\Omega}) \subset H_0^1(\Omega)$ equals v_2 along the edges and satisfies

$$\int_T v_{CR}\, dx = \int_T v_3\, dx \quad \text{for all } T \in \mathcal{T}. \tag{3.16}$$

An integration by parts shows

$$\int_T \nabla v_{CR}\, dx = \int_T \nabla v_3\, dx \quad \text{for all } T \in \mathcal{T}. \tag{3.17}$$

The approximation and stability properties of v_3 has been studied in former work of preconditioners for nonconforming FEM [4] (called enrichment therein). This along with standard arguments also proves approximation properties and stability in the sense that

$$||h_{\mathcal{T}}^{-1}(v_3 - v_{CR})|| + |||v_3|||_{NC} \le C_1\, |||v_{CR}|||_{NC}\,. \tag{3.18}$$

With (3.1), (2.2), (3.16)–(3.17) and the definition of Π_0, it follows that

$$\begin{aligned} a_{NC}(v_{CR}, \Phi) &- (g, v_{CR})_{L^2(\Omega)} \\ &= (\mathbf{A}\nabla\Phi, \nabla_{NC} v_{CR})_{L^2(\Omega)} + (\mathbf{b}\cdot\nabla\Phi + \gamma\Phi - g, v_{CR})_{L^2(\Omega)} \\ &= (\Pi_0(\mathbf{A}\nabla\Phi), \nabla v_3)_{L^2(\Omega)} + (\mathbf{b}\cdot\nabla\Phi + \gamma\Phi - g, v_{CR})_{L^2(\Omega)} \\ &= -((1-\Pi_0)(\mathbf{A}\nabla\Phi), \nabla v_3)_{L^2(\Omega)} \\ &\quad +((1-\Pi_0)(\mathbf{b}\cdot\nabla\Phi + \gamma\Phi - g), v_{CR} - v_3)_{L^2(\Omega)}. \end{aligned}$$

The Cauchy–Schwarz inequality with (3.18) yields

$$\begin{aligned} &a_{NC}(v_{CR}, \Phi) - (g, v_{CR})_{L^2(\Omega)} \\ &\quad \le \|(1-\Pi_0)(\mathbf{A}\nabla\Phi)\|\, \|v_3\|_1 + C_1 osc(g - \gamma\Phi - \mathbf{b}\cdot\nabla\Phi, \mathcal{T})\, |||v_{CR}|||_{NC}\,. \end{aligned}$$

This and the aforementioned stability $\|v_3\|_1 \le C_1\, |||v_{CR}|||_{NC}$ prove

$$\begin{aligned} \sup_{0\neq v_{CR}\in CR_0^1(\mathcal{T})} &\frac{|a_{NC}(v_{CR}, \Phi) - (g, v_{CR})_{L^2(\Omega)}|}{|||v_{CR}|||_{NC}} \\ &\le C_1\|(1-\Pi_0)(\mathbf{A}\nabla\Phi)\| + C_1 osc(g - \gamma\Phi - \mathbf{b}\cdot\nabla\Phi, \mathcal{T}). \end{aligned} \tag{3.19}$$

The approximation property of Π_0 proves that the first term on the right-hand side of (3.19) is bounded by

$$\begin{aligned} \|(1-\Pi_0)(\mathbf{A}\nabla\Phi)\| \le \|(\mathbf{A} - \Pi_0\mathbf{A})\nabla\Phi\| &+ \|\Pi_0\mathbf{A}(\nabla\Phi - \Pi_0\nabla\Phi)\| \\ &+ \|\Pi_0((\Pi_0\mathbf{A})\nabla\Phi) - \Pi_0(\mathbf{A}\nabla\Phi)\| \end{aligned}$$

$$\le 2\|(1-\Pi_0)\mathbf{A}\|_\infty \|\nabla\Phi\| + \|\mathbf{A}\|_\infty \|(1-\Pi_0)\nabla\Phi\|$$
$$\le 2C\|(1-\Pi_0)\mathbf{A}\|_\infty \|g\| + \|\mathbf{A}\|_\infty \|(1-\Pi_0)\nabla\Phi\|. \quad (3.20)$$

Given $\epsilon > 0$, from (3.12) there exists $h_3 = h_3(\epsilon) > 0$ such that for $0 < h \le h_3$

$$\|(1-\Pi_0)\nabla\Phi\| \le \|\Phi - \Phi_C\|_1 \le \frac{\epsilon}{4C_1\|\mathbf{A}\|_\infty}\|g\|,$$

and $\|(1-\Pi_0)\mathbf{A}\|_\infty \le \frac{\epsilon}{8CC_1}$. The boundedness of $\Phi \in H_0^1(\Omega)$ by $\|g\|$ shows

$$osc(g - \gamma\Phi - \mathbf{b}\cdot\nabla\Phi, \mathcal{T}) \le \|h(g - \gamma\Phi - \mathbf{b}\cdot\nabla\Phi)\| \le C_2 h\|g\|.$$

For $\epsilon > 0$, there exists an $h_4 > 0$ such that for $0 < h < h_4$, $osc(g-\gamma\Phi-\mathbf{b}\cdot\nabla\Phi, \mathcal{T}) \le \epsilon/2\|g\|$. Alltogether for $\epsilon > 0$, there exists $0 < h_2 \le \min\{h_3, h_4\}$ such that (3.15) holds. This concludes the proof. □

Proof of Theorem 3.1 The choice $v_{CR} = u_{CR}$ in (3.13), the Gårding's inequality (3.5), (3.3) and the discrete Friedrich inequality [3, pp 301] $\|u_{CR}\| \le C_{dF}\|u_{CR}\|_{H^1(\mathcal{T})}$ imply

$$\alpha \, |||u_{CR}|||_{NC}^2 \le \beta\|u_{CR}\|^2 + \frac{1}{\alpha_A}\Big(C_{dF}\|f_0\| + \|\mathbf{f}_1\|\Big)\, |||u_{CR}|||_{NC}\,, \quad (3.21)$$

Hence,

$$|||u_{CR}|||_{NC} \le \frac{C_{dF}\beta}{\alpha}\|u_{CR}\| + \frac{1}{\alpha\alpha_A}\Big(C_{dF}\|f_0\| + \|\mathbf{f}_1\|\Big). \quad (3.22)$$

The Aubin–Nitsche duality argument allows for an estimate of $\|u_{CR}\|$. Since $\mathcal{L}$ is an isomorphism, the dual problem (2.2) has a unique solution $\Phi \in H_0^1(\Omega)$, which satisfies $\|\Phi\|_1 \le C\|g\|$. The conforming finite element solution Φ_C of (2.2) satisfies (3.6) for all $g \in L^2(\Omega)$. Since $V(\mathcal{T}) \subset CR_0^1(\mathcal{T})$, (3.13) shows for $v_{CR} = \Phi_C$ that

$$a_{NC}(u_{CR}, \Phi_C) = (f_0, \Phi_C) + (\mathbf{f}_1, \nabla_{NC}\Phi_C). \quad (3.23)$$

Elementary algebra and (3.23) show

$$\begin{aligned}(g, u_{CR})_{L^2(\Omega)} &= a_{NC}(u_{CR}, \Phi - \Phi_C) + (g, u_{CR})_{L^2(\Omega)} - a_{NC}(u_{CR}, \Phi)\\ &\quad + (f_0, \Phi_C) + (\mathbf{f}_1, \nabla_{NC}\Phi_C)\\ &\le M\, |||u_{CR}|||_{NC}\, \|\Phi - \Phi_C\|_1 + \Big(C_{dF}\|f_0\| + \|\mathbf{f}_1\|\Big)\|\Phi_C\|_1\\ &\quad + |||u_{CR}|||_{NC} \sup_{0 \ne v_{CR} \in CR_0^1(\mathcal{T})} \frac{|a_{NC}(v_{CR}, \Phi) - (g, v_{CR})_{L^2(\Omega)}|}{|||v_{CR}|||_{NC}}.\end{aligned}$$

For $\epsilon > 0$, there exists an $h_5 = h_5(\epsilon) > 0$ such that the first term on the right-hand side is made $\leq \frac{\alpha}{2C_{dF}M\beta}\epsilon \,|||u_{CR}|||_{NC}\, \|g\|$ and from Lemma 3.2, the third term can be made $\leq \frac{\alpha}{2C_{dF}\beta}\epsilon\, |||u_{CR}|||_{NC}\, \|g\|$. The choice of $g = u_{CR}$ proves

$$\|u_{CR}\| \leq \frac{\alpha\epsilon}{C_{dF}\beta}\, |||u_{CR}|||_{NC} + C(C_{dF}\|f_0\| + \|\mathbf{f}_1\|).$$

For $0 < \epsilon < 1$, (3.22) results in

$$|||u_{CR}|||_{NC} \lesssim \|f_0\| + \|\mathbf{f}_1\|.$$

This proves the stability estimate (3.14) under the assumption that (3.13) has a solution. The bound (3.14) implies also the uniqueness of solution to (3.13). In fact, if the linear system of equations had a non-trivial kernel, there would exist unbounded solutions in contradiction to (3.14). □

3.4 *A priori* error estimates for NCFEM

This subsection discusses *a priori* error bounds for the non-conforming finite element solution. For related estimates, see [11] which discusses the existence, uniqueness and error estimates for general elliptic PDE with regularity assumption $u \in H^{1+\delta}$, where $\delta \in (\frac{1}{2}, 1]$ for non-conforming FEM, whereas we consider the case when $\delta \in (0, 1]$. The *a priori* error estimates in energy and L^2 norms under the regularity assumption that $u \in H^{1+\delta}$ for any $\delta > 0$ for self-adjoint and positive definite elliptic problems are discussed in [18]. The following L^2 error control for nonconforming FEMs has been observed in [10, Eq. (3.16)] but is left without a proof and stated under the restrictive assumption $\gamma \geq 0$.

Theorem 3.3 (L^2 and H^1 error estimates) *Let $u \in H_0^1(\Omega)$ be the unique weak solution to (2.1), let u_{CR} be the solution to (3.2). Then, for $\epsilon > 0$, there exists sufficiently small mesh-size h such that*

$$\|u - u_{CR}\| \leq \epsilon\, |||u - u_{CR}|||_{NC}\,, \tag{3.24}$$

and for $f \in L^2(\Omega)$,

$$|||u - u_{CR}|||_{NC} \leq \epsilon \|f\|. \tag{3.25}$$

Proof The Aubin–Nitsche duality technique for $g \in L^2(\Omega)$ plus (2.2) and (3.12) and some direct calculations prove, for any $v_C \in V(\mathcal{T})$, that

$$\begin{aligned}
&(g, u - u_{CR})_{L^2(\Omega)}\\
&\quad = a_{NC}(u - u_{CR}, \Phi - \Phi_C) + \Big(a_{NC}(u_{CR} - v_C, \Phi) - (g, u_{CR} - v_C)_{L^2(\Omega)}\Big)\\
&\quad \leq M\, |||u - u_{CR}|||_{NC}\, \|\Phi - \Phi_C\|_1
\end{aligned}$$

$$
\begin{aligned}
&+ \|u_{CR} - v_C\|_{NC} \sup_{0 \neq w_{CR} \in CR_0^1(\mathcal{T})} \frac{|a_{NC}(w_{CR}, \Phi) - (g, w_{CR})_{L^2(\Omega)}|}{\|w_{CR}\|_{NC}} \\
&\leq \frac{\epsilon}{2} \|u - u_{CR}\|_{NC} \ \|g\| \\
&+ \inf_{v_C \in V(\mathcal{T})} \|u_{CR} - v_C\|_{NC} \sup_{0 \neq w_{CR} \in CR_0^1(\mathcal{T})} \frac{|a_{NC}(w_{CR}, \Phi) - (g, w_{CR})_{L^2(\Omega)}|}{\|w_{CR}\|_{NC}}.
\end{aligned} \tag{3.26}
$$

Since [9]

$$\inf_{v_C \in V(\mathcal{T})} \|u_{CR} - v_C\|_{NC} \leq C_3 \|u - u_{CR}\|_{NC}$$

for sufficiently small mesh size h, the consistency condition (3.15) in (3.26) imply

$$(g, u - u_{CR})_{L^2(\Omega)} \leq \epsilon \ \|u - u_{CR}\|_{NC} \ \|g\|.$$

Hence,

$$\|u - u_{CR}\| = \sup_{0 \neq g \in L^2(\Omega)} \frac{|(g, u - u_{CR})_{L^2(\Omega)}|}{\|g\|} \leq \epsilon \ \|u - u_{CR}\|_{NC}. \tag{3.27}$$

This concludes the proof of (3.24).

Given any $v_C \in V(\mathcal{T}) \subset CR_0^1(\mathcal{T})$, the Gårding-type inequality (3.5) shows

$$
\begin{aligned}
\alpha \|u_{CR} - v_C\|_{NC}^2 - \beta \|u_{CR} - v_C\|^2 &\leq a_{NC}(u_{CR} - v_C, u_{CR} - v_C) \\
&= a_{NC}(u - v_C, u_{CR} - v_C) + \Big((f, u_{CR} - v_C)_{L^2(\Omega)} - a_{NC}(u, u_{CR} - v_C)\Big).
\end{aligned}
$$

The discrete Friedrichs inequality $\|u_{CR} - v_C\| \leq C_{dF} \|u_{CR} - v_C\|_{NC}$ leads to

$$
\begin{aligned}
\alpha \|u_{CR} - v_C\|_{NC} &\leq C_{dF} \beta \|u_{CR} - v_C\| + M \|u - v_C\|_1 \\
&\quad + \sup_{0 \neq w_{CR} \in CR_0^1(\mathcal{T})} \frac{|a_{NC}(u, w_{CR}) - (f, w_{CR})_{L^2(\Omega)}|}{\|w_{CR}\|_{NC}}.
\end{aligned}
$$

Write $u - u_{CR} := (u - v_C) - (u_{CR} - v_C)$ for an arbitrary v_C in $V(\mathcal{T})$. The preceding estimates plus triangle inequality show

$$
\begin{aligned}
\|u - u_{CR}\|_{NC} &\leq \frac{C_{dF} \beta}{\alpha} \|u - u_{CR}\| + \Big(\frac{C_{dF} \beta}{\alpha} + 1 + \frac{M}{\alpha}\Big) \inf_{v_C \in V(\mathcal{T})} \|u - v_C\|_1 \\
&\quad + \frac{1}{\alpha} \sup_{0 \neq w_{CR} \in CR_0^1(\mathcal{T})} \frac{|a_{NC}(u, w_{CR}) - (f, w_{CR})_{L^2(\Omega)}|}{\|w_{CR}\|_{NC}}.
\end{aligned} \tag{3.28}
$$

The last term is controlled with Lemma 3.2 which remains valid for $u \in H_0^1(\Omega)$ and for all $f \in L^2(\Omega)$.

