IMA Journal of Numerical Analysis (2017) **37**, 64–93
doi: 10.1093/imanum/drw005
Advance Access publication on May 10, 2016

Nonconforming FEM for the obstacle problem

C. Carstensen* and K. Köhler
Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6,
D-10099 Berlin, Germany
*Corresponding author: cc@mathematik.hu-berlin.de, koehlerk@mathematik.hu-berlin.de

[Received on 6 March 2015; revised on 13 November 2015]

The paper is dedicated to Professor Peter Wriggers on the occasion of his 65 birthday.

The main motivation for the application of the Crouzeix–Raviart nonconforming finite element method (NCFEM) to the obstacle problem in this paper is that it allows for fully computable guaranteed lower bounds of the energy and so for simple *a posteriori* error control. A further fully computable and guaranteed upper error bound follows from Braess' work, extended to the Crouzeix–Raviart NCFEM. This error bound competes with the error control from the lower energy bounds. Both *a posteriori* estimates are efficient with respect to the total error. The paper circumvents variational crimes through a medius analysis and the design of conforming companions. This leads to an improved *a priori* error analysis for the NCFEMs under minimal regularity assumptions on polyhedral domains. Numerical evidence supports the *a priori* convergence analysis and confirms guaranteed error control with moderate efficiency indices for uniform and adaptive mesh refinement.

Keywords: obstacle problem; variational inequality; nonconforming; finite elements; medius analysis; *a priori* error analysis; *a posteriori* error analysis; guaranteed upper error bounds; lower energy bounds; adaptive mesh refinement.

1. Introduction

The obstacle problem is the simplest mathematical model of a variational inequality, with countless applications and related models in free boundary value problems. The trial functions are restricted to some convex set K and any discretization replaces this set by a discrete approximation K_h. If $K_h \subseteq K$ the discretization is called conforming and it is called nonconforming otherwise.

The particular motivation for the application of the Crouzeix–Raviart nonconforming finite element method (NCFEM) in Wang (2003) remains less clear. Therein, compared with conforming finite element methods (CFEMs), higher regularity assumptions are made to prove linear convergence for convex domains $\Omega \subset \mathbb{R}^2$ with smooth boundary. The refined *a priori* error analysis of this paper shows, under the minimal regularity assumption (i.e., $\Delta u \in L^2(\Omega)$), that the NCFEM converges with optimal convergence rates for arbitrary polyhedral domains $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ and hence the Crouzeix–Raviart NCFEM becomes competitive with the CFEM.

This paper also explores the *a posteriori* error control from two different points of view. In the first place, the Crouzeix–Raviart NCFEM allows for the computation of guaranteed lower bounds for the energy. Some simple postprocessing leads to a computable estimate for the energy difference and hence also for the error in the energy norm. In the second place, the results in Braess (2005) for the CFEM are adapted to the Crouzeix–Raviart NCFEM. The two error estimates for NCFEM are comparable (up to unknown multiplicative constants).

Given a bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with boundary $\partial\Omega$, the energy product $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ on the Hilbert space $H^1(\Omega)$ reads

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \qquad \text{for all } u, v \in H^1(\Omega)$$

and induces the energy seminorm $|||\bullet||| := a(\bullet, \bullet)^{1/2}$, which is a norm on the vector space $V := H^1_0(\Omega) := \{v \in H^1(\Omega) |\, v = 0 \text{ on } \partial\Omega\}$, and the corresponding local variant $|||\bullet|||_\omega := \|\nabla\bullet\|_{L^2(\omega)}$ for $\omega \subseteq \Omega$. Given some source term $f \in L^2(\Omega)$, define $F \in L^2(\Omega)^*$ by

$$F(v) := \int_\Omega fv \,\mathrm{d}x, \qquad \text{for all } v \in L^2(\Omega).$$

We assume that the obstacle $\chi \in H^1(\Omega)$ and the Dirichlet boundary value $u_{\mathrm{D}} \in C(\partial\Omega) \cap H^{1/2}(\partial\Omega)$ satisfy $\chi \leqslant u_{\mathrm{D}}$ a.e. along $\partial\Omega$ in order to ensure that the closed and convex subset

$$K := \{v \in \mathscr{A} |\, \chi \leqslant v \text{ a.e.}\} \quad \text{of} \quad \mathscr{A} := \{v \in H^1(\Omega) |\, v = u_{\mathrm{D}} \text{ along } \partial\Omega\}$$

is nonempty. The well-established weak formulation of the obstacle problem leads to a unique $u \in K$, see Kinderlehrer & Stampacchia (1980, Chapter 2, Theorem 2.1), with

$$F(v - u) \leqslant a(u, v - u), \qquad \text{for all } v \in K. \tag{1.1}$$

The obstacle problem is also characterized by the minimization of the energy functional

$$E(v) := \tfrac{1}{2}\, a(v, v) - F(v) \qquad \text{over all } v \in K. \tag{1.2}$$

Throughout this paper, the exact solution $u \in K$ and the Lagrange multiplier $\Lambda := F - a(u, \bullet) \in V^*$ are approximated by the discrete solution u_{NC} in some discrete analogue K_{NC} of K and a certain novel discrete Lagrange multiplier Λ_{NC} in the discrete space $\mathrm{CR}^1(\mathscr{T})$ (cf. (2.1)).

The *first main result* of this paper establishes an *a priori* error estimate for the Crouzeix–Raviart NCFEM under the known regularity property $\Delta u \in L^2(\Omega)$ for $\Delta\chi \in L^2(\Omega)$ (cf. Rodrigues, 1987, Proposition 5:2.2) so that $\lambda := f + \Delta u \in L^2(\Omega)$. The *second main result* yields two guaranteed *lower* bounds for the minimal energy $E(u)$ and so allows for an *a posteriori* control of the error $|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}$ in the discrete (i.e., piecewise) energy norm $|||\bullet|||_{\mathrm{NC}} := \|\nabla_{\mathrm{NC}}\bullet\|_{L^2(\Omega)}$. The *third main result* is an explicit residual-based *a posteriori* error analysis with reliable and efficient control over the error $|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}} + |||\Lambda - \Lambda_{\mathrm{NC}}|||_*$ with the dual norm $|||\bullet|||_*$ in $H^{-1}(\Omega)$ up to data oscillation terms. This extends the *a posteriori* error analysis of Bartels & Carstensen (2004), Braess (2005) for the Courant FEM to the Crouzeix–Raviart NCFEM; cf. also the recent work Braess *et al.* (2008) on the *a posteriori* error analysis for mixed FEMs and Gudi & Porwal (2014) for dG FEMs. Numerical experiments confirm guaranteed error control with moderate over-estimation by a factor typically in the range 2–3.5 and support adaptive over uniform mesh refinement. An empirical comparison of the two *a posteriori* error estimates is included and shows that both converge with the same convergence rate but the residual-based *a posteriori* error is better by a small multiplicative constant.

The remaining parts of this paper are organized as follows. Section 2 introduces the discretization of the obstacle problem and discusses the approximation of the nonhomogeneous Dirichlet data and the design of a specific conforming companion to the discrete solution. Section 3 presents a new *a priori* error analysis under the reduced regularity assumption. Section 4 derives two guaranteed lower bounds for the minimal energy and two *a posteriori* error estimates, followed by the discussion of efficiency in Section 5. The paper concludes with three computational benchmark examples in Section 6 on uniform and adapted triangulations.

The paper applies standard notation for Lebesgue and Sobolev spaces and their norms $\|\bullet\|_{L^2(\Omega)}$, $|||\bullet||| := \|\nabla\bullet\|_{L^2(\Omega)}$, as well as their local variants $\|\bullet\|_{L^2(\omega)}$ and $|||\bullet|||_\omega$ for $\omega \subseteq \Omega$. The integral mean is denoted by $⨍$. Moreover $A \lesssim B$ abbreviates $A \leqslant CB$ for some generic constant C (which solely depends on the shape regularity of the underlying triangulation in Section 2.1) and $A \approx B$ abbreviates $A \lesssim B \lesssim A$.

The analysis in this paper is essentially carried out explicitly for the two-dimensional case but the generalization to three dimensions is straightforward (with additional explanations stated whenever necessary).

2. Preliminaries

This section introduces the discretization of the obstacle problem and discusses the approximation of the nonhomogeneous Dirichlet data and of some conforming companion to a nonconforming Crouzeix–Raviart function.

2.1 *Discretization*

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal Lipschitz domain partitioned in a shape-regular triangulation $\mathcal{T}$ into triangles in the sense of Ciarlet (1978), with nodes $\mathcal{N}$, interior nodes $\mathcal{N}(\Omega)$ and nodes on the boundary $\mathcal{N}(\partial\Omega)$. The set of edges is denoted by $\mathcal{E}$, with interior edges $\mathcal{E}(\Omega) := \{E \in \mathcal{E} | E \nsubseteq \partial\Omega\}$, and edges $\mathcal{E}(\partial\Omega)$ along the boundary $\partial\Omega$. Given any node $z \in \mathcal{N}$, let $\mathcal{T}(z)$ denote the set of all triangles T with $z \in \mathcal{N}(T)$ the set of the three vertices of a triangle T, let $|\mathcal{T}(z)| \approx 1$ denote the number of triangles in $\mathcal{T}(z)$ and let $\overline{\omega}_z := \cup_{T\in T(z)} T$ denote the node patch around z; $\overline{\omega}_T := \cup_{z\in\mathcal{N}(T)}\overline{\omega}_z$ denotes a patch around each triangle $T \in \mathcal{T}$. Any edge $E \in \mathcal{E}$ has length $|E|$, midpoint $\mathrm{mid}(E)$ and unit normal ν_E; $\mathrm{mid}(\mathcal{E}) := \{\mathrm{mid}(E) | \ E \in \mathcal{E}\}$ denotes the set of the midpoints of all edges. For any $k \in \mathbb{N}_0$, set

$$
\begin{aligned}
P_k(T) :=& \{v_k : T \to \mathbb{R} | \ v_k \text{ is a polynomial of degree} \leqslant k\};\\
P_k(\mathcal{T}) :=& \{v_k \in L^\infty(\Omega) | \ \forall T \in \mathcal{T}, \ v_k|_T \in P_k(T)\};\\
\mathrm{CR}^1(\mathcal{T}) :=& \{v_{\mathrm{NC}} \in P_1(\mathcal{T}) | \ v_{\mathrm{NC}} \text{ continuous at } \mathrm{mid}(\mathcal{E})\};\\
V_{\mathrm{NC}} :=& \{v_{\mathrm{NC}} \in \mathrm{CR}^1(\mathcal{T}) | \ \forall E \in \mathcal{E}(\partial\Omega), \ v_{\mathrm{NC}}(\mathrm{mid}(E)) = 0\};\\
V_1(\mathcal{T}) :=& P_1(\mathcal{T}) \cap C_0(\Omega);\\
V_2(\mathcal{T}) :=& P_2(\mathcal{T}) \cap C_0(\Omega);\\
\mathcal{A}_{\mathrm{NC}} :=& \{v_{\mathrm{NC}} \in \mathrm{CR}^1(\mathcal{T}) | \ \forall E \in \mathcal{E}(\partial\Omega), \ v_{\mathrm{NC}}(\mathrm{mid}(E)) = ⨍_E u_{\mathrm{D}} \,\mathrm{d}s\};\\
K_{\mathrm{NC}} :=& \{v_{\mathrm{NC}} \in \mathcal{A}_{\mathrm{NC}} | \ \forall E \in \mathcal{E}(\Omega), \ ⨍_E \chi \,\mathrm{d}s \leqslant v_{\mathrm{NC}}(\mathrm{mid}(E))\}.
\end{aligned} \tag{2.1}
$$

The triangulation $\mathcal{T}$ is regular in the sense that any two distinct triangles in $\mathcal{T}$ with nonempty intersection are either identical or share exactly one common node or one common edge. The triangulation $\mathcal{T}$ is shape regular in the sense that any interior angle of any triangle is bounded from below by some universal positive constant γ_0 and all the generic constants hidden in the notation $\lesssim$ (or $\approx$) solely depend on $\gamma_0 > 0$. Given the triangulation $\mathcal{T}$, define the (local) mesh size $h_{\mathcal{T}} \in P_0(\mathcal{T})$, the L^2 projection $\Pi_0 : L^2(\Omega) \to P_0(\mathcal{T})$ and the oscillation $\operatorname{osc}(f, \mathcal{T})$ of f by $h_{\mathcal{T}}|_T := h_T := \operatorname{diam}(T)$, $\Pi_0|_T f := \fint_T f\,\mathrm{d}x := \int_T f\,\mathrm{d}x/|T|$ for all $T \in \mathcal{T}$ and $f \in L^2(\Omega)$ (as well as for vector-valued functions in $L^2(\Omega;\mathbb{R}^2)$, etc.) and

$$\operatorname{osc}^2(f, \mathcal{T}) := \|h_{\mathcal{T}}(f - \Pi_0 f)\|^2_{L^2(\Omega)}.$$

For any subsets $\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{T}$, set

$$\int_{\mathcal{T}_1} \bullet\,\mathrm{d}x := \int_{\cup\mathcal{T}_1} \bullet\,\mathrm{d}x \quad \text{and} \quad \|\bullet\|_{L^2(\mathcal{T}_1\cup\mathcal{T}_2)} := \|\bullet\|_{L^2(\cup(\mathcal{T}_1\cup\mathcal{T}_2))}.$$

With the piecewise gradient $\nabla_{\mathrm{NC}} v_{\mathrm{NC}} \in P_0(\mathcal{T};\mathbb{R}^2)$ of any discrete function $v_{\mathrm{NC}} \in \mathrm{CR}^1(\mathcal{T})$, the discrete energy product $a_{\mathrm{NC}} : \mathrm{CR}^1(\mathcal{T}) \times \mathrm{CR}^1(\mathcal{T}) \to \mathbb{R}$ reads

$$a_{\mathrm{NC}}(u_{\mathrm{NC}}, v_{\mathrm{NC}}) := \int_\Omega \nabla_{\mathrm{NC}} u_{\mathrm{NC}} \cdot \nabla_{\mathrm{NC}} v_{\mathrm{NC}}\,\mathrm{d}x, \quad \text{for all } u_{\mathrm{NC}}, v_{\mathrm{NC}} \in \mathrm{CR}^1(\mathcal{T})$$

and induces the discrete energy seminorm $|||\cdot|||_{\mathrm{NC}} := a_{\mathrm{NC}}(\cdot,\cdot)^{1/2}$ in $\mathrm{CR}^1(\mathcal{T})$. With the discrete Friedrichs inequality $\|v_{\mathrm{NC}}\|_{L^2(\Omega)} \lesssim |||v_{\mathrm{NC}}|||_{\mathrm{NC}}$, for all $v_{\mathrm{NC}} \in V_{\mathrm{NC}}$ (see Brenner, 2003), this is a norm in V_{NC}. The local variant on $\omega \subset \Omega$ of this discrete energy norm reads $|||\bullet|||_{\mathrm{NC}(\omega)} := \|\nabla_{\mathrm{NC}}\bullet\|_{L^2(\omega)}$.

The discrete analogue to the variational inequality (1.1) seeks $u_{\mathrm{NC}} \in K_{\mathrm{NC}}$ with

$$F(v_{\mathrm{NC}} - u_{\mathrm{NC}}) \leqslant a_{\mathrm{NC}}(u_{\mathrm{NC}}, v_{\mathrm{NC}} - u_{\mathrm{NC}}), \quad \text{for all } v_{\mathrm{NC}} \in K_{\mathrm{NC}}. \tag{2.2}$$

As in the continuous case (1.2), the discrete solution $u_{\mathrm{NC}} \in K_{\mathrm{NC}}$ is also the minimizer of the analogous discrete energy functional

$$E_{\mathrm{NC}}(v_{\mathrm{NC}}) := \tfrac{1}{2} a_{\mathrm{NC}}(v_{\mathrm{NC}}, v_{\mathrm{NC}}) - F(v_{\mathrm{NC}}) \quad \text{over all } v_{\mathrm{NC}} \in K_{\mathrm{NC}}. \tag{2.3}$$

Each edge $E \in \mathcal{E}(\Omega)$ is associated with its edge-oriented basis function $\psi_E \in \mathrm{CR}^1(\mathcal{T})$, which satisfies $\psi_E \equiv 1$ along E and $\psi_E(\operatorname{mid}(F)) = 0$ for any other edge $F \in \mathcal{E} \setminus \{E\}$, and has support $\overline{\omega}_E := \cup\{T \in \mathcal{T} \mid E \in \mathcal{E}(T)\}$.

LEMMA 2.1 For each edge $E \in \mathcal{E}(\Omega)$, the solution u_{NC} to the discrete variational inequality (2.2) satisfies the discrete consistency condition ($\perp$ abbreviates orthogonality in $\mathbb{R}$, i.e., $a \perp b$ means $ab = 0$ for $a, b \in \mathbb{R}$)

$$0 \leqslant u_{\mathrm{NC}}(\operatorname{mid}(E)) - \fint_E \chi\,\mathrm{d}s \perp F(\psi_E) - a_{\mathrm{NC}}(u_{\mathrm{NC}}, \psi_E) \leqslant 0. \tag{2.4}$$

Proof. The discrete consistency condition follows from direct considerations with the degrees of freedom in (2.2). □

The discrete consistency conditions are the discrete analogue of the well-known (continuous) consistency condition (Kinderlehrer & Stampacchia, 1980) that the solution $u \in H^2_{\mathrm{loc}}(\Omega)$ to (1.1) satisfies

$$0 \leqslant u - \chi \perp f + \Delta u \leqslant 0 \quad \text{a.e. in } \Omega, \tag{2.5}$$

where $\perp$ abbreviates pointwise orthogonality.

2.2 *Two interpolation operators*

The conforming and nonconforming interpolation operators read

$$I_{\mathrm{C}} : C(\overline{\Omega}) \to P_1(\mathcal{T}) \cap C(\overline{\Omega}), \quad v \mapsto \sum_{z \in \mathcal{N}} v(z)\varphi_z,$$
$$I_{\mathrm{NC}} : H^1(\Omega) \to \mathrm{CR}^1(\mathcal{T}), \quad v \mapsto \sum_{E \in \mathcal{E}} \left(\fint_E v \, \mathrm{d}s \right) \psi_E.$$

Here and throughout this paper, φ_z denotes the (conforming) nodal basis function associated with the node $z \in \mathcal{N}$ and ψ_E is the edge-oriented basis function of $\mathrm{CR}^1(\mathcal{T})$ associated with the edge $E \in \mathcal{E}$. Known interpolation error estimates in two dimensions involve the constants

$$\kappa_{\mathrm{C}}(\gamma) := \sqrt{\frac{1/4 + 2/j_{1,1}^2}{1 - |\cos(\gamma)|}} \quad \text{and} \quad \kappa_{\mathrm{NC}} := \sqrt{1/48 + 1/j_{1,1}^2} \leqslant 0.2982 \tag{2.6}$$

for the maximal interior angle γ in the triangle K and the smallest positive root $j_{1,1} \geqslant 3.8317$ of the Bessel function of the first kind.

Lemma 2.2 (properties of the interpolation operators) Any $v \in H^1(\Omega)$ and its interpolation $I_{\mathrm{NC}}v$ satisfy

ⓐ $\left\| h_{\mathcal{T}}^{-1}(v - I_{\mathrm{NC}}v) \right\|_{L^2(\Omega)} \leqslant \kappa_{\mathrm{NC}} |||v - I_{\mathrm{NC}}v|||_{\mathrm{NC}}$;

ⓑ $\nabla_{\mathrm{NC}} I_{\mathrm{NC}} v = \Pi_0 \nabla v$.

Any $v \in H^2(K)$ on a triangle K with diameter h_K and $\kappa_{\mathrm{C}}(\gamma)$ from (2.6) satisfies

ⓒ $\left\| h_K^{-1} \nabla (v - I_{\mathrm{C}} v) \right\|_{L^2(K)} \leqslant \kappa_{\mathrm{C}}(\gamma) \left\| D^2 v \right\|_{L^2(K)}$.

Remark 2.3 The assertions ⓐ–ⓒ hold also in the three-dimensional case with different universal constants (Ciarlet, 1978).

Proof. The proof of ⓐ follows as in Carstensen *et al.* (2012) with the improved constant in Carstensen & Gallistl (2014, Theorem 4).

The proof of ⓑ follows from an integration by parts on each triangle and the integral mean property of I_{NC} along each edge $E \in \mathscr{E}$.