The error analysis of [21, Theorem 2], shows for any $\epsilon > 0$, that there exists an $h_6 = h_6(\epsilon) > 0$ such that for $0 < h \leq h_6$, the conforming finite element solution $u_C \in V(\mathcal{T})$ to (2.1) satisfies

$$\inf_{v_C \in V(\mathcal{T})} \|u - v_C\|_1 \leq \|u - u_C\|_1 \leq \epsilon \, \|f\|. \tag{3.29}$$

The combination of (3.27), (3.29) and (3.15) implies (3.25) for sufficiently small h. This concludes the proof. □

3.5 *A posteriori* error analysis for NCFEM

This subsection is devoted to *a posteriori* error analysis of NCFEM with the residual

$$\mathcal{R}es_{NC}(w) := (f, w)_{L^2(\Omega)} - a_{NC}(u_{CR}, w) \qquad \text{for all } w \in V + CR_0^1(\mathcal{T}). \tag{3.30}$$

Theorem 3.4 (*A posteriori* error control) *Provided the mesh-size is sufficiently small, it holds*

$$|||u - u_{CR}|||_{NC} \lesssim \|\mathcal{R}es_{NC}\|_{H^{-1}(\Omega)} + \min_{v \in V} |||u_{CR} - v|||_{NC}\,. \tag{3.31}$$

Proof The Gårding's inequality (3.5) for $e := u - u_{CR}$ plus elementary algebra with the bilinear forms a and a_{NC} plus (2.1) for $w := u - v_4$ with any $v_4 \in V$ satisfies

$$\alpha \, |||e|||_{NC}^2 - \beta \|e\|^2 \leq (f, w)_{L^2(\Omega)} - a_{NC}(u_{CR}, w) + a_{NC}(e, v_4 - u_{CR}). \tag{3.32}$$

The inequality (3.32) and the nonconforming residual $\mathcal{R}es_{NC}(w)$ of (3.30) show

$$|||e|||_{NC}^2 \leq \frac{\beta}{\alpha}\|e\|^2 + \frac{1}{\alpha}\mathcal{R}es_{NC}(w) + \frac{M}{\alpha}\, |||e|||_{NC}\, |||u_{CR} - v_4|||_{NC}\,. \tag{3.33}$$

The dual norm, (3.3) and triangle inequality imply

$$\mathcal{R}es_{NC}(w) = \mathcal{R}es_{NC}(u - v_4) \leq \frac{1}{\alpha_A}\, |||\mathcal{R}es_{NC}|||_{H^{-1}(\Omega)} \,(|||e|||_{NC} + |||u_{CR} - v_4|||_{NC}).$$

The inequality (3.33) with the above equation proves

$$|||e|||_{NC}^2 \leq \frac{2\beta}{\alpha}\|e\|^2 + \frac{3}{(\alpha\alpha_A)^2}\, |||\mathcal{R}es_{NC}|||_{H^{-1}(\Omega)}^2 + \left(\frac{2M^2}{\alpha^2} + 1\right) |||u_{CR} - v_4|||_{NC}^2\,.$$

Since $v_4 \in V$ is arbitrary and Theorem 3.3 shows $\|e\| \leq \frac{\alpha\epsilon}{2\beta}\, |||e|||_{NC}$, for $\epsilon > 0$ with $0 < \epsilon < 1$, there exists a sufficiently small mesh-size $\|h_{\mathcal{T}}\|_{L^\infty(\Omega)} << 1$ such that

$$|||e|||_{NC}^2 \leq \frac{3}{(\alpha\alpha_A)^2}\, |||\mathcal{R}es_{NC}|||_{H^{-1}(\Omega)}^2 + \left(\frac{2M^2}{\alpha^2} + 1\right) \min_{v \in V} |||u_{CR} - v|||_{NC}^2\,.$$

This implies (3.31) and concludes the proof. □

The analysis of the residual $\mathcal{R}es_{NC} \in H^{-1}(\Omega)$ with the kernel property $P_1(\mathcal{T}) \cap C_0(\bar{\Omega}) \subseteq CR_0^1(\mathcal{T}) \subseteq \mathcal{K}er\mathcal{R}es_{NC}$ is by now standard [7,8]. With $\mathbf{p}_{CR} := -(\mathbf{A}\nabla_{NC}u_{CR} + u_{CR}\mathbf{b})$, the explicit residual-based error estimator of [7] reads

$$\eta(\mathcal{T}) := \|h_{\mathcal{T}}(f - \gamma u_{CR} - \mathrm{div}_{NC}\mathbf{p}_{CR})\| + \|h_E^{1/2}[\mathbf{p}_{CR}]_E \cdot \nu_E\|_{L^2(\cup E)}. \quad (3.34)$$

Further details are, therefore, omitted. The residual $\min_{v\in V} |||u_{CR} - v|||_{NC}$ is easily estimated [9].

4 Mixed finite element methods

This section discusses the lowest-order Raviart–Thomas mixed finite element formulation and its equivalence to the NCFEM solution and derives *a priori* error estimates for the mixed method.

4.1 Raviart–Thomas finite element methods (RTFEM)

With respect to the shape-regular triangulation $\mathcal{T}$, the lowest-order Raviart–Thomas space reads

$$RT_0(\mathcal{T}) := \{\mathbf{q} \in H(\mathrm{div}, \Omega) : \forall T \in \mathcal{T}\ \exists \mathbf{c} \in \mathbb{R}^2\ \exists d \in \mathbb{R}\ \ \forall \mathbf{x} \in T,\ \mathbf{q}(\mathbf{x}) = \mathbf{c} + d\,\mathbf{x} \\ \text{and } \ \forall E \in \mathcal{E}(\Omega), [\mathbf{q}]_E \cdot \nu_E = 0\}.$$

Throughout this paper, $\mathbf{A}_h := \Pi_0\mathbf{A}$, $\mathbf{b}_h := \Pi_0\mathbf{b}$, $\mathbf{b}_h^* := \mathbf{A}_h^{-1}\mathbf{b}_h$, $\gamma_h := \Pi_0\gamma$, and $f_h := \Pi_0 f$ denote the respective piecewise constant approximations of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{b}^*$, γ and f. The discrete mixed finite element problem (RTFEM) for (2.5) seeks $(\mathbf{p_M}, u_M) \in RT_0(\mathcal{T}) \times P_0(\mathcal{T})$ with

$$(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*, \mathbf{q}_{RT})_{L^2(\Omega)} - (\mathrm{div}\,\mathbf{q}_{RT}, u_M)_{L^2(\Omega)} = 0 \ \text{ for all } \mathbf{q}_{RT} \in RT_0(\mathcal{T}), \quad (4.1)$$

$$(\mathrm{div}\,\mathbf{p_M}, v_h)_{L^2(\Omega)} + (\gamma_h u_M, v_h)_{L^2(\Omega)} = (f_h, v_h)_{L^2(\Omega)} \text{ for all } v_h \in P_0(\mathcal{T}). \quad (4.2)$$

4.2 Equivalence of RTFEM and NCFEM

The piecewise constant approximations $\mathbf{A}_h$ and $\mathbf{b_h}$ of $\mathbf{A}$ and $\mathbf{b}$ and

$$\tilde{u}_M(\mathbf{x}) = \left(1 + \frac{S(T)}{4}\gamma_h\right)^{-1}\left(\Pi_0\tilde{u}_{CR} + \frac{S(T)}{4}f_h\right) \quad \text{for } x \in T \in \mathcal{T}, \quad (4.3)$$

$$S(T) = \frac{1}{|T|}\int_T (\mathbf{x} - \mathrm{mid}(T)) \cdot \mathbf{A}_h^{-1}(\mathbf{x} - \mathrm{mid}(T))\,d\mathbf{x} \ \text{ for } T \in \mathcal{T}, \quad (4.4)$$

define a modified nonconforming FEM problem

$$
\begin{aligned}
&(\mathbf{A}_h \nabla_{NC} \tilde{u}_{CR} + \tilde{u}_M \mathbf{b}_h, \nabla_{NC} v_{CR}) + (\gamma_h \tilde{u}_M, v_{CR}) \\
&= (f_h, v_{CR}) \quad \text{for all } v_{CR} \in CR_0^1(\mathcal{T}).
\end{aligned}
\tag{4.5}
$$

Theorem 4.1 (Stability) *For sufficiently small mesh-size h, there exists a unique solution $\tilde{u}_{CR} \in CR_0^1(\mathcal{T})$ to the discrete problem (4.5) with*

$$
|||\tilde{u}_{CR}|||_{NC} \lesssim \|f_h\|. \tag{4.6}
$$

Proof A substitution of $\tilde{u}_M$ in (4.5) leads to

$$
\tilde{a}_{NC}(\tilde{u}_{CR}, v_{CR}) = (\tilde{f}_h, v_{CR}) \ \text{ for all } v_{CR} \in CR_0^1(\mathcal{T}) \tag{4.7}
$$

where

$$
\begin{aligned}
\tilde{a}_{NC}(\tilde{u}_{CR}, v_{CR}) :=& \left(\mathbf{A}_h \nabla_{NC} \tilde{u}_{CR} + \left(1 + \frac{S(\mathcal{T})}{4}\gamma_h\right)^{-1} (\Pi_0 \tilde{u}_{CR}) \mathbf{b}_h, \nabla_{NC} v_{CR} \right)_{L^2(\Omega)} \\
&+ (\gamma_h (1 + \frac{S(\mathcal{T})}{4}\gamma_h)^{-1} (\Pi_0 \tilde{u}_{CR}), v_{CR})_{L^2(\Omega)},
\end{aligned}
\tag{4.8}
$$

$S(\mathcal{T})|_T := S(T)$ for all $T \in \mathcal{T}$ and

$$
\begin{aligned}
(\tilde{f}_h, v_{CR})_{L^2(\Omega)} :=& (f_h, v_{CR})_{L^2(\Omega)} - \left(\left(1 + \frac{S(\mathcal{T})}{4}\gamma_h\right)^{-1} \frac{S(\mathcal{T})}{4} f_h \mathbf{b}_h, \nabla_{NC} v_{CR} \right)_{L^2(\Omega)} \\
&- (\gamma_h (1 + \frac{S(\mathcal{T})}{4}\gamma_h)^{-1} \frac{S(\mathcal{T})}{4} f_h, v_{CR})_{L^2(\Omega)}.
\end{aligned}
\tag{4.9}
$$

The stiffness matrix related to (4.7) is very similar to that of (3.2) except for some data perturbation and the substitution of $\Pi_0 \tilde{u}_{CR}$ instead of u_{CR} in two lower-order terms. The last substitution models one-point integration, and since the variable $\tilde{u}_{CR}$ is controlled in the energy norm $|||\cdot|||_{NC}$, it acts as some perturbation as well. All these perturbations tends to zero as the maximal mesh-size tends to zero and hence, the existence, uniqueness and stability results may be deduced as in Sect. 3.3.

To be more specific, the choice of $v_{CR} = \tilde{u}_{CR}$ in (4.7) and the Gårding-type inequality result in

$$
|||\tilde{u}_{CR}|||_{NC}^2 \lesssim \tilde{a}_{NC}(\tilde{u}_{CR}, \tilde{u}_{CR}) + \|\tilde{u}_{CR}\|^2 \tag{4.10}
$$

A use of (4.7) and $\|\tilde{u}_{CR}\| \lesssim |||\tilde{u}_{CR}|||$ yields

$$
|||\tilde{u}_{CR}|||_{NC} \lesssim \|\tilde{f}_h\| + \|\tilde{u}_{CR}\|. \tag{4.11}
$$

The Aubin–Nitsche duality argument allows for an estimate of $\|\tilde{u}_{CR}\|$. Recall that for given $g \in L^2(\Omega)$, $\Phi \in H_0^1(\Omega)$ is the unique solution to the dual problem $a(v, \Phi) =$

(g, v) from Sect. 3.3 and the conforming finite element solution Φ_C of (3.6) satisfies the estimate (3.12).

Since $V(\mathcal{T}) \subset CR_0^1(\mathcal{T})$, the choice of $v_{CR} = \Phi_C$ in (4.7) yields

$$\tilde{a}_{NC}(\tilde{u}_{CR}, \Phi_C) = (\tilde{f}_h, \Phi_C). \tag{4.12}$$

An elementary algebra with (4.12) and the discrete Friedrich inequality shows

$$\begin{aligned}(g, \tilde{u}_{CR})_{L^2(\Omega)} &= \tilde{a}_{NC}(\tilde{u}_{CR}, \Phi - \Phi_C) + (\tilde{f}_h, \Phi_C) + (g, \tilde{u}_{CR})_{L^2(\Omega)} - \tilde{a}_{NC}(\tilde{u}_{CR}, \Phi)\\ &\lesssim \|\!|\tilde{u}_{CR}|\!\|_{NC} \, \|\Phi - \Phi_C\|_1 + \|\tilde{f}_h\| \, \|\Phi_C\|_1\\ &\quad + \|\!|\tilde{u}_{CR}|\!\|_{NC} \sup_{0 \neq v_{CR} \in CR_0^1(\mathcal{T})} \frac{|\tilde{a}_{NC}(v_{CR}, \Phi) - (g, v_{CR})|}{\|\!|v_{CR}|\!\|_{NC}}.\end{aligned} \tag{4.13}$$

The last term on the right-hand side of (4.13) is

$$\begin{aligned}&\tilde{a}_{NC}(v_{CR}, \Phi) - (g, v_{CR})_{L^2(\Omega)}\\ &= a_{NC}(v_{CR}, \Phi) - (g, v_{CR})_{L^2(\Omega)} - (\nabla_{NC} v_{CR}, (\mathbf{A} - \mathbf{A}_h)\nabla\Phi)_{L^2(\Omega)}\\ &\quad - (v_{CR}, (\mathbf{b} - \mathbf{b}_h) \cdot \nabla\Phi + (\gamma - \gamma_h)\Phi)_{L^2(\Omega)} - (v_{CR} - \Pi_0 v_{CR}, \mathbf{b}_h \cdot \nabla\Phi\\ &\quad + \gamma_h \Phi)_{L^2(\Omega)} - \left(\frac{S(\mathcal{T})}{4}\gamma_h \left(1 + \frac{S(\mathcal{T})}{4}\gamma_h\right)^{-1} \Pi_0 v_{CR}, \mathbf{b}_h \cdot \nabla\Phi + \gamma_h \Phi\right)_{L^2(\Omega)}.\end{aligned}$$

The Cauchy–Schwarz inequality, the approximation property of Π_0 and $S(T) \approx h^2$ lead to

$$\begin{aligned}&\sup_{0 \neq v_{CR} \in CR_0^1(\mathcal{T})} \frac{|\tilde{a}_{NC}(v_{CR}, \Phi) - (g, v_{CR})_{L^2(\Omega)}|}{\|\!|v_{CR}|\!\|_{NC}}\\ &\qquad \lesssim \sup_{0 \neq v_{CR} \in CR_0^1(\mathcal{T})} \frac{|a_{NC}(v_{CR}, \Phi) - (g, v_{CR})_{L^2(\Omega)}|}{\|\!|v_{CR}|\!\|_{NC}}\\ &\qquad\quad + \left(h + \|\mathbf{A} - \mathbf{A_h}\|_\infty + \|\mathbf{b} - \mathbf{b_h}\|_\infty + \|\gamma - \gamma_h\|_\infty\right)\|\Phi\|_1.\end{aligned}$$

Lemma 3.2, $\|\mathbf{A} - \mathbf{A_h}\|_\infty \le \epsilon$, $\|\mathbf{b} - \mathbf{b_h}\|_\infty \le \epsilon$, $\|\gamma - \gamma_h\|_\infty \le \epsilon$ for $\epsilon > 0$ and $\|\Phi\|_1 \le C\|g\|$ result in

$$\sup_{0 \neq v_{CR} \in CR_0^1(\mathcal{T})} \frac{|\tilde{a}_{NC}(v_{CR}, \Phi) - (g, v_{CR})_{L^2(\Omega)}|}{\|\!|v_{CR}|\!\|_{NC}} \lesssim \epsilon\|g\|. \tag{4.14}$$

The combination with (3.12) and (4.13)–(4.14) leads to $(g, \tilde{u}_{CR}) \lesssim (\epsilon \ \|\!|\tilde{u}_{CR}|\!\|_{NC} + \|\tilde{f}_h\|) \ \|g\|$. Hence, the boundedness of $\|\tilde{f}_h\| \lesssim \|f_h\|$ yields

$$\|\tilde{u}_{CR}\| \lesssim \epsilon \ \|\!|\tilde{u}_{CR}|\!\|_{NC} + \|f_h\|.$$

A substitution in (4.11) for sufficiently small h results in

$$\|\!|\tilde{u}_{CR}|\!\|_{NC} \lesssim \|f_h\|.$$

Since $f_h = 0$ shows that $\tilde{u}_{CR} = 0$, uniqueness follows. This also implies existence of the discrete solution. □

Theorem 4.2 (Equivalence of RTFEM and NCFEM) *Recall $\tilde{u}_M$ from* (4.3) *and let $\tilde{u}_{CR} \in CR_0^1(\mathcal{T})$ solve (4.5). Then*

$$\tilde{\mathbf{p}}_M(\mathbf{x}) = -(\mathbf{A}_h \nabla_{NC}\tilde{u}_{CR} + \tilde{u}_M \mathbf{b}_h) + (f_h - \gamma_h \tilde{u}_M)\,\frac{(\mathbf{x} - \mathrm{mid}(T))}{2} \text{ for } \mathbf{x} \in T \in \mathcal{T} \tag{4.15}$$

defines $\tilde{\mathbf{p}}_M \in RT_0(\mathcal{T}) \subset H(\mathrm{div}, \Omega)$ and the pair $(\tilde{\mathbf{p}}_M, \tilde{u}_M)$ satisfies (4.1)–(4.2). Conversely, for any solution $(\tilde{\mathbf{p}}_M, \tilde{u}_M)$ in $RT_0(\mathcal{T}) \times P_0(\mathcal{T})$ of (4.1)–(4.2) the solution $\tilde{u}_{CR} \in CR_0^1(\mathcal{T})$ to (4.5) satisfies (4.3) and (4.15).