Assertion ⓒ is contained in Carstensen *et al.* (2012, Theorem 3.1). □

2.3 *Nonhomogeneous dirichlet data*

The nonhomogeneous Dirichlet data u_{D} leads to affine spaces, where u_{D} needs to be adapted to the corresponding discretization. Let the edge $E := \operatorname{conv}\{A, B\}$ and define the corresponding bubble function $b_E := 6\varphi_A\varphi_B$ with the nodal basis functions φ_A, φ_B for the nodes A, B, which satisfies $\operatorname{supp}(b_E) = \operatorname{supp}(\varphi_A) \cap \operatorname{supp}(\varphi_B)$ and $\fint_E b_E \mathrm{d}s = 1$.

Given the Dirichlet data $u_{\mathrm{D}} \in C(\partial\Omega)$, the functions $u_{\mathrm{D}1} \in P_1(\mathscr{T}) \cap C(\overline{\Omega})$ and $u_{\mathrm{D}2} \in P_2(\mathscr{T}) \cap C(\Omega)$ approximate the nonhomogeneous Dirichlet data. The function $u_{\mathrm{D}1}$ is defined by the nodal values

$$u_{\mathrm{D}1}(z) := \begin{cases} u_{\mathrm{D}}(z) & \text{for } z \in \mathscr{N}(\partial\Omega), \\ 0 & \text{for } z \in \mathscr{N}(\Omega) \end{cases}$$

and the function $u_{\mathrm{D}2}$ is given by

$$u_{\mathrm{D}2} := u_{\mathrm{D}1} + \sum_{E \in \mathscr{E}(\partial\Omega)} \fint_E (u_{\mathrm{D}} - u_{\mathrm{D}1})\, \mathrm{d}s\, b_E \in P_2(\mathscr{T}) \cap C(\overline{\Omega}).$$

Given $u_{\mathrm{D}1}$ and $u_{\mathrm{D}2}$, define the affine spaces

$$\mathscr{A}_1 := u_{\mathrm{D}1} + V_1(\mathscr{T}), \quad \mathscr{A}_2 := u_{\mathrm{D}2} + V_2(\mathscr{T}), \quad \text{and recall } \mathscr{A}_{\mathrm{NC}} := I_{\mathrm{NC}} u_{\mathrm{D}} + V_{\mathrm{NC}}.$$

LEMMA 2.4 Let $u_{\mathrm{D}} \in C(\partial\Omega) \cap H^2(\mathscr{E}(\partial\Omega))$. Then the quadratic approximation $u_{\mathrm{D}2} \in P_2(\mathscr{T}) \cap C(\overline{\Omega})$ satisfies $\fint_E u_{\mathrm{D}}\, \mathrm{d}s = \fint_E u_{\mathrm{D}2}\, \mathrm{d}s$ for any edge $E \in \mathscr{E}(\partial\Omega)$, $u_{\mathrm{D}2}(z) = u_{\mathrm{D}}(z)$ at any node $z \in \mathscr{N}(\partial\Omega)$ and

$$|||u_{\mathrm{D}2} - I_{\mathrm{NC}} u_{\mathrm{D}2}|||_{\mathrm{NC}} \lesssim \left\| h_{\mathscr{E}}^{3/2} \frac{\partial^2 u_{\mathrm{D}}}{\partial s^2} \right\|_{L^2(\partial\Omega)}.$$

Proof. The first two properties follow from the definition of $u_{\mathrm{D}2}$. Since the function $u_{\mathrm{D}2}$ satisfies $u_{\mathrm{D}2}|_E \in P_1(E)$ for any interior edge $E \in \mathscr{E}(\Omega)$, it follows that $u_{\mathrm{D}2} - I_{\mathrm{NC}} u_{\mathrm{D}2} \neq 0$ only on triangles $T \in \mathscr{T}$ with $\mathscr{E}(T) \cap \mathscr{E}(\partial\Omega) \neq \emptyset$. Let $T \in \mathscr{T}$ be such a triangle and assume first that $E = \partial T \cap \partial\Omega$. Then it holds that

$$u_{\mathrm{D}2}|_T = u_{\mathrm{D}1}|_T + \fint_E (u_{\mathrm{D}} - u_{\mathrm{D}1})\, \mathrm{d}s b_E.$$

The properties of the interpolation operator I_{NC} lead to

$$(u_{\mathrm{D}2} - I_{\mathrm{NC}} u_{\mathrm{D}2})|_T = \fint_E (u_{\mathrm{D}} - u_{\mathrm{D}1})\, \mathrm{d}s (b_E - I_{\mathrm{NC}} b_E).$$

The bubble function b_E and its nonconforming interpolation $I_{\mathrm{NC}} b_E$ satisfy

$$|||b_E - I_{\mathrm{NC}} b_E|||_{\mathrm{NC}(\mathcal{T})} \lesssim 1.$$

It remains to estimate $\fint_E (u_\mathrm{D} - u_{\mathrm{D}1})\,\mathrm{d}s$. Let the edge $E := \mathrm{conv}\{A, B\}$ of length $h = |E| = |A - B|$ have the vertices A and B. Without loss of generality suppose $A = (0, 0)$ and $B = (0, h)$ and write $u_\mathrm{D}(s)$ for $u_\mathrm{D}(s, 0)$. The definition of $u_{\mathrm{D}1}$ shows

$$I := \fint_E (u_\mathrm{D} - u_{\mathrm{D}1})\,\mathrm{d}s = \int_0^h \left(u_\mathrm{D}(s) - \frac{u_\mathrm{D}(0) + u_\mathrm{D}(h)}{2} \right) \mathrm{d}s/h.$$

The function

$$\zeta(s, t) := \begin{cases} 1 & \text{for } t < s, \\ -1 & \text{for } s < t \end{cases}$$

satisfies $\int_0^h \int_0^h \zeta(x, y)\,\mathrm{d}y\,\mathrm{d}x = 0$. For the constant $c := \int_0^h \partial u_\mathrm{D}(t)/\partial t\,\mathrm{d}t/h \in \mathbb{R}$, the fundamental theorem of calculus along $E \equiv (0,\ h)$ leads to

$$\begin{aligned} I &= \frac{1}{2h} \int_0^h \left(\int_0^s \frac{\partial u_\mathrm{D}(t)}{\partial t} \mathrm{d}t - \int_s^h \frac{\partial u_\mathrm{d}(t)}{\partial t} \mathrm{d}t \right) \mathrm{d}s \\ &= \frac{1}{2h} \int_0^h \int_0^h \zeta(s, t) \frac{\partial u_\mathrm{D}(t)}{\partial t} \mathrm{d}t\,\mathrm{d}s \\ &= \frac{1}{2h} \int_0^h \int_0^h \zeta(s, t) \left(\frac{\partial u_\mathrm{D}(t)}{\partial t} - c \right) \mathrm{d}t\,\mathrm{d}s. \end{aligned}$$

Since $u_\mathrm{D}|_E \in H^2(E)$, the Cauchy inequality in $L^2(0, h)$ and $\zeta^2 = 1$, followed by the Poincaré inequality, lead to

$$|I| \leqslant \frac{\sqrt{h}}{2} \left\| \frac{\partial u_\mathrm{D}}{\partial s} - c \right\|_{L^2(0,\ h)} \leqslant \frac{h^{3/2}}{2} \pi \left\| \frac{\partial^2 u_\mathrm{D}}{\partial s^2} \right\|_{L^2(0,\ h)}.$$

This proves

$$|||u_{\mathrm{D}2} - I_{\mathrm{NC}} u_{\mathrm{D}2}|||_{\mathrm{NC}(\mathcal{T})} \lesssim h_E^{3/2} \left\| \frac{\partial^2 u_\mathrm{D}}{\partial s^2} \right\|_{L^2(E)}.$$

In the second case where two edges of T belong to $\mathcal{E}(\partial\Omega)$, the triangle inequality is used to obtain the general result (with some hidden extra factor 2). The sum of all these estimates concludes the proof. □

REMARK 2.5 In three dimensions, the bubble functions are defined analogously and the above proof employs an interpolation error estimate for the nodal interpolation.

LEMMA 2.6 For any $v_D \in C(\partial\Omega) \cap H^2(\mathcal{E}(\partial\Omega))$ with $v_D = 0$ at $\mathcal{N}(\partial\Omega)$ there exists $w_D \in H^1(\Omega)$ with $w_D|_{\partial\Omega} = v_D|_{\partial\Omega}$, $w_D|_E = 0$ for $E \in \mathcal{E}(\Omega)$, $I_{NC} w_D = 0$, integral mean $\Pi_0 w_D = 0$ and

$$|||w_D||| \lesssim \left\| h_{\mathcal{E}}^{3/2} \frac{\partial^2 v_D}{\partial s^2} \right\|_{L^2(\partial\Omega)}.$$

Proof. The proof follows from Bartels *et al.* (2004, Theorem 4.2). Therein a function w_D is defined by harmonic extension. Denote this function by $\tilde{w}_D$. It can be modified to w_D to achieve the property $\Pi_0 w_D = 0$. To this end, define, for each $T \in \mathcal{T}$ with $T := \operatorname{conv}\{A, B, C\}$, the cubic bubble function $b_T := 60\varphi_A\varphi_B\varphi_C$, which satisfies $\operatorname{supp}(b_T) = T$ and $⨍_T b_T \, dx = 1$. Then the function w_D is given by $w_D|_T := \tilde{w}_D|_T - ⨍_T \tilde{w}_D \, dx b_T$, for all $T \in \mathcal{T}$. □

2.4 *Conforming companion*

The design of two conforming companions to any $v_{NC} \in V_{NC}$ in $V_1(\mathcal{T})$ and $V_2(\mathcal{T})$ starts with the map $J_1 : V_{NC} \to V_1(\mathcal{T})$ defined, for $v_{NC} \in V_{NC}$, by

$$J_1(v_{NC}) := \sum_{z \in \mathcal{N}(\Omega)} \left(\sum_{T \in \mathcal{T}(z)} \frac{v_{NC}|_T(z)}{|\mathcal{T}(z)|} \right) \varphi_z.$$

Recall that $\varphi_z \in P_1(\mathcal{T}) \cap C(\overline{\Omega})$ denotes the P_1 nodal basis function associated with the node z in $\mathcal{N}(\Omega)$. Define $J_2 : V_{NC} \to V_2(\mathcal{T})$ for $v_{NC} \in V_{NC}$ by

$$J_2(v_{NC}) := J_1(v_{NC}) + \sum_{E \in \mathcal{E}} \left(⨍_E \left(v_{NC} - J_1(v_{NC}) \right) ds \right) b_E \tag{2.7}$$

with the bubble function b_E associated with the edge E.