Proof Note that the continuity of the normal components on the boundaries of the triangles $T \in \mathcal{T}$ reflects the conformity $RT_0(\mathcal{T}) \subset H(\mathrm{div}, \Omega)$. Given an interior edge E shared by neighbouring triangles $T_+, T_- \in \mathcal{T}$ with unit normal ν_E pointing from T_- to T_+, let ψ_E denote the non-conforming basis function defined on an interior edge such that $\psi_E(\mathrm{mid}(E)) = 1$, while $\psi_E(\mathrm{mid}(F)) = 0$ for all $F \in \mathcal{E}\backslash\{E\}$. A piecewise integration by parts shows

$$\begin{aligned}(\tilde{\mathbf{p}}_M, \nabla_{NC}\psi_E)_{L^2(\Omega)} + (\mathrm{div}_{NC}\, \tilde{\mathbf{p}}_M, \psi_E)_{L^2(\Omega)} &= \int_{\partial T_+ \cup \partial T_-} \tilde{\mathbf{p}}_M \cdot \nu \psi_E \, ds \\ = \int_E (\tilde{\mathbf{p}}_M|_{T_+} \cdot \nu|_{T_+} + \tilde{\mathbf{p}}_M|_{T_-} \cdot \nu|_{T_-})\psi_E \, ds &= |E|[\tilde{\mathbf{p}}_M] \cdot \nu_E,\end{aligned} \tag{4.16}$$

where $\mathrm{div}_{NC} v|_T = \mathrm{div}\, v|_T$. The definition of $\tilde{\mathbf{p}}_M$, (4.5) and the fact

$$((f_h - \gamma_h \tilde{u}_M)\,(\mathbf{x} - \mathrm{mid}(T))/2, \nabla_{NC}\psi_E)_{L^2(\Omega)} = 0,$$

imply

$$(\tilde{\mathbf{p}}_M, \nabla_{NC}\psi_E)_{L^2(\Omega)} + (\mathrm{div}_{NC}\, \tilde{\mathbf{p}}_M, \psi_E)_{L^2(\Omega)} = 0.$$

Hence, (4.16) shows $|E|[\tilde{\mathbf{p}}_M] \cdot \nu = 0$. Since the edge E is arbitrary in $\mathcal{E}(\Omega)$, $\tilde{\mathbf{p}}_M \in RT_0(\mathcal{T}) \subset H(\mathrm{div}, \Omega)$. Since the distributional divergence is the piecewise one, (4.15) proves $\mathrm{div}_{NC}\, \tilde{\mathbf{p}}_M(\mathbf{x}) = f_h - \gamma_h \tilde{u}_M$. Hence, (4.2) is satisfied. A use of the definition of Π_0, an application of element-wise integration by parts, some elementary properties of elements in $RT_0(\mathcal{T})$, $CR_0^1(\mathcal{T})$, and (4.15) yield

$$\begin{aligned}&(\mathbf{A}_h^{-1}\tilde{\mathbf{p}}_M + \tilde{u}_M \mathbf{b}_h^*,\ \mathbf{q}_{RT})_{L^2(\Omega)} - (\mathrm{div}\, \mathbf{q}_{RT},\ \Pi_0 \tilde{u}_{CR})_{L^2(\Omega)} \\ &\quad = (\mathbf{A}_h^{-1}\tilde{\mathbf{p}}_M + \tilde{u}_M \mathbf{b}_h^*,\ \mathbf{q}_{RT})_{L^2(\Omega)} - (\mathrm{div}\, \mathbf{q}_{RT},\ \tilde{u}_{CR})_{L^2(\Omega)}\end{aligned}$$

$$= (\mathbf{A}_h^{-1}\tilde{\mathbf{p}}_M + \tilde{u}_M \mathbf{b}_h^*,\ \mathbf{q}_{RT})_{L^2(\Omega)} + (\nabla_{NC}\tilde{u}_{CR},\ \mathbf{q}_{RT})_{L^2(\Omega)}$$
$$= (\mathbf{A}_h^{-1}(f_h - \gamma_h \tilde{u}_M)(\bullet - \mathrm{mid}(\mathcal{T}))/2,\ \mathbf{q}_{RT})_{L^2(\Omega)}.$$

Recall $S(\mathcal{T})|_T = S(T)$ and the definition of $S(T)$ from (4.4). Some algebraic calculations with $\mathbf{q}_{RT} \in RT_0(\mathcal{T})$ and $\int_T (\mathbf{x} - \mathrm{mid}(T))\, dx = 0$ yield

$$(\mathbf{A}_h^{-1}\tilde{\mathbf{p}}_M + \tilde{u}_M \mathbf{b}_h^*,\ \mathbf{q}_{RT})_{L^2(\Omega)} - (\mathrm{div}\ \mathbf{q}_{RT}, \Pi_0 \tilde{u}_{CR})_{L^2(\Omega)}$$
$$= ((f_h - \gamma_h \tilde{u}_M)\mathbf{A}_h^{-1}(\bullet - \mathrm{mid}(\mathcal{T}))/2),\ (\bullet - \mathrm{mid}(\mathcal{T}))/2\ \mathrm{div}\ \mathbf{q}_{RT})_{L^2(\Omega)}$$
$$= \left(\frac{S(\mathcal{T})}{4}(f_h - \gamma_h \tilde{u}_M), \mathrm{div}\ \mathbf{q}_{RT}\right)_{L^2(\Omega)}.$$

An appropriate re-arrangement shows that the pair $(\tilde{\mathbf{p}}_M, \tilde{u}_M)$ satisfies (4.1). This concludes the proof of the first part.

To prove the converse implication, let $(\tilde{\mathbf{p}}_M, \tilde{u}_M)$ in $RT_0(\mathcal{T}) \times P_0(\mathcal{T})$ be some solution to (4.1)–(4.2). The discrete Helmholtz decomposition [1] states for the simply-connected domain Ω that the piecewise constant vector function $-\Pi_0(\mathbf{A}_h^{-1}\tilde{\mathbf{p}}_M + \tilde{u}_M \mathbf{b}_h^*) \in P_0(\mathcal{T}; \mathbb{R}^2)$ equals a discrete gradient $\nabla_{NC}\alpha_{CR}$ of some nonconforming function $\alpha_{CR} \in CR_0^1(\mathcal{T})$ plus the Curl β_c of some piecewise affine conforming function $\beta_c \in P_1(\mathcal{T}) \cap C(\bar{\Omega})$; that is,

$$-(\Pi_0 \mathbf{A}_h^{-1}\tilde{\mathbf{p}}_M + \tilde{u}_M \mathbf{b}_h^*) = \nabla_{NC}\alpha_{CR} + \mathrm{Curl}\ \beta_c.$$

The argument to verify this is to define α_{CR} as the solution to a Poisson problem of a nonconforming FEM with the right-hand side $-(\tilde{\mathbf{p}}_M + \tilde{u}_M \mathbf{b}_h, \mathbf{A}_h^{-1}\nabla_{NC} v_{CR})_{L^2(\Omega)}$ as a functional in $v_{CR} \in CR_0^1(\mathcal{T})$. Once α_{CR} is determined, the difference $\nabla_{NC}\alpha_{CR} + \Pi_0 \mathbf{A}_h^{-1}\tilde{\mathbf{p}}_M + \tilde{u}_M \mathbf{b}_h^*$ is $L^2(\Omega)$ orthogonal onto $\nabla_{NC} CR_0^1(\mathcal{T})$. Hence, it equals the Curl of some Sobolev functions so that Curl $\beta_c := (-\frac{\partial \beta_c}{\partial x_2}, \frac{\partial \beta_c}{\partial x_1})$ is piecewise constant. This concludes the proof of the above discrete Helmholtz decomposition.

Since Curl $\beta_c =: \mathbf{q}_{RT}$ is a divergence free Raviart–Thomas function, (4.1) implies

$$\|\mathrm{Curl}\ \beta_c\|^2 = -(\mathbf{A}_h^{-1}\tilde{\mathbf{p}}_M + \tilde{u}_M \mathbf{b}_h^*, \mathbf{q}_{RT})_{L^2(\Omega)} = 0.$$

Consequently,

$$\Pi_0 \tilde{\mathbf{p}}_M = -\mathbf{A}_h \nabla_{NC}\alpha_{CR} - \tilde{u}_M \mathbf{b}_h.$$

The Raviart–Thomas function allows for div $\tilde{\mathbf{p}}_M = \mathrm{div}_{NC}\ \tilde{\mathbf{p}}_M \in P_0(\mathcal{T})$ and hence (in 2D),

$$\tilde{\mathbf{p}}_M = \Pi_0 \tilde{\mathbf{p}}_M + (\mathrm{div}_{NC}\ \tilde{\mathbf{p}}_M)(\bullet - \mathrm{mid}(\mathcal{T}))/2.$$

The Eq. (4.2) is equivalent to $\mathrm{div}_{NC}\ \tilde{\mathbf{p}}_M = f_h - \gamma_h \tilde{u}_M$. The combination of the previous identities proves (4.15) for $\tilde{u}_{CR} := \alpha_{CR}$. A piecewise integration by parts of the product of $\tilde{\mathbf{p}}_M$ for (4.15) with $\nabla_{NC} v_{CR}$ leads to

$$-(\mathrm{div}_{NC}\, \tilde{\mathbf{p}}_M, v_{CR})_{L^2(\Omega)} = (\tilde{\mathbf{p}}_M, \nabla_{NC} v_{CR})_{L^2(\Omega)}.$$

The aforementioned identities for $\Pi_0 \tilde{\mathbf{p}}_M$ and $\mathrm{div}_{NC}\, \tilde{\mathbf{p}}_M$ show that this equals

$$-(f_h - \gamma_h \tilde{u}_M, v_{CR})_{L^2(\Omega)} = -(\mathbf{A}_h \nabla_{NC} \alpha_{CR} + \tilde{u}_M \mathbf{b}_h, \nabla_{NC} v_{CR})_{L^2(\Omega).}$$

This proves (4.5) for $\tilde{u}_{CR} \equiv \alpha_{CR}$. To verify (4.3), the identity (4.15) is substituted in (4.1) for some general

$$\mathbf{q}_{RT} = \Pi_0 \mathbf{q}_{RT} + (\mathrm{div}_{NC}\, \mathbf{q}_{RT})(\bullet - \mathrm{mid}(\mathcal{T}))/2 \in RT_0(\mathcal{T}).$$

This shows

$$(\mathrm{div}_{NC}\, \mathbf{q}_{RT}, \tilde{u}_M)_{L^2(\Omega)} = (-\nabla_{NC} \tilde{u}_{CR}, \mathbf{q}_{RT})_{L^2(\Omega)} + \left(f_h - \gamma_h \tilde{u}_M, \frac{S(\mathcal{T})}{4}\, \mathrm{div}_{NC}\, \mathbf{q}_{RT} \right)_{L^2(\Omega)}.$$

A piecewise integration by parts shows $(-\nabla_{NC} \tilde{u}_{CR}, \mathbf{q}_{RT})_{L^2(\Omega)} = (\tilde{u}_{CR}, \mathrm{div}_{NC}\, \mathbf{q}_{RT})_{L^2(\Omega)}$ and hence,

$$\left(\tilde{u}_M \left(1 + \gamma_h \frac{S(\mathcal{T})}{4}\right) - \frac{S(\mathcal{T})}{4} f_h - \tilde{u}_{CR},\ \mathrm{div}\, \mathbf{q}_{RT} \right)_{L^2(\Omega)} = 0.$$

Since the divergence operator is surjective from $RT_0(\mathcal{T})$ onto $P_0(\mathcal{T})$ and since the previous identity holds for all $\mathbf{q}_{RT} \in RT_0(\mathcal{T})$), it follows

$$\tilde{u}_M \left(1 + \gamma_h \frac{S(\mathcal{T})}{4}\right) = \frac{S(\mathcal{T})}{4} f_h + \Pi_0 \tilde{u}_{CR}.$$

This is equivalent to (4.3) and concludes the proof. □

4.3 *A priori* error estimates for RTFEM

This subsection establishes well-posedness of the mixed finite element method (4.1)–(4.2) and *a priori* error estimates for mixed formulation (2.5) via the equivalence of RTFEM and NCFEM.

The following theorem deals with the well-posedness of the mixed finite element method (4.1)–(4.2) with a more general right hand side.

For given $\mathbf{g_{RT}} \in RT_0(\mathcal{T})$, define $\mathbf{g} \in RT_0(\mathcal{T})^*$ by

$$\mathbf{g}(\mathbf{q}) := (\mathbf{A}_h^{-1} \mathbf{g}_{RT}, \mathbf{q})_{L^2(\Omega)} + (\mathrm{div}\, \mathbf{g}_{RT}, \mathrm{div}\, \mathbf{q})_{L^2(\Omega)} \quad \text{for all } \mathbf{q} \in RT_0(\mathcal{T}). \tag{4.17}$$

For $f_h \in P_0(\mathcal{T})$, and $\mathbf{g} \in RT_0(\mathcal{T})^*$ a modified mixed finite element method reads as: seek $(\mathbf{p_M}, u_M) \in RT_0(\mathcal{T}) \times P_0(\mathcal{T})$ such that

$$(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*, \mathbf{q}_{RT})_{L^2(\Omega)} - (\operatorname{div}\mathbf{q}_{RT}, u_M)_{L^2(\Omega)} = \mathbf{g}(\mathbf{q}_{RT}) \quad \text{for all } \mathbf{q}_{RT} \in RT_0(\mathcal{T}), \tag{4.18}$$

$$(\operatorname{div}\mathbf{p_M}, v_h)_{L^2(\Omega)} + (\gamma_h u_M, v_h)_{L^2(\Omega)} = (f_h, v_h)_{L^2(\Omega)} \quad \text{for all } v_h \in P_0(\mathcal{T}). \tag{4.19}$$

Theorem 4.3 (Stability) *For all* $\mathbf{g} \in RT_0(\mathcal{T})^*$ *given by (4.17) and* $f_h \in P_0(\mathcal{T})$, *the modified mixed finite element problem (4.18)–(4.19) has a unique solution* $(\mathbf{p_M}, u_M) \in RT_0(\mathcal{T}) \times P_0(\mathcal{T})$ *with*

$$\|(\mathbf{p_M}, u_M)\|_{H(\operatorname{div},\Omega)\times L^2(\Omega)} \lesssim \|(\mathbf{g}, f_h)\|_{H(\operatorname{div},\Omega)^*\times L^2(\Omega)}. \tag{4.20}$$

As in Sect. 4.2, the solution to modified RTFEM (4.18)–(4.19) is represented in terms of the solution to a suitable NCFEM.

Proof Since $\mathbf{g}(\mathbf{q})$ is given by (4.17), the Eq. (4.18) is written equivalently

$$(\mathbf{A}_h^{-1}(\mathbf{p_M} - \mathbf{g}_{RT}) + u_M\mathbf{b}_h^*, \mathbf{q}_{RT})_{L^2(\Omega)} = (\operatorname{div}\mathbf{q}_{RT}, u_M + \operatorname{div}\mathbf{g}_{RT})_{L^2(\Omega)} \quad \text{for all } \mathbf{q}_{RT} \in RT_0(\mathcal{T}). \tag{4.21}$$

Since $-\Pi_0(\mathbf{A}_h^{-1}(\mathbf{p_M} - \mathbf{g}_{RT}) + u_M\mathbf{b}_h^*) \in P_0(\mathcal{T}; \mathbb{R}^2)$, the discrete Helmholtz decomposition states

$$-\Pi_0(\mathbf{A}_h^{-1}(\mathbf{p_M} - \mathbf{g}_{RT}) + u_M\mathbf{b}_h^*) = \nabla_{NC}\alpha_{CR} + \operatorname{Curl}\beta_{\mathrm{C}} \tag{4.22}$$

for some nonconforming function $\alpha_{CR} \in CR_0^1(\mathcal{T})$ and some $\beta_C \in P_1(\mathcal{T}) \cap C(\overline{\Omega})$. The choice of $\mathbf{q}_{RT} = \operatorname{Curl}\beta_{\mathrm{C}}$ in (4.21) shows that $\operatorname{Curl}\beta_{\mathrm{C}} = 0$. Hence,

$$\Pi_0(\mathbf{p_M} - \mathbf{g}_{RT}) = -\Big(\mathbf{A}_h\nabla_{NC}\alpha_{CR} + u_M\mathbf{b}_h\Big).$$

Equation (4.19) implies

$$\operatorname{div}_{NC}(\mathbf{p}_M - \mathbf{g}_{RT}) = f_h - \gamma_h u_M - \operatorname{div}_{NC}\mathbf{g}_{RT}. \tag{4.23}$$

and

$$(\mathbf{p}_M - \mathbf{g}_{RT}) = \Pi_0(\mathbf{p}_M - \mathbf{g}_{RT}) + (\operatorname{div}_{NC}(\mathbf{p}_M - \mathbf{g}_{RT}))(\bullet - \operatorname{mid}(\mathcal{T}))/2.$$

Hence,

$$(\mathbf{p}_M - \mathbf{g}_{RT}) = -\Big(\mathbf{A}_h\nabla_{NC}\alpha_{CR} + u_M\mathbf{b}_h\Big) + (f_h - \gamma_h u_M - \operatorname{div}_{NC}\mathbf{g}_{RT})(\bullet - \operatorname{mid}(\mathcal{T}))/2. \tag{4.24}$$

For all $v_{CR} \in CR_0^1(\mathcal{T})$, the last term on the right hand-side of (4.24) is orthogonal to $\nabla_{NC} v_{CR}$ with respect to $L^2(\Omega)$ inner product. This leads to

$$(\mathbf{A}_h \nabla_{NC} \alpha_{CR} + u_M \mathbf{b}_h, \nabla_{NC} v_{CR}) = -(\mathbf{p}_M - \mathbf{g}_{RT}, \nabla_{NC} v_{CR}).$$

For the last term on the right-hand side, a piecewise integration with (4.23) yields

$$(\mathbf{A}_h \nabla_{NC} \alpha_{CR} + u_M \mathbf{b}_h, \nabla_{NC} v_{CR}) + (\gamma_h u_M, v_{CR}) = (f_h - \mathrm{div}_{NC}\, \mathbf{g}_{RT}, v_{CR}). \tag{4.25}$$

A substitution of (4.24) in (4.21) with $\mathbf{q}_{RT} := \Pi_0 \mathbf{q}_{RT} + (\mathrm{div}_{NC}\, \mathbf{q}_{RT})(\bullet - \mathrm{mid}(\mathcal{T}))/2$ and piecewise integration $(-\nabla_{NC} \alpha_{CR}, \mathbf{q}_{RT})_{L^2(\Omega)} = (\alpha_{CR}, \mathrm{div}_{NC}\, \mathbf{q}_{RT})_{L^2(\Omega)}$ yields after some direct calculation

$$\begin{aligned}&\left(\mathrm{div}_{NC}\, \mathbf{q}_{RT} \left(1 + \frac{S(\mathcal{T})}{4}\right), \mathrm{div}_{NC}\, \mathbf{g}_{RT}\right)_{L^2(\Omega)} + (u_M, \mathrm{div}_{NC}\, \mathbf{q}_{RT})_{L^2(\Omega)} \\ &\quad = (\alpha_{CR}, \mathrm{div}_{NC}\, \mathbf{q}_{RT})_{L^2(\Omega)} + \left(\frac{S(\mathcal{T})}{4}(f_h - \gamma_h u_M),\ \mathrm{div}_{NC}\, \mathbf{q}_{RT}\right)_{L^2(\Omega)}.\end{aligned}$$

Since this holds for all $\mathbf{q}_{RT} \in RT_0(\mathcal{T})$, it follows immediately

$$u_M = \left(1 + \gamma_h \frac{S(\mathcal{T})}{4}\right)^{-1} \left(-\left(1 + \frac{S(\mathcal{T})}{4}\right) \mathrm{div}_{NC}\, \mathbf{g}_{RT} + \frac{S(\mathcal{T})}{4} f_h + \Pi_0 \alpha_{CR}\right). \tag{4.26}$$

The stability result (3.14) of Theorem 3.1 applies to (4.25). This implies

$$|||\alpha_{CR}|||_{NC} \lesssim \|\mathbf{g}_{RT}\|_{H(\mathrm{div},\, \Omega)} + \|f_h\|. \tag{4.27}$$

From the representations (4.26) and (4.24) of u_M and $\mathbf{p_M}$, (4.27) proves stability result (4.20). This concludes the proof. □

Theorem 4.3 implies the well-posedness of the mixed finite element method (4.1)–(4.2).

Corollary 1 (Stability) *There exists a unique solution* $(\mathbf{p_M}, u_M) \in RT_0(\mathcal{T}) \times P_0(\mathcal{T})$ *to the problem (4.1)–(4.2) with*

$$\|(\mathbf{p_M}, u_M)\|_{H(\mathrm{div},\Omega)\times L^2(\Omega)} \lesssim \|f_h\|_{L^2(\Omega)}. \tag{4.28}$$

Below, the main theorem of this section is discussed.

Theorem 4.4 (*A priori* error control of RTFEM) *Under the assumption* **(A1)–(A2)** *with* $u \in H_0^1(\Omega)$ *for* $f \in L^2(\Omega)$ *and for* $\epsilon > 0$ *with sufficiently small maximal* *mesh-size h, there exists a unique solution* $(\mathbf{p}_M, u_M) \in RT_0(\mathcal{T}) \times P_0(\mathcal{T})$ *of the mixed method (4.1)–(4.2). Further, it holds*

$$\|u - u_M\| \lesssim (h + \epsilon^2)\|f\|, \tag{4.29}$$
$$\|\mathbf{p} - \mathbf{p_M}\| \lesssim (h + \epsilon)\|f\|, \tag{4.30}$$
$$\|\text{div }(\mathbf{p} - \mathbf{p_M})\| \lesssim \|f - f_h\| + (h + \epsilon^2)\|f\|. \tag{4.31}$$

The remaining parts of this subsection are devoted to the proof which starts with an error estimate of $\tilde{e} := u_{CR} - \tilde{u}_{CR}$.

Lemma 4.5 (An intermediate estimate) *Let* u_{CR} *and* $\tilde{u}_{CR}$ *be the solutions to (3.2) and (4.5), respectively. Then, for sufficiently small maximal mesh-size h,*

$$|||u_{CR} - \tilde{u}_{CR}|||_{NC} + \|u_{CR} - \tilde{u}_{CR}\| \lesssim h\|f\|. \tag{4.32}$$

Proof A substitution of $\tilde{u}_M$ in (4.5) leads to

$$\tilde{a}_{NC}(\tilde{u}_{CR}, v_{CR}) = (\tilde{f}_h, v_{CR}) \text{ for all } v_{CR} \in CR_0^1(\mathcal{T}) \tag{4.33}$$

with $S(\mathcal{T})|_T = S(T)$, $\tilde{a}_{NC}(\tilde{u}_{CR}, v_{CR})$ and $(\tilde{f}_h, v_{CR})$ as defined in (4.8) and (4.9), respectively. The Eq. (3.2) implies

$$\begin{aligned} a_{NC}(\tilde{e}, v_{CR}) &= a_{NC}(u_{CR} - \tilde{u}_{CR}, v_{CR}) \\ &= (f, v_{CR})_{L^2(\Omega)} - (\mathbf{A}\nabla_{NC}\tilde{u}_{CR} + \tilde{u}_{CR}\mathbf{b}, \nabla_{NC}v_{CR})_{L^2(\Omega)} \\ &\quad -(\gamma\tilde{u}_{CR}, v_{CR})_{L^2(\Omega)}. \end{aligned}$$

The addition of (4.7) as a zero term yields

$$\begin{aligned} a_{NC}(\tilde{e}, v_{CR}) &= (f - f_h, v_{CR})_{L^2(\Omega)} - (\mathbf{A}\nabla_{NC}\tilde{u}_{CR} + \tilde{u}_{CR}\mathbf{b}, \nabla_{NC}v_{CR})_{L^2(\Omega)} \\ &\quad -(\gamma\tilde{u}_{CR}, v_{CR})_{L^2(\Omega)} \\ &\quad +\left(\mathbf{A}_h\nabla_{NC}\tilde{u}_{CR} + \left(1 + \frac{S(\mathcal{T})}{4}\gamma_h\right)^{-1}(\Pi_0\tilde{u}_{CR})\mathbf{b}_h, \nabla_{NC}v_{CR}\right)_{L^2(\Omega)} \\ &\quad + \left(\gamma_h\left(1 + \frac{S(\mathcal{T})}{4}\gamma_h\right)^{-1}(\Pi_0\tilde{u}_{CR}), v_{CR}\right)_{L^2(\Omega)} \\ &\quad +\left(\left(1 + \frac{S(\mathcal{T})}{4}\gamma_h\right)^{-1}\frac{S(\mathcal{T})}{4}f_h\mathbf{b}_h, \nabla_{NC}v_{CR}\right)_{L^2(\Omega)} \\ &\quad +\left(\gamma_h\left(1 + \frac{S(\mathcal{T})}{4}\gamma_h\right)^{-1}\frac{S(\mathcal{T})}{4}f_h, v_{CR}\right)_{L^2(\Omega)}. \end{aligned} \tag{4.34}$$

Note that the first term on the right-hand side can be rewritten with Π_0 and then equals $(f - f_h, v_{CR} - \Pi_0 v_{CR})_{L^2(\Omega)}$. An addition and subtraction of some terms yields

$$
\begin{aligned}
a_{NC}(\tilde{e}, v_{CR}) &= (f - f_h, v_{CR} - \Pi_0 v_{CR})_{L^2(\Omega)} - ((\mathbf{A}-\mathbf{A}_h)\nabla_{NC}\tilde{u}_{CR}, \nabla_{NC} v_{CR})_{L^2(\Omega)} \\
&- ((\gamma - \gamma_h)\tilde{u}_{CR}, v_{CR})_{L^2(\Omega)} - (\gamma_h(\tilde{u}_{CR} - \Pi_0\tilde{u}_{CR}), v_{CR} - \Pi_0 v_{CR})_{L^2(\Omega)} \\
&- (\tilde{u}_{CR}(\mathbf{b}-\mathbf{b_h}), \nabla_{NC} v_{CR}) - \left(\tilde{u}_{CR}\left(\mathbf{b_h} - \left(1+\frac{S(\mathcal{T})}{4}\gamma_h\right)^{-1}\mathbf{b_h}\right), \nabla_{NC} v_{CR}\right)_{L^2(\Omega)} \\
&- \left(\left(\gamma_h - \gamma_h\left(1+\frac{S(\mathcal{T})}{4}\gamma_h\right)^{-1}\right)(\Pi_0\tilde{u}_{CR}), v_{CR}\right)_{L^2(\Omega)} \\
&+ \left(\left(1+\frac{S(\mathcal{T})}{4}\gamma_h\right)^{-1}\frac{S(\mathcal{T})}{4} f_h\mathbf{b}_h, \nabla_{NC} v_{CR}\right)_{L^2(\Omega)} \\
&+ \left(\gamma_h\left(1+\frac{S(\mathcal{T})}{4}\gamma_h\right)^{-1}\frac{S(\mathcal{T})}{4} f_h, v_{CR}\right)_{L^2(\Omega)}.
\end{aligned}
$$

With the choice of $v_{CR} = \tilde{e}$, use Gårding's inequality (3.5) to the left hand side term. For sufficiently small h, a use of $S(\mathcal{T}) \lesssim h^2$, $\|\tilde{u}_{CR}\| \lesssim \|\!|\tilde{u}_{CR}\|\!|_{NC}$ and $\|v_{CR} - \Pi_0 v_{CR}\| \lesssim h\,\|\!|v_{CR}\|\!|$ in the above equation results in

$$
\begin{aligned}
\alpha\,\|\!|\tilde{e}\|\!|_{NC}^2 - \beta\|\tilde{e}\|^2 \lesssim \Big(& osc(f,\mathcal{T}) + h^2(\|\gamma_h\|_\infty + \|\mathbf{b_h}\|_\infty\|\gamma_h\|_\infty + \|\gamma_h\|_\infty^2)\|\!|\mathbf{\tilde{u}_{CR}}\|\!|_{\mathbf{NC}} \\
&+ (\|\mathbf{A}-\mathbf{A}_h\|_\infty + \|\mathbf{b}-\mathbf{b}_h\|_\infty)\ \|\!|\tilde{u}_{CR}\|\!|_{NC} \\
&+ h^2(\|\mathbf{b}_h\|_\infty\|\gamma_h\|_\infty + \|\gamma_h\|_\infty^2)\|f_h\|\Big)\ \|\!|\tilde{e}\|\!|_{NC} \\
&+ \|\gamma - \gamma_h\|_\infty\|\tilde{u}_{CR}\|\ \|\tilde{e}\|.
\end{aligned} \tag{4.35}
$$

Since $\|\tilde{e}\| \lesssim \|\!|\tilde{e}\|\!|_{NC}$, an application of (4.6) shows

$$
\begin{aligned}
\|\!|\tilde{e}\|\!|_{NC} \lesssim osc(f,\mathcal{T}) &+ \left(h^2 + \|\mathbf{A}-\mathbf{A}_h\|_\infty + \|\mathbf{b}-\mathbf{b}_h\|_\infty + \|\gamma-\gamma_h\|_\infty\right)\ \|\!|\tilde{u}_{CR}\|\!|_{NC} \\
&+ h^2\|f_h\| + \|\tilde{e}\|.
\end{aligned} \tag{4.36}
$$