LEMMA 2.7 (Carstensen *et al.*, 2014, Proposition 2.3) Given $v_{NC} \in V_{NC}$, the function $v_2 := J_2(v_{NC}) \in V_2(\mathcal{T})$ satisfies

(a) $|||v_{NC} - v_2|||_{NC(T)} \lesssim \min_{v \in V} |||v - v_{NC}|||_{NC(\omega_T)}$, for any $T \in \mathcal{T}$;

(b) $|||v_{NC} - v_2|||_{NC} \lesssim \min_{v \in V} |||v - v_{NC}|||_{NC}$.

Any $w_{NC} \in \mathcal{A}_{NC}$ satisfies $w_{NC} = I_{NC} u_{D2} + (w_{NC} - I_{NC} u_{D2})$ and $w_{NC} - I_{NC} u_{D2} \in V_{NC}$. A conforming companion of w_{NC} is designed with the boundary approximation u_{D2} of u_D and the aforementioned operator J_2, namely

$$w_2 := u_{D2} + J_2(w_{NC} - I_{NC} u_{D2}) \in \mathcal{A}_2. \tag{2.8}$$

$$\begin{tikzcd}
V_{\mathrm{NC}} \arrow[r, "J_2"] \arrow[d, shift left, "+I_{\mathrm{NC}}u_{\mathrm{D2}}"] & V_2(\mathscr{T}) \arrow[r, hook] \arrow[d, shift left, "+u_{\mathrm{D2}}"] & V \arrow[d, shift left, "+u_{\mathrm{D}}"] \\
\mathscr{A}_{\mathrm{NC}} \arrow[u, shift left, "-I_{\mathrm{NC}}u_{\mathrm{D2}}"] \arrow[r, "u_{\mathrm{D2}}+J_2(\bullet-I_{\mathrm{NC}}u_{\mathrm{D2}})"'] & \mathscr{A}_2 \arrow[u, shift left, "-u_{\mathrm{D2}}"] \arrow[r, "+w_D"'] & \mathscr{A} \arrow[u, shift left, "-u_{\mathrm{D}}"]
\end{tikzcd}$$

FIG. 1. Commutative diagram between the discrete and continuous spaces related to the admissible functions; $w_{\mathrm{D}} \in H^1(\Omega)$ extends $u_{\mathrm{D}} - u_{\mathrm{D2}}$ (cf. Lemma 2.6).

Figure 1 illustrates the relation between the vector spaces V_{NC}, $V_2(\mathscr{T})$ and the affine spaces $\mathscr{A}_{\mathrm{NC}}$, $\mathscr{A}_2$.

LEMMA 2.8 (properties of the conforming companion) Let $u_{\mathrm{D}} \in C(\partial\Omega) \cap H^2(\mathscr{E}(\partial\Omega))$. Given any $w_{\mathrm{NC}} \in \mathscr{A}_{\mathrm{NC}}$, the conforming companion $w_2 \in \mathscr{A}_2$ from (2.8) satisfies

ⓐ $I_{\mathrm{NC}} w_2 = w_{\mathrm{NC}}$;

ⓑ $$|||w_2 - I_{\mathrm{NC}} w_2|||_{\mathrm{NC}(T)} \lesssim \min_{v\in\mathscr{A}} |||w_{\mathrm{NC}} - v|||_{\mathrm{NC}(\omega_T)} + \left\| h_{\mathscr{E}}^{3/2} \frac{\partial^2 u_{\mathrm{D}}}{\partial s^2} \right\|_{L^2(\partial\omega_T\cap\partial\Omega)}, \qquad \text{for all } T \in \mathscr{T};$$

ⓒ $$|||w_2 - I_{\mathrm{NC}} w_2|||_{\mathrm{NC}} \lesssim \min_{v\in\mathscr{A}} |||w_{\mathrm{NC}} - v|||_{\mathrm{NC}} + \left\| h_{\mathscr{E}}^{3/2} \frac{\partial^2 u_{\mathrm{D}}}{\partial s^2} \right\|_{L^2(\partial\Omega)}.$$

Proof. A direct integration of (2.8) along any edge $E \in \mathscr{E}$ shows

$$\begin{aligned}
\fint_E w_2 \,\mathrm{d}x = \fint_E (u_{\mathrm{D2}} + J_2(w_{\mathrm{NC}} - I_{\mathrm{NC}}u_{\mathrm{D2}}))\,\mathrm{d}s \\
&= \fint_E u_{\mathrm{D2}}\,\mathrm{d}s + \fint_E J_1(w_{\mathrm{NC}} - I_{\mathrm{NC}}u_{\mathrm{D2}})\,\mathrm{d}s \\
&\quad + \fint_E ((w_{\mathrm{NC}} - I_{\mathrm{NC}}u_{\mathrm{D2}}) - J_1(w_{\mathrm{NC}} - I_{\mathrm{NC}}u_{\mathrm{D2}}))\,\mathrm{d}s \fint_E b_E\,\mathrm{d}s \\
&= \fint_E u_{\mathrm{D2}}\,\mathrm{d}x + \fint_E (w_{\mathrm{NC}} - I_{\mathrm{NC}}u_{\mathrm{D2}})\,\mathrm{d}s \\
&= I_{\mathrm{NC}}u_{\mathrm{D2}}(\mathrm{mid}(E)) + w_{\mathrm{NC}}(\mathrm{mid}(E)) - I_{\mathrm{NC}}u_{\mathrm{D2}}(\mathrm{mid}(E)) \\
&= \fint_E w_{\mathrm{NC}}\,\mathrm{d}s.
\end{aligned}$$

This implies ⓐ.

The proof of ⓑ utilises Lemmas 2.4–2.7. The triangle inequality yields

$$|||w_{\mathrm{NC}} - w_2|||_{\mathrm{NC}} \leqslant |||u_{\mathrm{D}2} - I_{\mathrm{NC}}u_{\mathrm{D}2}|||_{\mathrm{NC}} + |||w_{\mathrm{NC}} - I_{\mathrm{NC}}u_{\mathrm{D}2} - w_2 + u_{\mathrm{D}2}|||_{\mathrm{NC}}.$$

Lemma 2.4 shows $|||u_{\mathrm{D}2} - I_{\mathrm{NC}}u_{\mathrm{D}2}|||_{\mathrm{NC}} \lesssim \left\| h_{\mathcal{E}}^{3/2}\partial^2 u_{\mathrm{D}}/\partial s^2 \right\|_{L^2(\partial\Omega)}$. Recall $v_{\mathrm{NC}} := w_{\mathrm{NC}} - I_{\mathrm{NC}}u_{\mathrm{D}2} \in V_{\mathrm{NC}}$ and $v_2 := J_2(v_{\mathrm{NC}}) = -u_{\mathrm{D}2} + w_2 \in V_2(\mathcal{T})$ from (2.7). Lemma 2.7ⓐ implies

$$|||w_{\mathrm{NC}} - I_{\mathrm{NC}}u_{\mathrm{D}2} - w_2 + u_{\mathrm{D}2}|||_{\mathrm{NC}(T)} \lesssim \min_{v\in V} |||v - w_{\mathrm{NC}} + I_{\mathrm{NC}}u_{\mathrm{D}2}|||_{\mathrm{NC}(\omega_T)}.$$

With the approximations $u_{\mathrm{D}2}$ and $I_{\mathrm{NC}}u_{\mathrm{D}2}$ of the Dirichlet data u_{D}, Lemma 2.6 shows the existence of some function $w_{\mathrm{D}} \in H^1(\Omega)$ which satisfies $w_{\mathrm{D}}|_{\partial\Omega} = (u_{\mathrm{D}} - u_{\mathrm{D}2})|_{\partial\Omega}$, $I_{\mathrm{NC}}w_{\mathrm{D}} = 0$ and $|||w_{\mathrm{D}}||| \lesssim \left\| h_{\mathcal{E}}^{3/2}\partial^2 u_{\mathrm{D}}/\partial s^2 \right\|_{L^2(\partial\Omega)}$. The triangle inequality yields

$$\min_{v\in V} |||v - w_{\mathrm{NC}} + I_{\mathrm{NC}}u_{\mathrm{D}2}|||_{\mathrm{NC}(\omega_T)} \leqslant \min_{w\in\mathcal{A}} |||w - w_{\mathrm{NC}}|||_{\mathrm{NC}(\omega_T)} + |||u_{\mathrm{D}2} - I_{\mathrm{NC}}u_{\mathrm{D}2}|||_{\mathrm{NC}(\omega_T)} + |||w_{\mathrm{D}}|||_{\mathrm{NC}(\omega_T)}.$$

Lemmas 2.4 and 2.6 estimate the second and third terms to prove ⓑ.

The proof of ⓒ follows from the summation of the squares of inequality ⓑ and the finite overlap of the element patches ω_T. □

3. *A priori* error analysis

This section proves an *a priori* error estimate for the error $|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}$ for the solutions $u \in K$ and $u_{\mathrm{NC}} \in K_{\mathrm{NC}}$ to the continuous and discrete obstacle problem (1.1) and (2.2). The result uses only the regularity property $\Delta u \in L^2(\Omega)$ guaranteed for $f \in L^2(\Omega)$ and $\Delta\chi \in L^2(\Omega)$ (see Rodrigues, 1987, Proposition 5:2.2 for a proof) and generalizes Wang (2003) to singular solutions (e.g., as in the example of Section 6.4). The *a priori* result employs four subsets of the triangulation $\mathcal{T}$:

$$\begin{aligned} &\mathcal{T}_+ := \{T \in \mathcal{T} \mid u > \chi \text{ a.e. in } T\}, \qquad \mathcal{T}_0 := \{T \in \mathcal{T} \mid u = \chi \text{ a.e. in } T\},\\ &\mathcal{T}_M := \mathcal{T} \setminus (\mathcal{T}_+ \cup \mathcal{T}_0) \quad \text{and} \quad \mathcal{T}_{\partial\Omega} := \{T \in \mathcal{T} \mid |\partial T \cap \partial\Omega| > 0\}. \end{aligned} \tag{3.1}$$

In other words, $\mathcal{T}_+$ denotes the triangles without contact, $\mathcal{T}_0$ those with full contact, $\mathcal{T}_M$ contains the triangles at the interface and $\mathcal{T}_{\partial\Omega}$ the triangles with at least one edge on the boundary ($|\partial T \cap \partial\Omega| > 0$ means that $\partial T \cap \partial\Omega$ has positive length).

THEOREM 3.1 (*A priori* error estimate) Let $\chi \in H^2(\Omega)$. Then the continuous and discrete solutions $u \in K$ and $u_{\mathrm{NC}} \in K_{\mathrm{NC}}$ to the obstacle problem satisfy

$$\begin{aligned} |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}} \lesssim\ & |||u - I_{\mathrm{NC}}u|||_{\mathrm{NC}} + \|h_{\mathcal{T}}(|f| + |\lambda|)\|_{L^2(\Omega)} \\ & + \left\|h_{\mathcal{T}}D^2\chi\right\|_{L^2(\mathcal{T}_0\cup\mathcal{T}_M)} + \left\| h_{\mathcal{E}}^{3/2}\partial^2 u_{\mathrm{D}}/\partial s^2 \right\|_{L^2(\partial\Omega)}. \end{aligned}$$

The theorem above shows that the nonconforming FEM requires weaker regularity than the conforming Courant FEM, as it requires only that $\Delta u \in L^2(\Omega)$. This is in contrast to the *a priori* error analysis for the Courant FEM presented in Falk (1974), where full H^2 regularity is assumed and extra work is required for reduced elliptic regularity with $u \in H^{1+s}(\Omega)$ for $0 \leqslant s < 1$.

Proof of Theorem 3.1. **Step 1** of the proof utilises $a_{\mathrm{NC}}(u_{\mathrm{NC}}, u) = a_{\mathrm{NC}}(u_{\mathrm{NC}}, I_{\mathrm{NC}} u)$ and $u_2 := u_{\mathrm{D2}} + J_2(u_{\mathrm{NC}} - I_{\mathrm{NC}} u_{\mathrm{D2}})$ with $I_{\mathrm{NC}} u_2 = u_{\mathrm{NC}}$ from Section 2.4. With the abbreviation

$$\Lambda_{\mathrm{NC}}(v_{\mathrm{NC}}) := F(v_{\mathrm{NC}}) - a_{\mathrm{NC}}(u_{\mathrm{NC}}, v_{\mathrm{NC}}) \quad \text{for } v_{\mathrm{NC}} \in V_{\mathrm{NC}}, \tag{3.2}$$

the discrete variational inequality (2.2) shows for $I_{\mathrm{NC}} u \in K_{\mathrm{NC}}$ that

$$0 \leqslant \Lambda_{\mathrm{NC}}(u_{\mathrm{NC}} - I_{\mathrm{NC}} u).$$

Since $u_{\mathrm{NC}} = I_{\mathrm{NC}} u_2$, Lemma 2.2.ⓑ shows $\Pi_0 \nabla_{\mathrm{NC}}(u_2 - u_{\mathrm{NC}}) = \Pi_0 \nabla_{\mathrm{NC}}(u - I_{\mathrm{NC}} u) = 0$. This leads to

$$\begin{aligned} |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}^2 \leqslant \mathrm{LHS}^2 &:= |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}^2 + \Lambda_{\mathrm{NC}}(u_{\mathrm{NC}} - I_{\mathrm{NC}} u) \\ &= a(u, u - u_2) + a(u - I_{\mathrm{NC}} u, u_2 - u_{\mathrm{NC}}) + F(u_{\mathrm{NC}} - I_{\mathrm{NC}} u). \end{aligned}$$

With $\lambda := f + \Delta u \in L^2(\Omega)$, the consistency conditions (2.5) read $0 \leqslant u - \chi \perp \lambda \leqslant 0$ ($\perp$ abbreviates pointwise orthogonality a.e.). This, an integration by parts and $w_{\mathrm{D}} \in H^1(\Omega)$, designed as in Lemma 2.6 with $w_{\mathrm{D}}|_{\partial\Omega} = (u_{\mathrm{D}} - u_{\mathrm{D2}})|_{\partial\Omega}$, imply

$$\begin{aligned} a(u, u - u_2) &= a(u, u - u_2 - w_{\mathrm{D}}) + a(u, w_{\mathrm{D}}) \\ &= \int_\Omega (-\lambda)(u - u_2)\,\mathrm{d}x + F(u - u_2) + \int_\Omega w_{\mathrm{D}} \Delta u\,\mathrm{d}x + a(u, w_{\mathrm{D}}) \\ &= \int_\Omega (-\lambda)(\chi - u_{\mathrm{NC}})\,\mathrm{d}x + \int_\Omega (-\lambda)(u_{\mathrm{NC}} - u_2)\,\mathrm{d}x + F(u - u_2) + \int_\Omega w_{\mathrm{D}} \Delta u\,\mathrm{d}x + a(u, w_{\mathrm{D}}). \end{aligned}$$

A Cauchy inequality for the term $\int_\Omega w_{\mathrm{D}} \Delta u\,\mathrm{d}x$ followed by a Poincaré inequality (recall $\Pi_0 w_{\mathrm{D}} = 0$ and the Poincaré constant $h_{\mathscr{T}}/j_{1,1}$ from Laugesen & Siudeja, 2010) prove

$$\int_\Omega w_{\mathrm{D}} \Delta u\,\mathrm{d}x = \sum_{T \in \mathscr{T}_{\partial\Omega}} \int_T (w_{\mathrm{D}} - \Pi_0 w_{\mathrm{D}})(\Delta u - \Pi_0 \Delta u)\,\mathrm{d}x \leqslant \mathrm{osc}(\Delta u, \mathscr{T}_{\partial\Omega})/j_{1,1} |||w_{\mathrm{D}}|||.$$

The design of w_{D} yields $\int_E w_{\mathrm{D}}\,\mathrm{d}s = 0$ for any edge $E \in \mathscr{E}$ and hence for any $T \in \mathscr{T}$, $\int_T \nabla w_{\mathrm{D}}\,\mathrm{d}x = 0$. This, an integration by parts, and a Cauchy inequality yield, for $T \in \mathscr{T}_{\partial\Omega}$, that

$$\int_T \nabla u \cdot \nabla w_{\mathrm{D}}\,\mathrm{d}x = \int_T \nabla(u - u_{\mathrm{NC}}) \cdot \nabla w_{\mathrm{D}}\,\mathrm{d}x \leqslant |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}(T)} |||w_{\mathrm{D}}|||_T.$$

The combination of the aforementioned estimates proves

$$\begin{aligned}\mathrm{LHS}^2 \leqslant &\int_\Omega (-\lambda)(\chi - u_{\mathrm{NC}})\,\mathrm{d}x + \int_\Omega (-\lambda)(u_{\mathrm{NC}} - u_2)\,\mathrm{d}x\\ &+ \mathrm{osc}(\Delta u, \mathscr{T}_{\partial\Omega})/j_{1,1}|||w_{\mathrm{D}}||| + |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}|||w_{\mathrm{D}}|||\\ &+ a_{\mathrm{NC}}(u - I_{\mathrm{NC}}u, u_2 - u_{\mathrm{NC}}) + F(u - u_2 + u_{\mathrm{NC}} - I_{\mathrm{NC}}u).\end{aligned}$$

Since $I_{\mathrm{NC}}u_2 = u_{\mathrm{NC}}$, a Cauchy inequality, Lemma 2.2ⓐ, and Lemma 2.8ⓒ show for $A := |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}} + \left\| h_{\mathscr{E}}^{3/2} \partial^2 u_{\mathrm{D}}/\partial s^2 \right\|_{L^2(\partial\Omega)}$ that

$$\begin{aligned}\int_\Omega (-\lambda)(u_{\mathrm{NC}} - u_2)\,\mathrm{d}x &\lesssim \|h_{\mathscr{T}}\lambda\|_{L^2(\Omega)} A,\\ a_{\mathrm{NC}}(u - I_{\mathrm{NC}}u, u_2 - u_{\mathrm{NC}}) &\lesssim |||u - I_{\mathrm{NC}}u|||_{\mathrm{NC}} A,\\ F(u - u_2 + u_{\mathrm{NC}} - I_{\mathrm{NC}}u) &\lesssim \|h_{\mathscr{T}} f\|_{L^2(\Omega)} A.\end{aligned}$$

With $\mathrm{RHS}_1 := |||u - I_{\mathrm{NC}}u|||_{\mathrm{NC}} + \|h_{\mathscr{T}}(|f| + |\lambda|)\|_{L^2(\Omega)} + \left\| h_{\mathscr{E}} \partial^2 u_{\mathrm{D}}/\partial s^2 \right\|_{L^2(\partial\Omega)}$, the combination of the previous estimates and Lemma 2.6 results in

$$\mathrm{LHS}^2 - \int_\Omega (-\lambda)(\chi - u_{\mathrm{NC}})\,\mathrm{d}x \lesssim \mathrm{RHS}_1\Big(|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}} + \mathrm{RHS}_1\Big). \tag{3.3}$$

Step 2 is the analysis of

$$I_T := \int_T (-\lambda)(\chi - u_{\mathrm{NC}})\,\mathrm{d}x \quad \text{for any triangle } T \in \mathscr{T}.$$

It resolves the subtle third case, where $T \in \mathscr{T}_M$ is neither fully in the contact zone nor in the noncontact zone. Recall the three subsets $\mathscr{T}_+$, $\mathscr{T}_0$ and $\mathscr{T}_M$ from (3.1).

Case 1: For any $T \in \mathscr{T}_+$ it holds that $u > \chi$ a.e. on T and the consistency condition (2.5) reveals $\lambda = 0$ and hence $I_T = 0$.