It therefore, remains to estimate $\|\tilde{e}\|$. An appeal to Aubin–Nitsche duality argument applied to the dual problem (2.2) plus (3.12) and (3.15) lead to

$$
\begin{aligned}
(g,\tilde{e})_{L^2(\Omega)} &= a_{NC}(\tilde{e}, \Phi - \Phi_C) + (g,\tilde{e})_{L^2(\Omega)} - a_{NC}(\tilde{e},\Phi) + a_{NC}(\tilde{e},\Phi_C) \\
&\lesssim \|\!|\tilde{e}\|\!|_{NC}\ \|\Phi - \Phi_C\|_1 + |a_{NC}(\tilde{e},\Phi_C)| \\
&\quad + \|\!|\tilde{e}\|\!|_{NC} \sup_{0\neq w_{CR}\in CR_0^1(\mathcal{T})} \frac{|a_{NC}(w_{CR},\Phi) - (g, w_{CR})_{L^2(\Omega)}|}{\|\!|w_{CR}\|\!|_{NC}}.
\end{aligned}
$$

For the second last term on the right-hand side, recall (4.34) with $v_{CR} = \Phi_C$ and proceed as in the proof of the estimate (4.35) to obtain

$$|a_{NC}(\tilde{e}, \Phi_C)| \lesssim \Big(osc(f, \mathcal{T}) + (h^2 + \|\mathbf{A} - \mathbf{A}_h\|_\infty + \|\mathbf{b} - \mathbf{b}_h\|_\infty + \|\gamma - \gamma_h\|_\infty) \,|\!|\!|\tilde{u}_{CR}|\!|\!|_{NC} + h^2 \|f_h\|\Big) \|\Phi_C\|_1. \tag{4.37}$$

Since $\|\Phi_C\|_1 \lesssim \|\Phi\|_1 \lesssim \|g\|$, a substitution of (3.12), (3.15) and (4.37) in the previous estimates yields

$$\|\tilde{e}\| = \sup_{0 \neq g \in L^2(\Omega)} \frac{|(g, \tilde{e})_{L^2(\Omega)}|}{\|g\|} \lesssim osc(f, \mathcal{T}) + \epsilon \,|\!|\!|\tilde{e}|\!|\!|_{NC} + h^2 \|f_h\| + \Big(h^2 + \|\mathbf{A} - \mathbf{A}_h\|_\infty + \|\mathbf{b} - \mathbf{b}_h\|_\infty + \|\gamma - \gamma_h\|_\infty\Big) |\!|\!|\tilde{u}_{CR}|\!|\!|_{NC}. \tag{4.38}$$

Since $|\!|\!|\tilde{u}_{CR}|\!|\!|_{NC} \lesssim \|f_h\|$ with $\|f_h\| \lesssim \|f\|$, (4.36) results in

$$|\!|\!|\tilde{e}|\!|\!|_{NC} \lesssim osc(f, \mathcal{T}) + \Big(h^2 + \|\mathbf{A} - \mathbf{A}_h\|_\infty + \|\mathbf{b} - \mathbf{b}_h\|_\infty + \|\gamma - \gamma_h\|_\infty\Big) \|f\| + \|\tilde{e}\|. \tag{4.39}$$

For sufficiently small h, $\|\mathbf{A} - \mathbf{A}_h\|_\infty \lesssim h$, $\|\mathbf{b} - \mathbf{b}_h\|_\infty \lesssim h$, $\|\gamma - \gamma_h\|_\infty \lesssim h$ in (4.38) leads to

$$\|\tilde{e}\| \lesssim \epsilon \,|\!|\!|\tilde{e}|\!|\!|_{NC} + h\, \|f\|. \tag{4.40}$$

A substitution of (4.40) in (4.36) results for sufficiently small h in

$$|\!|\!|\tilde{e}|\!|\!|_{NC} \lesssim h\, \|f\|.$$

This and (4.40) prove (4.32). □

Proof of Theorem 4.4 Uniqueness of a discrete solution follows from the stability result (4.28) with $f_h = 0$. In order to estimate $\|u - u_M\|$, the definition of u_M in (4.3) implies

$$\|u - u_M\| = \left\| (1 + \gamma_h S(\mathcal{T})/4)^{-1} \left((1 + \gamma_h S(\mathcal{T})/4) u - \left(\Pi_0 \tilde{u}_{CR} + \frac{S(\mathcal{T})}{4} f_h \right) \right) \right\|$$
$$\lesssim \|u - u_{CR}\| + \|u_{CR} - \tilde{u}_{CR}\| + \|\tilde{u}_{CR} - \Pi_0 \tilde{u}_{CR}\| + \|\frac{S(\mathcal{T})}{4}(f_h - \gamma_h u)\|.$$

Since $\|\tilde{u}_{CR} - \Pi_0 \tilde{u}_{CR}\| \lesssim h\, |\!|\!|\tilde{u}_{CR}|\!|\!|_{NC}$ and $S(\mathcal{T}) \lesssim h^2$, this yields

$$\|u - u_M\| \lesssim \|u - u_{CR}\| + \|u_{CR} - \tilde{u}_{CR}\| + h\, |\!|\!|\tilde{u}_{CR}|\!|\!|_{NC} + h^2 \|f_h - \gamma_h u\|. \tag{4.41}$$

A substitution of (4.6) in (4.41) with Lemma 4.5 and Theorem 3.3 results in

$$\|u-u_M\| \lesssim osc(f,\mathcal{T}) + (\epsilon^2+h)\ \|f\|.$$

The definition of $\mathbf{p}$ and (4.15) imply

$$\mathbf{p}-\mathbf{p_M} = -(\mathbf{A}\nabla u + u\mathbf{b}) + (\mathbf{A}_h\nabla_{NC}\tilde{u}_{CR} + u_M\mathbf{b}_h) - (f_h - \gamma_h u_M)(\bullet - \text{mid}(\mathcal{T}))/2.$$

Hence,

$$\|\mathbf{p}-\mathbf{p_M}\| \le \|-(\mathbf{A}-\mathbf{A}_h)\nabla u - u(\mathbf{b}-\mathbf{b}_h) - \mathbf{A}_h(\nabla u - \nabla_{NC}\tilde{u}_{CR}) - (u-u_M)\mathbf{b}_h\| + h\|f_h - \gamma_h u_M\|. \tag{4.42}$$

The substitution of $u-\tilde{u}_{CR} = (u-u_{CR}) + (u_{CR}-\tilde{u}_{CR})$ in (4.42) results in

$$\|\mathbf{p}-\mathbf{p_M}\| \lesssim \|\mathbf{A}-\mathbf{A}_h\|_\infty\|u\|_1 + \|\mathbf{b}-\mathbf{b}_h\|_\infty\|u\| + |||u-u_{CR}|||_{NC} + |||u_{CR}-\tilde{u}_{CR}|||_{NC} + \|u-u_M\| + h\|f_h-\gamma_h u\| + h\|u-u_M\|.$$

For sufficiently small h, Lemma 4.5, Theorem 3.3, and (4.3) imply

$$\|\mathbf{p}-\mathbf{p_M}\| \lesssim osc(f,\mathcal{T}) + \epsilon\ \|f\|.$$

In order to prove the estimate of $\|\text{div}(\mathbf{p}-\mathbf{p}_M)\|$, (2.4) and (4.2) together lead to

$$\text{div}(\mathbf{p}-\mathbf{p_M}) = f - f_h - \gamma u + \gamma_h u_M.$$

Hence,

$$\|\text{div}(\mathbf{p}-\mathbf{p_M})\| \le \|f-f_h\| + \|\gamma-\gamma_h\|_\infty\|u\| + \|\gamma_h\|_\infty\|u-u_M\|. \tag{4.43}$$

A substitution of (4.29) in (4.43) yields (4.31) and this concludes the proof. □

Remark 4.6 With the regularity result $u \in H^{1+\delta}(\Omega)\cap H_0^1(\Omega)$, one obtains $\epsilon = O(h^\delta)$ in Lemma 3.2, Theorems 3.3 and 4.4 by (3.10) and (3.12). Then, the error estimates in Theorem 4.4 read

$$\|u-u_M\| \lesssim h^{\min(1,2\delta)}\ \|f\|, \tag{4.44}$$

$$\|\mathbf{p}-\mathbf{p_M}\| \lesssim h^\delta\|f\|, \tag{4.45}$$

$$\|\text{div}\ (\mathbf{p}-\mathbf{p_M})\| \lesssim \|f-f_h\| + h^{\min(1,2\delta)}\ \|f\|. \tag{4.46}$$

For $\delta = 1$, results (4.44)–(4.45) are proved in [13] and [14].

Remark 4.7 Note that for our analysis, only regularity estimate for the dual problem in the broken Sobolev $H^{1+\delta}(\mathcal{T})$, for some δ with $0<\delta<1$, is required and hence, the assumptions on $\mathbf{A}$, $\mathbf{b}$ and γ may be weakened in the sense that $\mathbf{A} \in W^{1,\infty}(\mathcal{T};\mathbb{R}^{2\times 2}_{sym})$,

$\mathbf{b} \in W^{1,\infty}(\mathcal{T}; \mathbb{R}^2)$ and $\gamma \in W^{1,\infty}(\mathcal{T}; \mathbb{R})$. Such conditions are more relevant for elliptic interface problems, when the interfaces are aligned to element faces, (cf. [20, Sect. 2.4]).

5 *A posteriori* error control

This section is devoted to the *a posteriori* error analysis of the mixed finite element scheme (4.1)–(4.2) to generalize [6] via the unified approach of [7].

Define $\mathbf{e_p} := \mathbf{p} - \mathbf{p_M}$, and $e_u := u - u_M$. Then, (2.5) and (4.1)–(4.2) lead to

$$(\mathbf{A}^{-1}\mathbf{e_p} + e_u\mathbf{b}^*, \mathbf{q})_{L^2(\Omega)} - (\operatorname{div} \mathbf{q}, e_u)_{L^2(\Omega)} = \mathcal{R}_1(\mathbf{q}) \quad \text{for all } \mathbf{q} \in H(\operatorname{div}, \Omega), \tag{5.1}$$

$$(\operatorname{div} \mathbf{e_p}, v)_{L^2(\Omega)} + (\gamma e_u, v)_{L^2(\Omega)} = \mathcal{R}_2(v) \quad \text{for all } v \in L^2(\Omega). \tag{5.2}$$

Here and throughout this paper $\mathcal{R}_1(\mathbf{q})$ and $\mathcal{R}_2(v)$ read

$$\mathcal{R}_1(\mathbf{q}) := \mathcal{R}_{11}(\mathbf{q}) + \mathcal{R}_{12}(\mathbf{q}), \tag{5.3}$$

$$\mathcal{R}_2(v) := (f - \operatorname{div} \mathbf{p_M} - \gamma u_M, v)_{L^2(\Omega)} \tag{5.4}$$

where

$$\mathcal{R}_{11}(\mathbf{q}) := -(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*, \mathbf{q})_{L^2(\Omega)} + (\operatorname{div} \mathbf{q}, u_M)_{L^2(\Omega)},$$
$$\mathcal{R}_{12}(\mathbf{q}) := -((\mathbf{A}^{-1} - \mathbf{A}_h^{-1})\mathbf{p_M} + u_M(\mathbf{b}^* - \mathbf{b}_h^*), \mathbf{q})_{L^2(\Omega)}.$$

5.1 Unified *a posteriori* analysis

Theorem 2.1–2.2 imply the well-posedness of the system (2.5) and so the residuals $\mathcal{R}_1$, $\mathcal{R}_2$ of (5.3)–(5.4) allow for the equivalence [7]

$$\|\mathbf{p} - \mathbf{p_M}\|_{H(\operatorname{div},\Omega)} + \|u - u_M\|_{L^2(\Omega)} \approx \|\mathcal{R}_1\|_{H(\operatorname{div},\Omega)^*} + \|\mathcal{R}_2\|_{L^2(\Omega)}. \tag{5.5}$$

The estimate for $\mathcal{R}_2(v)$ reads

$$\|\mathcal{R}_2\| = \|f - \operatorname{div} \mathbf{p_M} - \gamma\, u_M\|. \tag{5.6}$$

Recall that $f_h = \operatorname{div} \mathbf{p_M} + \gamma_h\, u_M$ denotes a piecewise polynomial approximation of f.

Fortin interpolation operator [5, pp. 124,128]. There exists an interpolation operator

$$I_F : H^1(\Omega; \mathbb{R}^2) \longrightarrow RT_0(\mathcal{T})$$

with the orthogonality condition

$$\int_\Omega u_M \operatorname{div}(\boldsymbol{\Phi} - I_F \boldsymbol{\Phi}) dx = 0 \qquad \text{for all } \boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^2) \tag{5.7}$$

and the approximation property

$$\|h_{\mathcal{T}}^{-1}(\boldsymbol{\Phi} - I_F \boldsymbol{\Phi})\| \lesssim \|\boldsymbol{\Phi}\|_{H^1(\Omega)}. \tag{5.8}$$

Lemma 5.1 (Regular split) *For any* $\mathbf{q} \in H(\operatorname{div}, \Omega)$*, there exist* $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^2)$ *and* $\psi \in H^1(\Omega)$ *such that* $\mathbf{q} = \boldsymbol{\Phi} + \operatorname{Curl} \psi$ *in* Ω *and*

$$\|\nabla \boldsymbol{\Phi}\| + \|\nabla \psi\| \lesssim \|\mathbf{q}\|_{H(\operatorname{div}, \Omega)}. \tag{5.9}$$

Proof Let $\mathbf{q} \in H(\operatorname{div}, \Omega)$. Extend $\operatorname{div} \mathbf{q}|_\Omega$ by zero in some ball $\mathcal{B} \supset\supset \Omega$. Let $z \in H^2(\mathcal{B}) \cap H_0^1(\mathcal{B})$ be the unique solution to $-\Delta z = \operatorname{div} \mathbf{q}$ in Ω with $z|_{\partial\mathcal{B}} = 0$. Also, let $\boldsymbol{\Phi} = -\nabla z$, so that

$$\|\nabla \boldsymbol{\Phi}\| \lesssim \|z\|_2 \lesssim \|\operatorname{div} \mathbf{q}\| \leq \|\mathbf{q}\|_{H(\operatorname{div}, \Omega)}.$$

Since $\boldsymbol{\Phi} = -\nabla z$, $\operatorname{div}(\mathbf{q} - \boldsymbol{\Phi}) = 0$ in Ω, and hence, $\mathbf{q} = \boldsymbol{\Phi} + \operatorname{Curl} \psi$ with $\|\nabla \psi\| = \|\operatorname{Curl} \psi\| = \|\mathbf{q} - \boldsymbol{\Phi}\| \lesssim \|\mathbf{q}\|_{H(\operatorname{div}, \Omega)}$. □

Lemma 5.2 *It holds*

$$\|\mathcal{R}_1\|_{H(\operatorname{div}, \Omega)^*} \lesssim \|h_{\mathcal{T}}(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M \mathbf{b}_h^*)\| + \min_{v \in H_0^1(\Omega)} \|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M \mathbf{b}_h^* - \nabla v\|$$
$$+ \|(\mathbf{A^{-1}} - \mathbf{A}_h^{-1})\mathbf{p_M}\| + \|u_M(\mathbf{b}^* - \mathbf{b}_h^*)\|.$$

Proof For the residual $\mathcal{R}_{11}(\mathbf{q})$ from (5.3), the regular decomposition of $\mathbf{q} \in H(\operatorname{div}, \Omega)$ from Lemma 5.1 and the interpolation operator $I_F \boldsymbol{\Phi} \in RT_0(\mathcal{T}) \subset \operatorname{Ker} \mathcal{R}_{11}$, lead to

$$\begin{aligned}
\mathcal{R}_{11}(\mathbf{q}) &= \mathcal{R}_{11}(\boldsymbol{\Phi} + \operatorname{Curl} \psi) = \mathcal{R}_{11}(\boldsymbol{\Phi} - I_F \boldsymbol{\Phi} + \operatorname{Curl} \psi) \\
&= -(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M \mathbf{b}_h^*, \boldsymbol{\Phi} - I_F \boldsymbol{\Phi})_{L^2(\Omega)} + (u_M, \operatorname{div}(\boldsymbol{\Phi} - I_F \boldsymbol{\Phi}))_{L^2(\Omega)} \\
&\quad - (\operatorname{Curl} \psi, \mathbf{A}_h^{-1}\mathbf{p_M} + u_M \mathbf{b}_h^*)_{L^2(\Omega)} + (u_M, \operatorname{div}(\operatorname{Curl} \psi))_{L^2(\Omega)}.
\end{aligned}$$

This and (5.7) imply

$$\begin{aligned}
\mathcal{R}_{11}(\mathbf{q}) = &-(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M \mathbf{b}_h^*, \boldsymbol{\Phi} - I_F \boldsymbol{\Phi})_{L^2(\Omega)} \\
&-(\operatorname{Curl} \psi, \mathbf{A}_h^{-1}\mathbf{p_M} + u_M \mathbf{b}_h^*)_{L^2(\Omega)}.
\end{aligned} \tag{5.10}$$

The first term on the right-hand side of (5.10) is bounded by

$$|(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M \mathbf{b}_h^*, \boldsymbol{\Phi} - I_F \boldsymbol{\Phi})_{L^2(\Omega)}| \leq \|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M \mathbf{b}_h^*\| \|\boldsymbol{\Phi} - I_F \boldsymbol{\Phi}\|.$$

The approximation property (5.8) and Lemma 5.1 result in

$$|(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*, \mathbf{\Phi} - I_F\mathbf{\Phi})_{L^2(\Omega)}| \lesssim \|h_{\mathcal{T}}(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*)\|\|\nabla\mathbf{\Phi}\| \\ \lesssim \|h_{\mathcal{T}}(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*)\|\|\mathbf{q}\|_{H(\mathrm{div},\Omega)}. \quad (5.11)$$

Given any $v \in H_0^1(\Omega)$, the second term on the right-hand side of (5.10) is bounded by

$$-(\mathrm{Curl}\,\psi, \mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*)_{L^2(\Omega)} = -(\mathrm{Curl}\,\psi, \mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*)_{L^2(\Omega)} \\ + (\mathrm{Curl}\,\psi, \nabla v)_{L^2(\Omega)} \\ \le \|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^* - \nabla v\|\ \|\mathrm{Curl}\,\psi\| \\ \lesssim \|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^* - \nabla v\|\ \|\mathbf{q}\|_{H(\mathrm{div},\,\Omega)}. \quad (5.12)$$

The combination of (5.11)–(5.12) shows

$$\mathcal{R}_{11}(\mathbf{q}) \lesssim \Big(\|h_{\mathcal{T}}(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*)\| \\ + \min_{v\in H_0^1(\Omega)} \|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^* - \nabla v\|\Big)\ \|\mathbf{q}\|_{H(\mathrm{div},\,\Omega)}. \quad (5.13)$$

The Cauchy–Schwartz inequality leads to

$$\mathcal{R}_{12}(\mathbf{q}) \lesssim \Big(\|(\mathbf{A}^{-1} - \mathbf{A}_h^{-1})\mathbf{p_M}\| + \|u_M(\mathbf{b}^* - \mathbf{b}_h^*)\|\Big)\|\mathbf{q}\|_{H(\mathrm{div},\,\Omega)}. \quad (5.14)$$

The estimate (5.3) follows from (5.13)–(5.14) as

$$\mathcal{R}_1(\mathbf{q}) \lesssim \Big(\|h_{\mathcal{T}}(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*)\| + \min_{v\in H_0^1(\Omega)} \|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^* - \nabla v\| \\ + \|(\mathbf{A}^{-1} - \mathbf{A}_h^{-1})\mathbf{p_M}\| + \|u_M(\mathbf{b}^* - \mathbf{b}_h^*)\|\Big)\|\mathbf{q}\|_{H(\mathrm{div},\,\Omega)}.$$

□

Lemma 5.2 and Eq. (5.6) result in the following reliable a posteriori estimate η.

Theorem 5.3 (*A posteriori* error control) *Let* $(\mathbf{p}, u)$ *and* $(\mathbf{p_M}, u_M)$ *solve (2.5) and (4.1)–(4.2). Then, it holds*

$$\|\mathbf{p} - \mathbf{p_M}\|_{H(\mathrm{div},\Omega)} + \|u - u_M\| \lesssim \eta := \|f - \mathrm{div}\,\mathbf{p_M} - \gamma\ \mathrm{u_M}\| \\ + \|h_{\mathcal{T}}(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*)\| + \min_{v\in H_0^1(\Omega)} \|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^* - \nabla v\| \\ + \|(\mathbf{A}^{-1} - \mathbf{A}_h^{-1})\mathbf{p_M}\| + \|u_M(\mathbf{b}^* - \mathbf{b}_h^*)\|. \quad (5.15)$$

The following lemma enables a refined *a posteriori* error analysis for $\|u - u_M\|$ and $\|\mathbf{p} - \mathbf{p_M}\|$.

Lemma 5.4 *Let $\tilde{u}_{CR}$ and $(\mathbf{p_M}, u_M)$ solve (4.5) and (4.1)–(4.2), respectively. Then it holds*

$$\max\left\{\|\nabla_{NC}\tilde{u}_{CR}\|, \|(f_h - \gamma_h u_M)\,\mathbf{A}_h^{-1}\frac{(\mathbf{x}-\mathrm{mid}(T))}{2}\|\right\} \le \|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*\|.$$

Proof From (4.15),

$$\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^* = -\nabla_{NC}\tilde{u}_{CR} + (f_h - \gamma_h u_M)\,\mathbf{A}_h^{-1}\frac{(\mathbf{x}-\mathrm{mid}(T))}{2}.$$

Since $((f_h - \gamma_h\tilde{u}_M)\,(\mathbf{x}-\mathrm{mid}(T))\,/2, \nabla_{NC}\tilde{u}_{CR}) = 0$, the Pythagoras theorem yields

$$\|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*\|^2 = \|\nabla_{NC}\tilde{u}_{CR}\|^2 + \|(f_h - \gamma_h u_M)\,\mathbf{A}_h^{-1}\frac{(\mathbf{x}-\mathrm{mid}(T))}{2}\|^2.$$

□

A consequence of the Lemma 5.4 and the structure of $\mathbf{p_M}$ and u_M is the following bound.

Corollary 2 *It holds*

$$\|h_{\mathcal{T}}\,\mathbf{p_M}\| + \|h_{\mathcal{T}}\,u_M\| \lesssim \|h_{\mathcal{T}}^2\,f_h\| + \|h_{\mathcal{T}}(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*)\|.$$

The following theorem concerns on an improved error estimate of $e_u := u - u_M$ in L^2-norm.

Theorem 5.5 (Refined error estimates) *Let $u \in H_0^1(\Omega)$ be the unique weak solution to (2.1) and let $(\mathbf{p}_M, u_M)$ be the solution to (4.1)–(4.2). For sufficiently small maximum mesh size h, it holds*

$$\begin{aligned}
&\|\mathbf{A}^{-1/2}(\mathbf{p}-\mathbf{p_M})\| \lesssim osc(f,\mathcal{T}) + osc(f-\gamma\,u_M,\mathcal{T})\\
&+\min_{v\in H_0^1(\Omega)}\|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^* - \nabla v\| + \Big(1 + \|h_{\mathcal{T}}^{-1}(\mathbf{A}-\mathbf{A}_h)\|_\infty + \|h_{\mathcal{T}}^{-1}(\mathbf{b}-\mathbf{b}_h)\|_\infty\\
&+\|h_{\mathcal{T}}^{-1}(\gamma-\gamma_h)\|_\infty\Big)\|h_{\mathcal{T}}(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*)\| + \|h_{\mathcal{T}}^2\,f_h\| + \|h_{\mathcal{T}}\,(f_h-\gamma_h\,u_M)\|\\
&+(\|(\mathbf{A}^{-1}-\mathbf{A}_h^{-1})\mathbf{p_M}\| + \|u_M(\mathbf{b}^*-\mathbf{b}_h^*)\|).
\end{aligned} \tag{5.16}$$

Provided $u \in H^{1+\delta}(\Omega)$ for some $0 < \delta < 1$, it holds

$$\begin{aligned}
\|u-u_M\| \lesssim\; & osc(f,\mathcal{T}) + osc(f-\gamma\,u_M,\mathcal{T})\\
&+\min_{v\in H_0^1(\Omega)}\|h_{\mathcal{T}}^{\delta}(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^* - \nabla v)\|\\
&+\Big(1 + \|h_{\mathcal{T}}^{-1}(\mathbf{A}-\mathbf{A}_h)\|_\infty + \|h_{\mathcal{T}}^{-1}(\mathbf{b}-\mathbf{b}_h)\|_\infty + \|h_{\mathcal{T}}^{-1}(\gamma-\gamma_h)\|_\infty\Big)\\
&\times\|h_{\mathcal{T}}(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*)\| + \|h_{\mathcal{T}}^2\,f_h\| + \|h_{\mathcal{T}}^{1+\delta}\,(f_h-\gamma_h\,u_M)\|\\
&+(\|h_{\mathcal{T}}^{\delta}(\mathbf{A}^{-1}-\mathbf{A}_h^{-1})\mathbf{p_M}\| + \|h_{\mathcal{T}}^{\delta}\,u_M(\mathbf{b}^*-\mathbf{b}_h^*)\|).
\end{aligned} \tag{5.17}$$

Proof Consider the Helmholtz decomposition $\mathbf{e_p} = \mathbf{A}\nabla z + \mathrm{Curl}\ \beta$ for $z \in H_0^1(\Omega)$ and $\beta \in H^1(\Omega)/\mathbb{R}$ with $\mathbf{e_p} = \mathbf{p} - \mathbf{p_M}$

$$(\mathbf{A}^{-1}\mathbf{e_p}, \mathbf{e_p})_{L^2(\Omega)} = (\mathbf{e_p}, \nabla z)_{L^2(\Omega)} + (\mathbf{A}^{-1}\mathbf{e_p}, \mathrm{Curl}\ \beta)_{L^2(\Omega)}. \tag{5.18}$$

For the first term on the right-hand side of (5.18), an integration by parts plus (5.2) lead to

$$\begin{aligned}(\mathbf{e_p}, \nabla z)_{L^2(\Omega)} &= (\mathrm{div}\ \mathbf{e_p}, z) = \mathcal{R}_2(z) - (\gamma(u - u_M), z)_{L^2(\Omega)} \\ &= (f - f_h - (\gamma - \gamma_h)u_M, z - \Pi_0 z)_{L^2(\Omega)} - (\gamma e_u, z)_{L^2(\Omega)}, \\ &\lesssim osc(f - \gamma u_M, \mathcal{T})\ \|z\|_1 + \|e_u\|\|z\|. \end{aligned} \tag{5.19}$$

Given any $v \in H_0^1(\Omega)$, Eq. (2.5) shows

$$\begin{aligned}&(\mathbf{A}^{-1}\mathbf{e_p},\ \mathrm{Curl}\ \beta)_{L^2(\Omega)} \\ &\quad = -(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*, \mathrm{Curl}\ \beta)_{L^2(\Omega)} - (e_u\mathbf{b}^*, \mathrm{Curl}\ \beta)_{L^2(\Omega)} \\ &\quad\quad - ((\mathbf{A}^{-1} - \mathbf{A}_h^{-1})\mathbf{p_M} + u_M(\mathbf{b}^* - \mathbf{b}_h^*),\ \mathrm{Curl}\ \beta)_{L^2(\Omega)} + (\nabla v,\ \mathrm{Curl}\ \beta)_{L^2(\Omega)} \\ &\quad \lesssim \min_{v\in H_0^1(\Omega)} \|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^* - \nabla v\|\|\mathrm{Curl}\ \beta\| + \|e_u\|\|\mathrm{Curl}\ \beta\| \\ &\quad\quad + (\|(\mathbf{A}^{-1} - \mathbf{A}_h^{-1})\mathbf{p_M}\| + \|u_M(\mathbf{b}^* - \mathbf{b}_h^*)\|)\|\mathrm{Curl}\ \beta\|. \end{aligned} \tag{5.20}$$

The substitution of (5.19)–(5.20) in (5.18) plus $\|z\| \lesssim \|z\|_1 \lesssim \|\mathbf{e_p}\| \lesssim \|\mathbf{A^{-1/2}e_p}\|$ with $\|\mathrm{Curl}\ \beta\| \lesssim \|\mathbf{e_p}\| \lesssim \|\mathbf{A^{-1/2}e_p}\|$ result in

$$\begin{aligned}\|\mathbf{A}^{-1/2}\mathbf{e_p}\| \lesssim osc(f - \gamma u_M, \mathcal{T}) + \min_{v\in H_0^1(\Omega)} \|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^* - \nabla v\| + \|e_u\| \\ + \|(\mathbf{A}^{-1} - \mathbf{A}_h^{-1})\mathbf{p_M}\| + \|u_M(\mathbf{b}^* - \mathbf{b}_h^*)\|. \end{aligned} \tag{5.21}$$

The estimate of $\|e_u\|$ starts with a triangle inequality

$$\|e_u\| \leq \|u - \tilde{u}_{CR}\| + \|\tilde{u}_{CR} - u_M\|. \tag{5.22}$$

With $\tilde{e} = u_{CR} - \tilde{u}_{CR}$, (4.38) and (4.36) yield (for sufficiently small mesh size h) that

$$\begin{aligned}|||\tilde{e}|||_{NC} + \|\tilde{e}\| \lesssim osc(f, \mathcal{T}) + \Big(\|h_\mathcal{T}\|_\infty + \|h_\mathcal{T}^{-1}(\mathbf{A} - \mathbf{A}_h)\|_\infty + \|h_\mathcal{T}^{-1}(\mathbf{b} - \mathbf{b}_h)\|_\infty \\ + \|h_\mathcal{T}^{-1}(\gamma - \gamma_h)\|_\infty\Big)\ |||h_\mathcal{T}\tilde{u}_{CR}|||_{NC} + \|h_\mathcal{T}^2\ f_h\|. \end{aligned} \tag{5.23}$$

The estimates for $\|u - \tilde{u}_{CR}\|$ are derived with the help of (3.24) and (5.23) and a repeated use of triangle inequality. This proves

$$\begin{aligned}\|u-\tilde u_{CR}\| &\le \|u-u_{CR}\|+\|u_{CR}-\tilde u_{CR}\|\\ &\lesssim \epsilon(|||u-\tilde u_{CR}|||_{NC}+|||\tilde u_{CR}-u_{CR}|||_{NC})+\|u_{CR}-\tilde u_{CR}\|\\ &\lesssim \epsilon\|\nabla_{NC}(u-\tilde u_{CR})\|+osc(f,\mathcal{T})+\|h_{\mathcal{T}}^2\ f_h\|\\ &\quad+\Big(\|h_{\mathcal{T}}\|_\infty+\|h_{\mathcal{T}}^{-1}(\mathbf{A}-\mathbf{A}_h)\|_\infty+\|h_{\mathcal{T}}^{-1}(\mathbf{b}-\mathbf{b}_h)\|_\infty\\ &\quad+\|h_{\mathcal{T}}^{-1}(\gamma-\gamma_h)\|_\infty\Big)\ |||h_{\mathcal{T}}\tilde u_{CR}|||_{NC}\,. \end{aligned}\tag{5.24}$$