Case 2: For any $T \in \mathscr{T}_0$, $u \equiv \chi$ a.e. on T. Since $\lambda := (\Delta u + f) \leqslant 0$, $\lambda_T := \fint_T \lambda\,\mathrm{d}x \leqslant 0$. Since $I_{\mathrm{NC}}\chi - u_{\mathrm{NC}}$ is affine on T and $\fint_E (I_{\mathrm{NC}}\chi - u_{\mathrm{NC}})\,\mathrm{d}x \leqslant 0$, for all edges $E \in \mathscr{E}(T)$, it follows that

$$\int_T (I_{\mathrm{NC}}\chi - u_{\mathrm{NC}})\,\mathrm{d}x \leqslant 0, \quad \text{whence} \quad 0 \leqslant \int_T \lambda_T (I_{\mathrm{NC}}\chi - u_{\mathrm{NC}})\,\mathrm{d}x.$$

With $e := u - u_{\mathrm{NC}} = \chi - u_{\mathrm{NC}}$ on $T \in \mathscr{T}_0$ and $e_T := \fint_T e\,\mathrm{d}x$, this leads to

$$\begin{aligned}I_T &= \int_T (\lambda_T - \lambda)e\,\mathrm{d}x - \int_T \lambda_T (I_{\mathrm{NC}}\chi - u_{\mathrm{NC}})\,\mathrm{d}x - \int_T \lambda_T(\chi - I_{\mathrm{NC}}\chi)\,\mathrm{d}x\\ &\leqslant \int_T (\lambda_T - \lambda)(e - e_T)\,\mathrm{d}x - \int_T \lambda_T(\chi - I_{\mathrm{NC}}\chi)\,\mathrm{d}x.\end{aligned}$$

Since $\Pi_0 \nabla \chi = \nabla_{\mathrm{NC}} I_{\mathrm{NC}} \chi$, a Poincaré inequality with the constant $h_{\mathscr{T}}/j_{1,1}$ from Laugesen & Siudeja (2010), and Lemma 2.2ⓐ (with $\kappa_{\mathrm{NC}} \leqslant 0.2982$) in the previous estimate shows

$$I_T \leqslant \operatorname{osc}(\lambda, T)/j_{1,1} |||e|||_{\mathrm{NC}(T)} + h_T^2 \kappa_{\mathrm{NC}}/j_{1,1} \, \|\lambda_T\|_{L^2(T)} \left\| D^2 \chi \right\|_{L^2(T)}.$$

Case 3: In the remaining case $T \in \mathscr{T}_M$, $\left|\{x \in T |\, u(x) = \chi(x)\}\right| < |T|$ and so the compact subset

$$\mathscr{C} := \{x \in T |\, u(x) = \chi(x) \geqslant u_{\mathrm{NC}}(x)\}$$

has a measure $|\mathscr{C}| < |T|$. (Note that u, χ, u_{NC} are continuous on T and so $\mathscr{C}$ is closed.) For $x \in T \setminus \mathscr{C}$ it follows that either $\chi(x) < u(x)$ or $u_{\mathrm{NC}}(x) < \chi(x)$. The continuous consistency conditions (2.5) show that $\lambda(x) = 0$ for $\chi(x) < u(x)$ and since $\lambda \leqslant 0$ a.e. in Ω, it follows that for $x \in T$ with $u_{\mathrm{NC}}(x) < \chi(x)$,

$$(-\lambda(x))(\chi(x) - u_{\mathrm{NC}}(x)) \leqslant 0.$$

These observations and a Cauchy inequality imply

$$I_T \leqslant \int_{\mathscr{C}} (-\lambda)(\chi - u_{\mathrm{NC}}) \, \mathrm{d}x \leqslant \|\lambda\|_{L^2(\mathscr{C})} |\mathscr{C}|^{1/2} \, \|\chi - u_{\mathrm{NC}}\|_{L^\infty(\mathscr{C})}. \tag{3.4}$$

For any edge $E \in \mathscr{E}$,

$$\int_E (\chi - u_{\mathrm{NC}}) \, \mathrm{d}s = \int_E (I_{\mathrm{NC}} \chi - u_{\mathrm{NC}}) \, \mathrm{d}x \leqslant 0.$$

(This follows for all interior edges $E \in \mathscr{E}(\Omega)$ since $u_{\mathrm{NC}} \in K_{\mathrm{NC}}$, and from $\chi \leqslant u_{\mathrm{D}}$ on $\partial\Omega$, also for any edge on the boundary $E \in \mathscr{E}(\partial\Omega)$.) Hence, the continuous function $w := \chi - u_{\mathrm{NC}}$ satisfies $w \leqslant 0$ at least at one point on ∂T.

Since $w \geqslant 0$ on $\mathscr{C}$, the function $w \in H^2(T)$ has some zero $x_0 \in T$. Therefore, for any $x \in T$, let $F := \operatorname{conv}\{x, x_0\}$. A nondegenerate triangle $K := \operatorname{conv}\{x_0, x, P\}$ can be defined as follows (cf. Fig. 2): Any straight line s_0 through x and x_0 has parallels through the three vertices of T. Two of them, s_1 and s_2, have a maximal distance d and the interior of the triangle T lies between them. Hence $d \geqslant \operatorname{width}(T)$ and elementary considerations prove $\operatorname{width}(T) = \min\{\text{height of a vertex of } T\}$. The distance of the straight line s_0 to one of the lines s_1 or s_2 is d' and the other is $d - d'$. Without loss of generality let d' be the bigger distance with $d' \geqslant d/2$. Therefore there exists a vertex P of T such that the height d' of P onto the line s_0, which includes $F := \operatorname{conv}\{x, x_0\}$, satisfies $d' \geqslant d/2$. Then, $K := \operatorname{conv}\{x_0, x, P\}$ satisfies $|K| \geqslant |F| \operatorname{width}(T)/4$. The triangle T satisfies

$$|T| = h_T \operatorname{width}(T)/2 = h_T \operatorname{width}(T)^2/(2\operatorname{width}(T)).$$

Since all angles γ in the triangle T satisfy $\gamma_0 \leqslant \gamma$ (by shape regularity), the definition of the tangent shows $h_T/2 \tan \gamma_0 \leqslant \operatorname{width}(T)$. This proves

$$|T|^{1/2} \tan^{1/2}(\gamma_0) \leqslant \operatorname{width}(T).$$

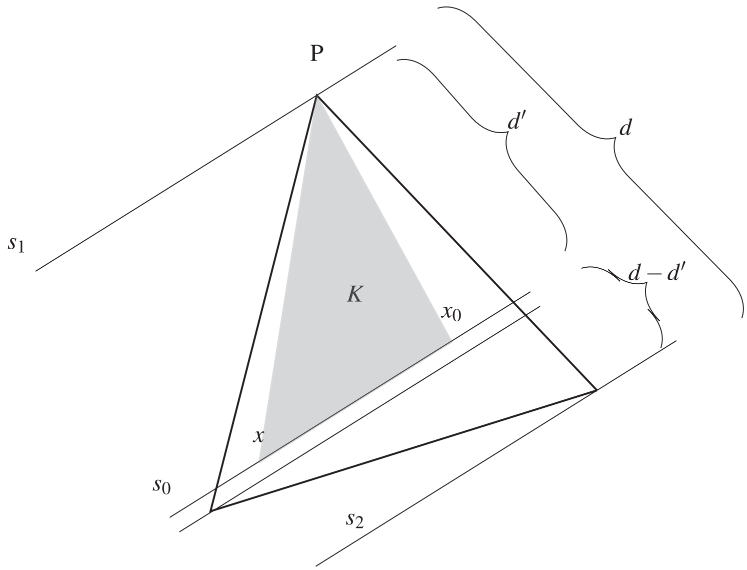

FIG. 2. Design of the triangle K in Theorem 3.1.

Since $|K| \leqslant |T|$ it follows that

$$|F| \tan^{1/2}(\gamma_0) \frac{|K|^{1/2}}{4} \leqslant |F| \tan^{1/2}(\gamma_0) \frac{|T|^{1/2}}{4} \leqslant |K|.$$

In other words,

$$\frac{|F|}{|K|^{1/2}} \leqslant 4 \cot^{1/2}(\gamma_0). \tag{3.5}$$

Suppose for the moment that $w \in C^1(\Omega)$ with $w(x_0) = 0$. Then

$$|w(x)| \leqslant w(x_0) + \int_0^1 |\nabla w(x_0 + t(x - x_0)) \cdot (x - x_0)| \mathrm{d}t \leqslant |F| \fint_F |\nabla w| \, \mathrm{d}s.$$

The trace identity (Carstensen *et al.*, 2012, Lemma 2.1) yields

$$\fint_F |\nabla w| \, \mathrm{d}s = \fint_K |\nabla w| \, \mathrm{d}x + \frac{1}{2} \fint_K D(|\nabla w|) \cdot (x - P) \, \mathrm{d}x.$$

The combination of the previous two displayed formulas shows

$$|w(x)| \leqslant |F| \fint_K |\nabla w| \, \mathrm{d}x + \frac{|F|}{2} \fint_K D(|\nabla w|) \cdot (x - P) \, \mathrm{d}x.$$

Notice that the derivative of the modulus function is bounded by the modulus of the derivative. The estimate (3.5) and a Cauchy inequality imply

$$\cot^{1/2}(\gamma_0)|w(x)| \leqslant |||w|||_K + h_T \left\|D^2 w\right\|_{L^2(K)}.$$

Since x is arbitrary in T and since $D^2 w = D^2\chi$ it follows that

$$\|\chi - u_{\mathrm{NC}}\|_{L^\infty(T)} \leqslant \tan^{1/2}(\gamma_0)(|||\chi - u_{\mathrm{NC}}|||_{\mathrm{NC}(T)} + h_T \left\|D^2\chi\right\|_{L^2(T)}).$$

A triangle and a Poincaré inequality (with $\Pi_0\nabla\chi = \nabla_{\mathrm{NC}} I_{\mathrm{NC}}\chi$) show

$$\|\chi - u_{\mathrm{NC}}\|_{L^\infty(T)} \leqslant \tan^{1/2}(\gamma_0)(|||I_{\mathrm{NC}}\chi - u_{\mathrm{NC}}|||_{\mathrm{NC}(T)} + h_T(1 + 1/j_{1,1}) \left\|D^2\chi\right\|_{L^2(T)}). \tag{3.6}$$

Estimate (3.6) holds for all $w := \chi - u_{\mathrm{NC}} \in C^1(T) \cap H^2(T)$. A density argument reveals that it also holds for all $\chi \in H^2(T)$. Since $\nabla(I_{\mathrm{NC}}\chi - u_{\mathrm{NC}})$ is constant on the triangle T, it follows that

$$\begin{aligned}
|||I_{\mathrm{NC}}\chi - u_{\mathrm{NC}}|||_{\mathrm{NC}(T)} &= |T|^{1/2}|\mathscr{C}|^{-1/2}| \,\|\nabla(I_{\mathrm{NC}}\chi - u_{\mathrm{NC}})\|_{L^2(\mathscr{C})} \\
&\leqslant |T|^{1/2}|\mathscr{C}|^{-1/2}\Big(\|\nabla(\chi - I_{\mathrm{NC}}\chi)\|_{L^2(\mathscr{C})} \\
&\qquad + \|\nabla(\chi - u)\|_{L^2(\mathscr{C})} + \|\nabla(u - u_{\mathrm{NC}})\|_{L^2(\mathscr{C})}\Big).
\end{aligned}$$

A Poincaré inequality (with the Poincaré constant $h_T/j_{1,1}$ from Laugesen & Siudeja, 2010) yields

$$\|\nabla(\chi - I_{\mathrm{NC}}\chi)\|_{L^2(\mathscr{C})} \leqslant \|\nabla(\chi - I_{\mathrm{NC}}\chi)\|_{L^2(T)} \leqslant h_T \left\|D^2\chi\right\|_{L^2(T)}/j_{1,1}.$$

Since $u - \chi = 0$ on the compact set $\mathscr{C}$, it holds that $\|\nabla(u - \chi)\|_{L^2(\mathscr{C})} = 0$. Therefore

$$|||I_{\mathrm{NC}}\chi - u_{\mathrm{NC}}|||_{\mathrm{NC}(T)} \leqslant |T|^{1/2}|\mathscr{C}|^{-1/2}\Big(h_T/j_{1,1} \left\|D^2\chi\right\|_{L^2(T)} + |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}(T)}\Big).$$

Combination with (3.4) and (3.6) results in

$$I_T \leqslant \tan^{1/2}(\gamma_0)\, \|h_T\lambda\|_{L^2(\mathscr{C})} \Big(|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}(T)} + \left\|h_T D^2\chi\right\|_{L^2(\mathscr{T})}(1 + 2/j_{1,1})\Big).$$

This concludes Case 3.

The summary of the three cases leads to

$$\int_\Omega (-\lambda)(\chi - u_{\mathrm{NC}})\,\mathrm{d}x \lesssim \|h_{\mathscr{T}}\lambda\|_{L^2(\Omega)} (|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}} + \left\|h_{\mathscr{T}} D^2\chi\right\|_{L^2(\mathscr{T}_0 \cup \mathscr{T}_M)}). \tag{3.7}$$

The combination of (3.3) and (3.7) reads

$$\mathrm{LHS}^2 \lesssim \mathrm{RHS}_1 \Big(\mathrm{RHS}_1 + \left\|h_{\mathscr{T}} D^2\chi\right\|_{L^2(\mathscr{T}_0 \cup \mathscr{T}_M)} + |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}\Big).$$

A Young inequality and the absorption of $|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}} \leqslant \mathrm{LHS}$ conclude the proof. □

4. *A posteriori* error analysis

This section provides guaranteed lower bounds for the exact energy $E(u)$ in (1.2) based on the discrete energy $E_{\mathrm{NC}}(u_{\mathrm{NC}})$ in (2.3), as well as reliable error estimators for the obstacle problem. Given the discrete Crouzeix–Raviart solution $u_{\mathrm{NC}} \in K_{\mathrm{NC}}$ of (2.2), define some discrete Lagrange multiplier

$$\lambda_{\mathrm{NC}} := \sum_{E\in\mathscr{E}(\Omega)} \rho_E \psi_E / \|\psi_E\|^2_{L^2(\Omega)} \qquad \text{with } \rho_E := F(\psi_E) - a_{\mathrm{NC}}(u_{\mathrm{NC}}, \psi_E) \tag{4.1}$$

with the edge-oriented basis function $\psi_E \in V_{\mathrm{NC}}$ associated with the edge $E \in \mathscr{E}(\Omega)$. Recall Λ_{NC} from (3.2) and notice that

$$\Lambda_{\mathrm{NC}}(v_{\mathrm{NC}}) = \int_\Omega \lambda_{\mathrm{NC}} v_{\mathrm{NC}}\,\mathrm{d}x, \qquad \text{for all } v_{\mathrm{NC}} \in V_{\mathrm{NC}}.$$

In the sequel, $\Lambda_{\mathrm{NC}}(v)$ will always denote the L^2 scalar product of any Lebesgue function $v \in L^2(\Omega)$ with $\lambda_{\mathrm{NC}} \in V_{\mathrm{NC}}$. The residual-based *a posteriori* error estimate involves the continuous Lagrange multiplier

$$\Lambda := F - a(u,\cdot) \in V^*$$

with the L^2 representation $\lambda = f + \Delta u$. The dual norm $|||\Lambda - \Lambda_{\mathrm{NC}}|||_*$ reads

$$|||\Lambda - \Lambda_{\mathrm{NC}}|||_* := \sup_{v\in V\setminus\{0\}} \int_\Omega v(\lambda - \lambda_{\mathrm{NC}})\,\mathrm{d}x / |||v|||.$$

The subset $\mathscr{T}' := \left\{T \in \mathscr{T} \,\middle|\, 0 < |\{x \in T \,|\, \lambda_{\mathrm{NC}}(x) > 0\}|\right\}$ of $\mathscr{T}$ is employed in Theorems 4.1 and 4.4.

The lower bounds for the exact energy $E(u)$ are given in the following theorem with the constant $\kappa_{\mathrm{NC}} \leqslant 0.2983$ from Lemma 2.2ⓐ.

THEOREM 4.1 (Lower energy bounds) The discrete solution u_{NC} and the continuous solution u to the obstacle problem satisfy

$$ⓐ\ E_{\mathrm{NC}}(u_{\mathrm{NC}}) - \frac{\kappa^2_{\mathrm{NC}}}{2}\|h_{\mathscr{T}} f\|^2_{L^2(\Omega)} \leqslant E(u) - \Lambda_{\mathrm{NC}}(u_{\mathrm{NC}} - I_{\mathrm{NC}} u) \leqslant E(u)$$

$$\begin{aligned} ⓑ\ & E_{\mathrm{NC}}(u_{\mathrm{NC}}) - \left(\kappa_{\mathrm{NC}}\|h_{\mathscr{T}}(f - \lambda_{\mathrm{NC}})\|_{L^2(\Omega)} + \mathrm{osc}(\lambda_{\mathrm{NC}}, \mathscr{T}')\right)^2/2 \\ & - \int_{\mathscr{T}'} (\chi - u_{\mathrm{NC}})\Pi_0 \lambda_{\mathrm{NC}}\,\mathrm{d}x + \int_{\mathscr{T}\setminus\mathscr{T}'} (I_{\mathrm{NC}}\chi - \chi)\lambda_{\mathrm{NC}}\,\mathrm{d}x \\ & \leqslant E(u) - \int_{\mathscr{T}\setminus\mathscr{T}'} (\chi - u)\lambda_{\mathrm{NC}}\,\mathrm{d}x - \int_{\mathscr{T}'} (\chi - u)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x \leqslant E(u). \end{aligned}$$

The lower energy bounds of Theorem 4.1 are of separate interest, but may be used for error control of any $v \in K$ based on the identity

$$\frac{1}{2}|||u - v|||^2 + \Lambda(u - v) = E(v) - E(u). \tag{4.2}$$

Combination with Theorem 4.1ⓑ leads to the error bound in the following result where all summands on the left-hand side are non-negative and all summands on the right-hand side are computable.

COROLLARY 4.2 Any $v \in K$ satisfies

$$\begin{aligned}\frac{1}{2}|||u-v|||^2 + \Lambda(u-v) + \int_{\mathcal{T}\setminus\mathcal{T}'}(\chi-u)\lambda_{\mathrm{NC}}\,\mathrm{d}x + \int_{\mathcal{T}'}(\chi-u)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x \\ \leqslant E(v) - E_{\mathrm{NC}}(u_{\mathrm{NC}}) + \frac{1}{2}\Big(\kappa_{\mathrm{NC}}\,\|h_{\mathcal{T}}(f-\lambda_{\mathrm{NC}})\|_{L^2(\Omega)} + \mathrm{osc}(\lambda_{\mathrm{NC}},\mathcal{T}')\Big)^2 \\ + \int_{\mathcal{T}'}(\chi-u_{\mathrm{NC}})\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x + \int_{\mathcal{T}\setminus\mathcal{T}'}(\chi-I_{\mathrm{NC}}\chi)\lambda_{\mathrm{NC}}\,\mathrm{d}x.\end{aligned}$$

□

REMARK 4.3 In practice, $v \in K$ can be chosen with the help of any conforming companion, for example J_2 from (2.7). Then $v = \max\{\chi, w_{\mathrm{D}} + u_{\mathrm{D2}} + J_2(u_{\mathrm{NC}} - I_{\mathrm{NC}}u_{\mathrm{D2}})\} \in K$ for $w_{\mathrm{D}} \in H^1(\Omega)$ as in Lemma 2.6 with $w_{\mathrm{D}}|_{\partial\Omega} = u_{\mathrm{D}} - u_{\mathrm{D2}}$ is an admissible function.

The following residual-based *a posteriori* error analysis generalizes Braess (2005).

THEOREM 4.4 (Guaranteed upper error bound) Any $v \in K$ satisfies

$$\begin{aligned}ⓐ\ \frac{1}{2}|||u-u_{\mathrm{NC}}|||_{\mathrm{NC}}^2 + \Lambda(u-v) + \int_{\mathcal{T}'}(\chi-u)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x + \int_{\mathcal{T}\setminus\mathcal{T}'}(\chi-u)\lambda_{\mathrm{NC}}\,\mathrm{d}x \\ \leqslant \frac{1}{2}|||v-u_{\mathrm{NC}}|||_{\mathrm{NC}}^2 + \int_{\mathcal{T}'}(\chi-v)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x + \int_{\mathcal{T}\setminus\mathcal{T}'}(\chi-v)\lambda_{\mathrm{NC}}\,\mathrm{d}x \\ + \frac{1}{2}\left(\kappa_{\mathrm{NC}}\,\|h_{\mathcal{T}}(f-\lambda_{\mathrm{NC}})\|_{L^2(\Omega)} + \mathrm{osc}(\lambda_{\mathrm{NC}},\mathcal{T}')/j_{1,1}\right)^2\end{aligned}$$

$$\begin{aligned}ⓑ\ |||\Lambda-\Lambda_{\mathrm{NC}}|||_* \leqslant |||u-u_{\mathrm{NC}}|||_{\mathrm{NC}} + \mathrm{osc}(f-\lambda_{\mathrm{NC}},\mathcal{T})/j_{1,1} \\ + \frac{1}{2}\left\|\Pi_0(f-\lambda_{\mathrm{NC}})(\bullet-\mathrm{mid}(\mathcal{T}))\right\|_{L^2(\Omega)} + |||u_{\mathrm{NC}}-v|||_{\mathrm{NC}} + |||w_{\mathrm{D}}|||.\end{aligned}$$

REMARK 4.5 The comparison of Corollary 4.2 with Theorem 4.4ⓐ is possible through the following formula (which follows from straightforward algebra):

$$E(v) - E_{\mathrm{NC}}(u_{\mathrm{NC}}) = \frac{1}{2}|||v-u_{\mathrm{NC}}|||_{\mathrm{NC}}^2 + \Lambda_{\mathrm{NC}}(u_{\mathrm{NC}}-v) + (\Lambda_{\mathrm{NC}}-F)(v-I_{\mathrm{NC}}v). \tag{4.3}$$

Theorem 4.1 plus (4.2) and (4.3) imply Theorem 4.4 with different constants in the form

$$\begin{aligned}|||u-u_{\mathrm{NC}}|||_{\mathrm{NC}}^2 \leqslant 3|||u_{\mathrm{NC}}-v|||_{\mathrm{NC}} + \Lambda_{\mathrm{NC}}(u_{\mathrm{NC}}-v) + 3\kappa_{\mathrm{NC}}^2\,\|h_{\mathcal{T}}(f-\lambda_{\mathrm{NC}})\|_{L^2(\Omega)}^2 + 2\,\mathrm{osc}(\lambda_{\mathrm{NC}},\mathcal{T}')^2 \\ + 2\int_{\mathcal{T}'}(\chi-u_{\mathrm{NC}})\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x + 2\Lambda_{\mathrm{NC}}(u_{\mathrm{NC}}-v) + 2\int_{\mathcal{T}\setminus\mathcal{T}'}(\chi-I_{\mathrm{NC}}\chi)\lambda_{\mathrm{NC}}\,\mathrm{d}x.\end{aligned}$$

The remaining parts of this section are devoted to the proofs of Theorems 4.1 and 4.4. The analysis of the nonlinearity utilizes the subsequent lemma.