Define $\tilde{\mathbf{p}}_{CR}:=-(\mathbf{A}_h\nabla_{NC}\tilde u_{CR}+u_M\mathbf{b}_h)$ and $\mathbf{p}=-(\mathbf{A}\nabla u+\mathbf{b}u)$ along with an addition and subtraction of the term $\mathbf{p_M},\ u_M\mathbf{b}^*,\ \mathbf{A}_h^{-1}p_M$. This shows

$$\begin{aligned}\|\nabla_{NC}(u-\tilde u_{CR})\| \le \|\mathbf{A}^{-1}\mathbf{e_p}\|+\|(\mathbf{A}^{-1}-\mathbf{A}_h^{-1})\mathbf{p_M}\|+\|\mathbf{A}_h^{-1}(\mathbf{p_M}-\tilde{\mathbf{p}}_{CR})\|\\ +\|e_u\mathbf{b}^*\|+\|u_M(\mathbf{b}^*-\mathbf{b}_h^*)\|.\end{aligned}\tag{5.25}$$

For the third term on the right-hand side of (5.25), (4.15) leads to

$$\|\mathbf{p_M}-\tilde{\mathbf{p}}_{CR}\|\le\|(f_h-\gamma_h u_M)(\mathbf{x}-\mathrm{mid}(T))\|\lesssim\|h_{\mathcal{T}}(f_h-\gamma_h\ u_M)\|.\tag{5.26}$$

The combination of (5.24)–(5.26) results in

$$\begin{aligned}\|u-\tilde u_{CR}\| \lesssim\ & osc(f,\mathcal{T})+\epsilon\Big(\|\mathbf{A}^{-1/2}\mathbf{e_p}\|+\|e_u\|\Big)+\|h_{\mathcal{T}}^2\ f_h\|+\epsilon\|h_{\mathcal{T}}(f_h-\gamma_h\ u_M)\|\\ &+\Big(\|h_{\mathcal{T}}\|_\infty+\|h_{\mathcal{T}}^{-1}(\mathbf{A}-\mathbf{A}_h)\|_\infty+\|h_{\mathcal{T}}^{-1}(\mathbf{b}-\mathbf{b}_h)\|_\infty\\ &+\|h_{\mathcal{T}}^{-1}(\gamma-\gamma_h)\|_\infty\Big)\\ &\times\ |||h_{\mathcal{T}}\tilde u_{CR}|||_{NC}+\epsilon(\|(\mathbf{A}^{-1}-\mathbf{A}_h^{-1})\mathbf{p_M}\|+\|u_M(\mathbf{b}^*-\mathbf{b}_h^*)\|).\end{aligned}$$

To bound $\|\tilde u_{CR}-u_M\|$ in (5.22), use (4.3) to obtain

$$\begin{aligned}\|\tilde u_{CR}-u_M\| &\le \left(1+\frac{S(\mathcal{T})}{4}\gamma_h\right)^{-1}\|\tilde u_{CR}-\Pi_0\tilde u_{CR}+\frac{S(\mathcal{T})}{4}(\gamma_h\tilde u_{CR}-f_h)\|,\\ &\lesssim \|h_{\mathcal{T}}\nabla_{NC}\tilde u_{CR}\|+\left|\left|\left|h_{\mathcal{T}}^2\tilde u_{CR}\right|\right|\right|_{NC}+\|h_{\mathcal{T}}^2\ f_h\|.\end{aligned}$$

The combination of the previous estimates with (5.22) and Lemma 5.4 leads to

$$\begin{aligned}\|e_u\| \lesssim\ & osc(f,\mathcal{T})+\epsilon\Big(\|\mathbf{A}^{-1/2}\mathbf{e_p}\|+\|e_u\|\Big)+\|h_{\mathcal{T}}^2\ f_h\|+\epsilon\|h_{\mathcal{T}}(f_h-\gamma_h\ u_M)\|\\ &+\Big(1+\|h_{\mathcal{T}}^{-1}(\mathbf{A}-\mathbf{A}_h)\|_\infty+\|h_{\mathcal{T}}^{-1}(\mathbf{b}-\mathbf{b}_h)\|_\infty+\times\|h_{\mathcal{T}}^{-1}(\gamma-\gamma_h)\|_\infty\Big)\\ &\times\ \|h_{\mathcal{T}}(\mathbf{A}_h^{-1}\mathbf{p_M}+u_M\mathbf{b}_h^*)\|+\epsilon\Big(\|(\mathbf{A}^{-1}-\mathbf{A}_h^{-1})\mathbf{p_M}\|+\|u_M(\mathbf{b}^*-\mathbf{b}_h^*)\|\Big).\end{aligned}\tag{5.27}$$

For sufficiently small mesh size h, (5.27) and (5.21) prove (5.16). Under extra regularity assumptions on u, it is easy to establish (5.17). □

Remark 5.6 Corollary 2 yields

$$\begin{aligned}
&\|(\mathbf{A}^{-1} - \mathbf{A}_h^{-1})\mathbf{p_M}\| + \|u_M(\mathbf{b}^* - \mathbf{b}_h^*)\| \\
&\lesssim \left(\|h_{\mathcal{T}}^{-1}(\mathbf{A}^{-1} - \mathbf{A}_h^{-1})\|_\infty + \|h_{\mathcal{T}}^{-1}(\mathbf{b}^* - \mathbf{b}_h^*)\|_\infty\right) \\
&\times\left(\|h_{\mathcal{T}}^2\ f_h\| + \|h_{\mathcal{T}}(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*)\|\right).
\end{aligned}$$

Those estimates can be used in (5.16)–(5.17) to improve the estimates in Theorem 5.5.

5.2 Efficiency

This section is devoted to prove that the error estimator η yields lower bounds for the error in the mixed finite element approximation.

Theorem 5.7 (Efficiency) *Under the assumptions (***A1***)–(***A2***) it holds*

$$\begin{aligned}
\eta \lesssim\ &\|u - u_M\| + \|\mathbf{p} - \mathbf{p_M}\| + \|(\mathbf{A}^{-1} - \mathbf{A}_h^{-1})\mathbf{p_M}\| \\
&+ \|u_M(\mathbf{b}^* - \mathbf{b}_h^*)\| + \|\mathrm{div}\ (\mathbf{p} - \mathbf{p_M})\|. \qquad (5.28)
\end{aligned}$$

Proof Step 1 of the proof utilizes $v := -u$, and the definition $\mathbf{p} = -\mathbf{A}\nabla u + \mathbf{b}u$ to verify

$$\begin{aligned}
&\min_{v\in H_0^1(\Omega)} \|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^* - \nabla v\| \le \|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^* + \nabla u\| \\
&= \|\mathbf{A}_h^{-1}(\mathbf{p} - \mathbf{p_M})\| + \|(u - u_M)\mathbf{b}^*\| + \|(\mathbf{A}^{-1} - \mathbf{A}_h^{-1})\mathbf{p_M}\| + \|u_M(\mathbf{b}^* - \mathbf{b}_h^*)\| \\
&\lesssim \|\mathbf{p} - \mathbf{p_M}\| + \|u - u_M\| + \|(\mathbf{A}^{-1} - \mathbf{A}_h^{-1})\mathbf{p_M}\| + \|u_M(\mathbf{b}^* - \mathbf{b}_h^*)\|.
\end{aligned}$$

In step 2, define the function $\mathbf{q}_T := b_T(\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*) \in P_4(T) \cap W_0^{1,\infty}(T)$ and the cubic bubble function $b_T = 27\lambda_1\lambda_2\lambda_3 \in P_3(T) \cap C_0(T)$ in terms of the barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$ of $T \in \mathcal{T}$[22]. Since $\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*$ is affine on $T \in \mathcal{T}$, an equivalence of norm argument shows

$$\|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*\|_{L^2(T)}^2 \lesssim \int_T \mathbf{q}_T \cdot (\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*)dx.$$

The definition of $\mathbf{p}$ and (2.4) show that

$$\begin{aligned}
\|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*\|_{L^2(T)}^2 \lesssim &\int_T \mathbf{q}_T \cdot \big(\mathbf{A}^{-1}(\mathbf{p_M} - \mathbf{p}) - (u - u_M)\mathbf{b}^*\big)dx \\
&+ \int_T \mathbf{q}_T \cdot \left((\mathbf{A}_h^{-1} - \mathbf{A}^{-1})\mathbf{p_M} - u_M(\mathbf{b}^* - \mathbf{b}_h^*)\right) dx \\
&- \int_T \mathbf{q}_T \cdot \nabla u\ dx.
\end{aligned}$$

The Cauchy inequality and $\|\mathbf{q}_T\|_{L^2(T)} \lesssim \|\mathbf{A}_h^{-1}\mathbf{p_M} + u_M\mathbf{b}_h^*\|_{L^2(T)}$ is employed in the first two terms. An integration by parts with $\nabla u_M|_T = 0$ shows in the last term that

$$
\begin{aligned}
h_T^2\|\mathbf{A}_h^{-1}\mathbf{p_M}+u_M\mathbf{b}_h\|_{L^2(T)}^2 \lesssim h_T\|\mathbf{A}_h^{-1}\mathbf{p_M}+u_M\mathbf{b}_h^*\|_{L^2(T)}\Big(&h_T\|\mathbf{p}-\mathbf{p_M}\|_{L^2(T)}\\
&+h_T\|u-u_M\|_{L^2(T)}+h_T\|(\mathbf{A}^{-1}-\mathbf{A}_h^{-1})\mathbf{p_M}\|\\
&+h_T\|u_M(\mathbf{b}^*-\mathbf{b}_h^*)\|_{L^2(T)}\Big)\\
&+h_T^2\int_T (u-u_M)\mathrm{div}\ \mathbf{q}_T\,dx.
\end{aligned}
$$

Since $\mathbf{q}_T \in P_4(T)$, an inverse estimate yields

$$
h_T\|\mathrm{div}\ \mathbf{q}_T\|_{L^2(T)} \lesssim \|\mathbf{q}_T\|_{L^2(T)} \lesssim \|\mathbf{A}_h^{-1}\mathbf{p_M}+u_M\mathbf{b}_h^*\|_{L^2(T)}.
$$

Since $h_T \lesssim 1$, it follows

$$
\begin{aligned}
h_T\|\mathbf{A}_h^{-1}\mathbf{p_M}+u_M\mathbf{b}_h^*\|_{L^2(T)} \lesssim\ &\|u-u_M\|_{L^2(T)}+\|\mathbf{p}-\mathbf{p_M}\|_{L^2(T)}\\
&+\|(\mathbf{A}^{-1}-\mathbf{A}_h^{-1})\mathbf{p_M}\|_{L^2(T)}+\|u_M(\mathbf{b}^*-\mathbf{b}_h^*)\|_{L^2(T)}.
\end{aligned}
$$

The sum over all triangles implies

$$
\begin{aligned}
h_{\mathcal{T}}\|\mathbf{A}_h^{-1}\mathbf{p_M}+u_M\mathbf{b}_h^*\| \lesssim\ &\|u-u_M\|+\|\mathbf{p}-\mathbf{p_M}\|\\
&+\|(\mathbf{A}^{-1}-\mathbf{A}_h^{-1})\mathbf{p_M}\|+\|u_M(\mathbf{b}^*-\mathbf{b}_h^*)\|.
\end{aligned}
$$

In Step 3, the term $\|f-\mathrm{div}\ \ \mathbf{p_M}-\gamma u_M\|$ is estimated as

$$
\begin{aligned}
\|f-\mathrm{div}\ \ \mathbf{p_M}-\gamma u_M\| &= \|\mathrm{div}\ \ \mathbf{p}+\gamma u-\mathrm{div}\ \ \mathbf{p_M}-\gamma u_M\|\\
&\lesssim \|\mathrm{div}\ (\mathbf{p}-\mathbf{p_M})\|+\|u-u_M\|.
\end{aligned}
$$

This concludes the rest of the proof. □

6 Computational experiments

This section is devoted to validation of theoretical results by numerical experiments and to test the performance of the adaptive algorithm.

6.1 Practical implementation

The adaptive finite element algorithm starts with the initial coarse triangulation $\mathcal{T}_0$, followed by the procedures **SOLVE, ESTIMATE, MARK and REFINE** for different levels $\ell = 0, 1, 2, \ldots$.

SOLVE The discrete solution $(\mathbf{p}_\ell, u_\ell) \in RT_0(\mathcal{T}_\ell) \times P_0(\mathcal{T}_\ell)$ of (4.1–4.2) is computed on each level ℓ with the corresponding triangulation $\mathcal{T}_\ell$ and basis functions as prescribed in [2].

ESTIMATE The estimator η_ℓ is defined in (5.15). In the estimator term $\|\mathbf{A}_h^{-1}\mathbf{p}_\ell + u_\ell \mathbf{b}_h^* - \nabla v\|$, the function v is chosen by post processing $\tilde{u}_{CR}$, that is $v = -\mathcal{A}\tilde{u}_{CR}$, where the averaging operator $\mathcal{A} : CR^1(\mathcal{T}) \to P_1(\mathcal{T})$ [9] is defined by

$$v(z) := \mathcal{A}\tilde{u}_{CR}(z) := \sum_{T\in\mathcal{T}(z)} \frac{\tilde{u}_{CR}|_T(z)}{|\mathcal{T}(z)|} \qquad \text{for all } z \in \mathcal{N}.$$

$|\mathcal{T}(z)|$ denote the cardinality of the triangles sharing node z.

MARK For $0 < \theta \le 1$, compute a minimal subset $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ for red refinement such that

$$\theta\eta_\ell^2 \le \eta_\ell^2(\mathcal{M}_\ell) = \sum_{T\in\mathcal{M}_\ell} \eta_{T,\ell}^2.$$

REFINE The new triangulation $\mathcal{T}_{\ell+1}$ is generated using red-blue-green refinement of the marked elements.

Table 1 Errors and the experimental convergence rates for uniform and adaptive mesh refinement

Uniform refinement					Adaptive refinement				
N	e	$CR(e)$	η	$CR(\eta)$	N	e	$CR(e)$	η	$CR(\eta)$
68	0.66234		1.25235		68	0.66234		1.25235	
256	0.35809	0.4639	0.64497	0.5005	149	0.48702	0.3919	0.90745	0.4106
992	0.19558	0.4457	0.33468	0.4843	359	0.30569	0.4837	0.57511	0.4702
3904	0.10808	0.4336	0.17602	0.4690	1442	0.15641	0.4800	0.29166	0.4469
15488	0.06055	0.4203	0.09440	0.4521	2275	0.12555	0.4820	0.23609	0.4636
61696	0.034469	0.4077	0.05185	0.4339	7788	0.06666	0.5226	0.12703	0.5079
246272	0.019915	0.3963	0.02924	0.4139	24706	0.037423	0.5074	0.07095	0.5185

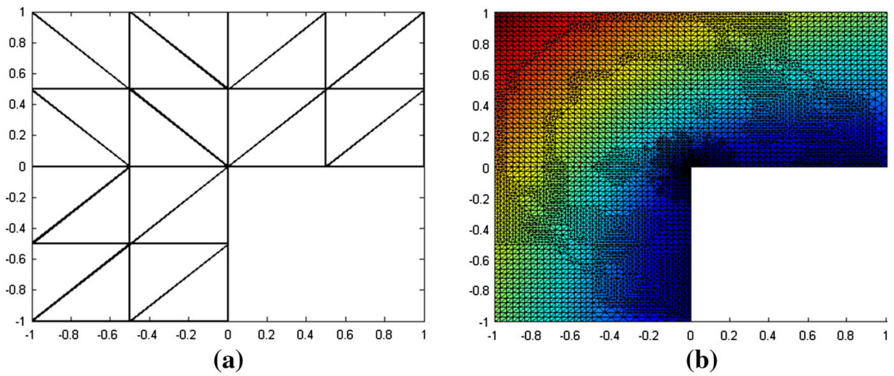


Fig. 1 **a** Initial triangulation $\mathcal{T}_0$, **b** Discrete solution u_M for adaptive mesh-refinement

Remark 6.1 In the process of computation of the solution, the given function f over each element is approximated by the integral mean $f_h = \frac{1}{|T|}\int_T f(x)dx$. The integrals $\int_T f(x)dx$ are computed by one-point numerical quadrature rule over the element, that is, $|T|f(\text{mid(T)})$, where $|T|$ denotes an area of element T and mid(T) is the centroid of the element. For the edge integral with Dirichlet condition u_D simple one point integration reads $\int_E u_D ds \approx |E|u_D(\text{mid}(E))$, where $|E|$ denotes the length of edge and $(\text{mid}(E))$, the midpoint of the edge.

Remark 6.2 Let $(\mathbf{p}, u)$ and $(\mathbf{p_M}, u_M)$ solve (2.5) and (4.1)–(4.2) and let $e := \|\mathbf{p} - \mathbf{p_M}\|_{H(\text{div},\Omega)} + \|u - u_M\|$. With the number of unknowns Ndof (ℓ) and the error $e(\ell)$ on the level ℓ, the experimental order of convergence is defined by

$$CR(e) = \frac{\log(e(\ell-1)/e(\ell))}{\log(\text{Ndof}(\ell)/\text{Ndof}(\ell-1))} \quad \text{for } e \text{ and } \eta.$$

Example 6.1 Consider the PDE (1.1) with coefficients $\mathbf{A} = I$, $\mathbf{b} = (r\cos\theta, r\sin\theta)$ and $\gamma = -4$ with Dirichlet boundary condition on the L-shaped domain $\Omega = (-1,1)\times(-1,1)\backslash[0,1]\times[-1,0]$ and the exact solution (given in polar coordinates)

$$u(r,\theta) = r^{2/3}\sin\big(2\theta/3\big).$$

For the given parameters, conditions of [12, Theorem3.1] are *not* satisfied. Utilizing their notation, $b_1(\mathbf{q}, v) := -(v, \text{div }\mathbf{q})_{L^2(\Omega)} + (\tilde{\mathbf{b}}v, \mathbf{q})_{L^2(\Omega)}$ with $\tilde{\mathbf{b}} = \mathbf{A}^{-1}\mathbf{b}$, for $v = |\Omega|^{-\frac{1}{2}}$ with $\|v\| = 1$,

$$\beta_1 \le \sup_{\mathbf{q}\in H(\text{div},\Omega)/\{0\}} \frac{\|\text{div }\mathbf{q}\| + \|\tilde{\mathbf{b}}v\|\ \|\mathbf{q}\|}{\|\mathbf{q}\|_{H(\text{div},\Omega)}} \le \sqrt{1 + \int_\Omega |\mathbf{x}|^2 dx} \le \sqrt{3} \tag{6.1}$$

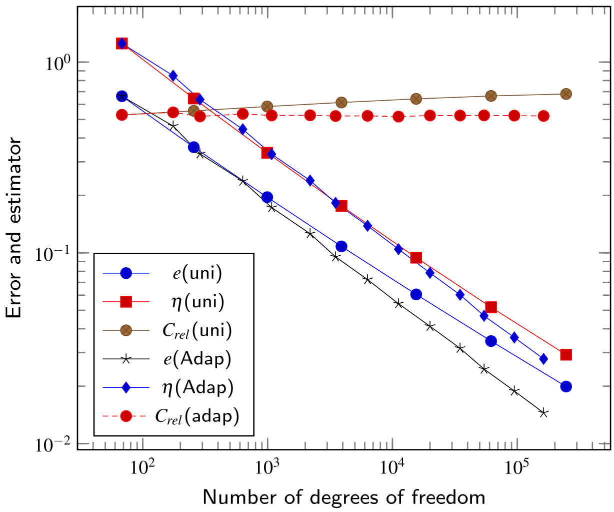


Fig. 2 Ndof versus e, η and C_{rel}

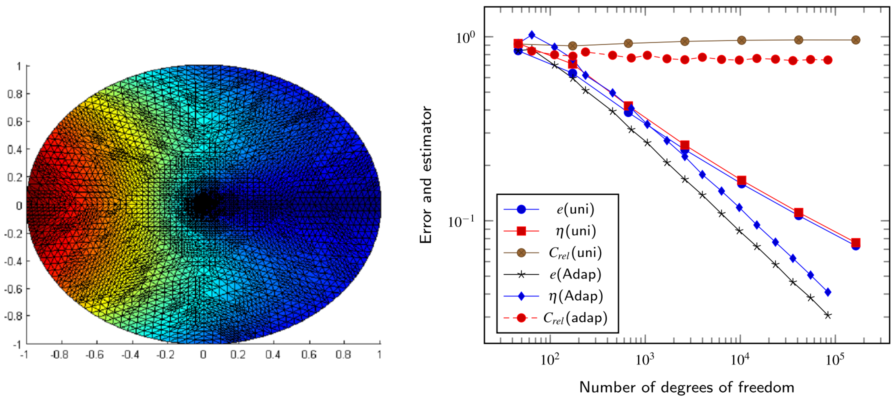
since $|\mathbf{x}| \le \sqrt{2}$ for all $x \in \Omega$. It is relatively straightforward to verify $\alpha \le \|a\| = 1$ (in the notation of [12]) and hence $\alpha\|a\|^{-2}\beta_1^2 - \gamma \le 3 - 4 < 0$ (notice that the coefficient $\gamma = -4$ in [12, pp. 224–225] is different from the parameter $\gamma = 4$ in [12, Eq. (3.3)] and this might give reasons for confusion). This violates the (implicit) condition $\delta_1 \ge 0$ in [12, Eq. (3.1)].

Tables 1 shows the errors and experimental convergence rate for uniform and adaptive mesh-refinements. Figure 1a denotes the initial triangulation $\mathcal{T}_0$ with $h \approx 0.5$. Figure 1b depicts the discrete solution u_M and illustrates the adaptive mesh-refinement near the singularity. In Fig. 2, a convergence history for the error e and the estimator η is plotted as a function of the number of degrees of freedom for the cases of uniform and adaptive mesh-refinement of the non-convex L-shaped domain. Adaptive mesh refinement gives an optimal empirical convergence rate of order 0.5 for e, while standard uniform refinement achieves suboptimal empirical convergence rate as expected from the theory. For both the cases, C_{rel}, the ratio between the error e and the estimator is also plotted.

Example 6.2 Crack problem: Consider the PDE (1.1) with coefficients $\mathbf{A} = I,\ \mathbf{b} = (x-1, y+1)$ and $\gamma = 0$ on $\Omega = \{(x, y) \in \mathbb{R}^2 : |\mathbf{x}| \le 1\}\backslash[0,\ 1] \times \{0\}$ with Dirichlet boundary condition and exact solution $u(r,\theta) = r^{1/2}\sin\theta/2 - r^2/2\sin^2(\theta)$ (in polar coordinates).

The problem is called non-coercive [19], since $(\gamma - \frac{1}{2}\nabla\cdot\mathbf{b}) < 0$. Figure 3a shows the discrete solution u_M along with the adaptive mesh-refinement. Note that the mesh is strongly refined near the singularity at the origin. The results are summarized in Figure 3b and displays convergence rates for the error e and the *a posteriori* estimator η. It is observed that a suboptimal empirical convergence rate of 0.25 for uniform mesh-refinement and an improved optimal empirical convergence rate of 0.5 for adaptive mesh-refinement are achieved. In this case, C_{rel} is close to 0.95.

Example 6.3 Consider the PDE (1.1) with coefficients $\mathbf{A} = I,\ \mathbf{b} = (0, 0)$ for different values of γ and Dirichlet boundary conditions on the L-shaped domain.



(a) **(b)**

Fig. 3 **a** Discrete solution u_M for adaptive refinement. **b** Ndof versus e, η and C_{rel}

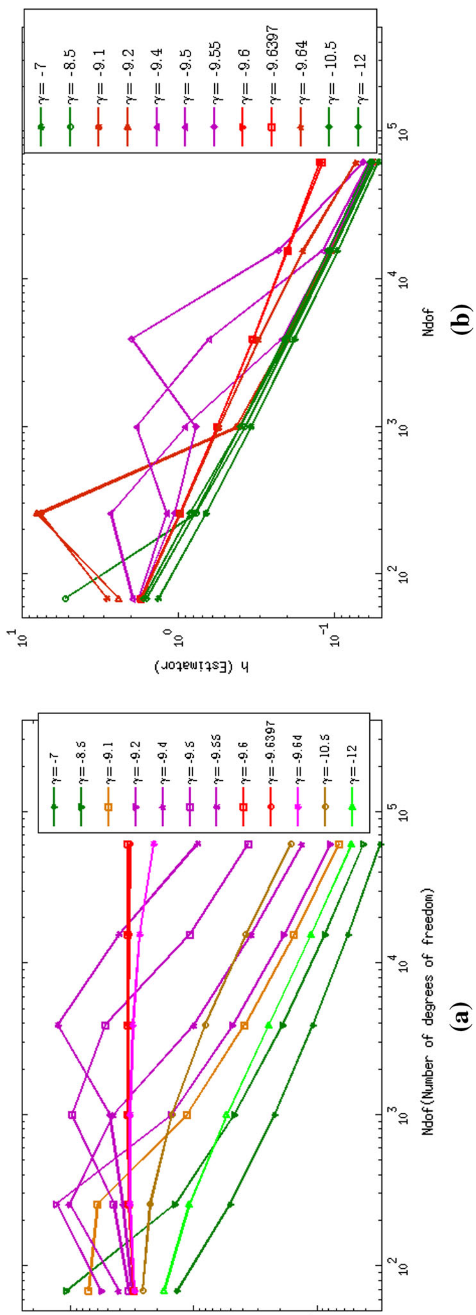


Fig. 4 **a** e and **b** η for different γ with uniform refinement

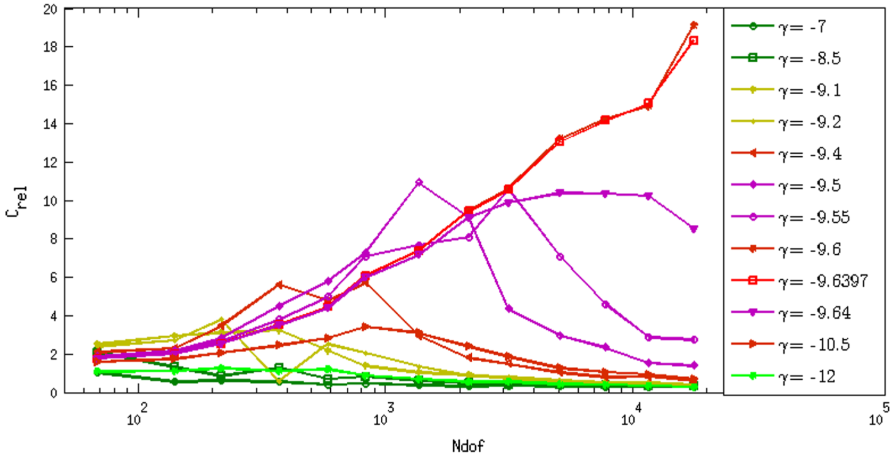


Fig. 5 C_{rel} for different γ with adaptive refinement

Since the first Laplace eigenvalue for the L-shaped domain $\lambda_1 \approx 9.6397238440219$, the coefficients lead to the Laplace operator with positive and negative eigenvalues.

The fact that the convergence is sensitive to the smallness of the discretization parameter h is clearly observed in Fig. 4a. This observation holds true for conforming, nonconforming and mixed finite element methods. Figure 4b depicts that the estimator mirrors the error behavior. This is also true for the case of adaptive refinement.

Figure 5 plots the reliability constant C_{rel} for various values of γ close to the eigenvalue λ_1 versus the number of degrees of freedom. Note that C_{rel} is sensitive to the discretization parameter h especially when γ is closer to λ_1. Thus, a sufficiently small mesh-size is a crucial requirement for the well-posedness and the convergence of the solution.

6.2 Conclusions

From the numerical experiments, it is observed that efficiency index lies between 1 and 2 for both uniform and adaptive triangulations. This confirms the efficiency of *a posteriori* error control for non-smooth problems defined in non-convex domains.

The overall assumption on the mesh-size to be sufficiently small is in fact crucial in practice, as shown in the third example empirically.

Acknowledgments The first author acknowledges the support of National Program on Differential Equations: Theory, Computation & Applications (NPDE-TCA) vide Department of Science & Technology (DST) Project No. SR/S4/MS:639/09 during his visit to IIT Bombay. The second author acknowledges the financial support of Council of Scientific and Industrial Research (CSIR), Government of India. The authors sincerely thank the two anonymous referees for careful reviews and several remarks which improved the paper.

References

1. Arnold, D.N., Falk, R.S.: A uniformly accurate finite element method for the Reissner–Mindlin plate. SIAM J. Numer. Anal. **26**, 1276–1290 (1989)

2. Bahriawati, C., Carstensen, C.: Three matlab implementations of the lowest order Raviart–Thomas MFEM with a posteriori error control. Comput. Methods Appl. Math. **5**, 333–1361 (2005)
3. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods. Springer, New York (2008)
4. Brenner, S.C.: Two-level additive Schwarz preconditioners for nonconforming finite element methods. Math. Comput. **65**, 897–921 (1996)
5. Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Springer, New York (1991)
6. Carstensen, C.: A posteriori error estimate for the mixed finite element method. Math. Comput. **66**, 465–476 (1997)
7. Carstensen, C.: A unifying theory of a posteriori finite element error control. Numer. Math. **100**, 617–637 (2005)
8. Carstensen, C., Hu, J.: A unifying theory of a posteriori error control for nonconforming finite element methods. Numer. Math. **107**, 473–502 (2007)
9. Carstensen, C., Eigel, M., Löbhard, C., Hoppe, R.H.W.: A review of unified a posteriori finite element error control. Numer. Math. Theory Methods Appl. **5**, 509–558 (2012)
10. Chen, H., Xu, X., Hoppe, R.H.W.: Convergence and quasi-optimality of adaptive nonconforming finite element methods for some nonsymmetric and indefinite problems. Numer. Math. **116**, 383–419 (2010)
11. Chen, J., Li, L.: Convergence and domain decomposition algorithm for nonconforming and mixed methods for nonselfadjoint and indefinite problems. Comput. Methods Appl. Mech. Eng. **173**, 1–20 (1999)
12. Ciarlet Jr, P., Huang, J., Zou, J.: Some observations on generalized saddle-point problems. SIAM J. Matrix Anal. Appl. **25**, 224–236 (2003)
13. Douglas Jr, J., Roberts, J.E.: Global estimates for mixed methods for second order elliptic equations. Math. Comput. **44**, 39–52 (1985)
14. Douglas Jr, J., Roberts, J.E.: Mixed finite element methods for second order elliptic problems. Mat. Apl. Comput. **1**, 91–103 (1982)
15. Dauge, M.: Elliptic boundary problems on corner domains. Lecture Notes in Mathematics, vol. 1341. Springer, Berlin (1988)
16. Evans, L.C.: Partial Differential Equations. American Mathematical Society, Providence, Rhode Island (1998)
17. Gilberg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (1983)
18. Mao, S., Shi, Z.: On the error bounds of nonconforming finite elements. Sci. China Math. **53**, 2917–2926 (2010)
19. Mekchay, K., Nochetto, R.H.: Convergence of adaptive finite element methods for general second order linear elliptic PDEs. SIAM J. Numer. Anal. **43**, 1803–1827 (2006)
20. Nicaise, S.: Polygonal Interface Problems. Verlag Peter D Lang, Frankfurt am Main (1993)
21. Schatz, A.H., Wang, J.: Some new error estimates for Ritz–Galerkin methods with minimal regularity assumptions. Math. Comput. **65**, 19–27 (1996)
22. Verfürth, R.: A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques. Wiley-Teubner, New York (1996)