LEMMA 4.6 The continuous and discrete solutions u and u_{NC}, and the discrete Lagrange multiplier Λ_{NC}, satisfy

$$\begin{aligned}\Lambda_{\mathrm{NC}}(u-u_{\mathrm{NC}}) = &\int_{\mathcal{T}'}(\chi-u_{\mathrm{NC}})\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x - \int_{\mathcal{T}\setminus\mathcal{T}'}\lambda_{\mathrm{NC}}(I_{\mathrm{NC}}\chi-u)\,\mathrm{d}x\\ &-\int_{\mathcal{T}'}(\chi-u)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x + \int_{\mathcal{T}'}(1-\Pi_0)\lambda_{\mathrm{NC}}(1-\Pi_0)(u-u_{\mathrm{NC}})\,\mathrm{d}x.\end{aligned}$$

Proof. The discrete consistency conditions (2.4) show that the product

$$\lambda_{\mathrm{NC}}(I_{\mathrm{NC}}\chi-u_{\mathrm{NC}})$$

vanishes at the midpoint $\mathrm{mid}(E)$ of any edge $E\in\mathcal{E}$. Since the three-point quadrature formula at the edge midpoints is exact for quadratic polynomials $P_2(T)$ on the triangle T, it follows that

$$\int_T \lambda_{\mathrm{NC}}(I_{\mathrm{NC}}\chi-u_{\mathrm{NC}})\,\mathrm{d}x = 0 \quad \text{for any } T\in\mathcal{T}.$$

Hence, with $\int_{\mathcal{T}\setminus\mathcal{T}'}\bullet\,\mathrm{d}x := \sum_{T\in\mathcal{T}\setminus\mathcal{T}'}\int_T\bullet\,\mathrm{d}x$, it holds that

$$\int_{\mathcal{T}\setminus\mathcal{T}'}\lambda_{\mathrm{NC}}(u-u_{\mathrm{NC}})\,\mathrm{d}x = \int_{\mathcal{T}\setminus\mathcal{T}'}\lambda_{\mathrm{NC}}(u-I_{\mathrm{NC}}\chi)\,\mathrm{d}x.$$

It remains to perform the analysis for $T\in\mathcal{T}'$. Recall that the L^2 projection Π_0 onto $P_0(\mathcal{T})$ is the piecewise integral mean operator with respect to the triangulation $\mathcal{T}$. The inequalities

$$\Pi_0\lambda_{\mathrm{NC}} \leqslant 0 \leqslant \Pi_0(u_{\mathrm{NC}}-I_{\mathrm{NC}}\chi) \text{ a.e.}$$

and algebraic transformations motivate the split

$$\begin{aligned}\int_T \lambda_{\mathrm{NC}}(u-u_{\mathrm{NC}})\,\mathrm{d}x &= \int_T (u-u_{\mathrm{NC}})\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x + \int_T (1-\Pi_0)\lambda_{\mathrm{NC}}(u-u_{\mathrm{NC}})\,\mathrm{d}x\\ &= \int_T (u-\chi)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x + \int_T (\chi-u_{\mathrm{NC}})\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x\\ &\quad + \int_T (1-\Pi_0)\lambda_{\mathrm{NC}}(1-\Pi_0)(u-u_{\mathrm{NC}})\,\mathrm{d}x.\end{aligned}$$

The combination of the aforementioned estimates concludes the proof. □

Proof of Theorem 4.1. The proof is divided into seven steps.

Step 1. The properties of the nonconforming interpolation operator and a direct calculation lead to

$$\frac{1}{2}|||u-I_{\mathrm{NC}}u|||_{\mathrm{NC}}^2 = E(u) - E_{\mathrm{NC}}(I_{\mathrm{NC}}u) + F(u-I_{\mathrm{NC}}u).$$

Step 2. Some algebra with the definition of Λ_{NC} in (3.2) leads to

$$\Lambda_{\mathrm{NC}}(u_{\mathrm{NC}} - I_{\mathrm{NC}}u) + \frac{1}{2}|||I_{\mathrm{NC}}u - u_{\mathrm{NC}}|||_{\mathrm{NC}}^2 = E_{\mathrm{NC}}(I_{\mathrm{NC}}u) - E_{\mathrm{NC}}(u_{\mathrm{NC}}).$$

Step 3. Since $\int_T \nabla_{\mathrm{NC}}(u - I_{\mathrm{NC}}u)\,\mathrm{d}x = 0$ on each triangle $T \in \mathcal{T}$, the Pythagorean theorem yields

$$|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}^2 = |||u - I_{\mathrm{NC}}u|||_{\mathrm{NC}}^2 + |||I_{\mathrm{NC}}u - u_{\mathrm{NC}}|||_{\mathrm{NC}}^2.$$

Step 4. The combination of Steps 1–3 leads to

$$\Lambda_{\mathrm{NC}}(u_{\mathrm{NC}} - I_{\mathrm{NC}}u) + \frac{1}{2}|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}^2 = E(u) - E_{\mathrm{NC}}(u_{\mathrm{NC}}) + F(u - I_{\mathrm{NC}}u). \tag{4.4}$$

Step 5 is the proof of assertion ⓐ. Lemma 2.2ⓐ shows

$$\left\|h_{\mathcal{T}}^{-1}(u - I_{\mathrm{NC}}u)\right\|_{L^2(\Omega)} \leqslant \kappa_{\mathrm{NC}}|||u - I_{\mathrm{NC}}u|||_{\mathrm{NC}} \leqslant \kappa_{\mathrm{NC}}|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}.$$

This, a Cauchy inequality and some Young inequality lead to

$$F(u - I_{\mathrm{NC}}u) \leqslant \frac{\kappa_{\mathrm{NC}}^2}{2}\,\|h_{\mathcal{T}}f\|_{L^2(\Omega)}^2 + \frac{1}{2}|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}^2. \tag{4.5}$$

The combination of (4.4) and (4.5) concludes the proof of ⓐ.

Step 6. Identity (4.4) also reads

$$\Lambda_{\mathrm{NC}}(u_{\mathrm{NC}} - u) + \frac{1}{2}|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}^2 = E(u) - E_{\mathrm{NC}}(u_{\mathrm{NC}}) + (F - \Lambda_{\mathrm{NC}})(u - I_{\mathrm{NC}}u).$$

Lemma 2.2.ⓐ leads to

$$\Lambda_{\mathrm{NC}}(u_{\mathrm{NC}} - u) + \frac{1}{2}|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}^2 \leqslant E(u) - E_{\mathrm{NC}}(u_{\mathrm{NC}}) + \kappa_{\mathrm{NC}}\,\|h_{\mathcal{T}}(f - \lambda_{\mathrm{NC}})\|_{L^2(\Omega)}\,|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}. \tag{4.6}$$

Step 7 concludes the proof of ⓑ. Lemma 4.6 shows

$$\begin{aligned}\Lambda_{\mathrm{NC}}(u - u_{\mathrm{NC}}) = {}& \int_{\mathcal{T}'}(\chi - u_{\mathrm{NC}})\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x - \int_{\mathcal{T}\setminus\mathcal{T}'}\lambda_{\mathrm{NC}}(I_{\mathrm{NC}}\chi - \chi)\,\mathrm{d}x \\ & - \int_{\mathcal{T}\setminus\mathcal{T}'}\lambda_{\mathrm{NC}}(\chi - u)\,\mathrm{d}x - \int_{\mathcal{T}'}(\chi - u)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x \\ & + \int_{\mathcal{T}'}(1 - \Pi_0)\lambda_{\mathrm{NC}}(1 - \Pi_0)(u - u_{\mathrm{NC}})\,\mathrm{d}x.\end{aligned} \tag{4.7}$$

The Poincaré inequality with constant $h_{\mathcal{T}}/j_{1,1}$ from Laugesen & Siudeja (2010) for each triangle $T \in \mathcal{T}'$ yields

$$\int_{\mathcal{T}'} (1-\Pi_0)\lambda_{\mathrm{NC}}(1-\Pi_0)(u-u_{\mathrm{NC}})\,\mathrm{d}x \leqslant \operatorname{osc}(\lambda_{\mathrm{NC}},\mathcal{T}')/j_{1,1}|||u-u_{\mathrm{NC}}|||_{\mathrm{NC}(\mathcal{T}')}. \tag{4.8}$$

The combination of this with (4.7) and (4.6) plus the absorption of $|||u-u_{\mathrm{NC}}|||_{\mathrm{NC}}$ conclude the proof of ⓑ in Theorem 4.1. □

Proof of Theorem 4.4. Given any $v \in K$, the definitions of Λ and Λ_{NC} plus elementary algebra show for $e := u - u_{\mathrm{NC}}$ that

$$\begin{aligned}\Lambda(u-v)+a_{\mathrm{NC}}(e,u-v) &= F(1-I_{\mathrm{NC}})(u-v)+\Lambda_{\mathrm{NC}}(I_{\mathrm{NC}}(u-v))\\ &= \int_\Omega (f-\lambda_{\mathrm{NC}})(1-I_{\mathrm{NC}})(u-v)\,\mathrm{d}x+\Lambda_{\mathrm{NC}}(u-v).\end{aligned}$$

The binomial theorem shows

$$2a_{\mathrm{NC}}(e,u-v)=|||e|||^2_{\mathrm{NC}}+|||u-v|||^2-|||v-u_{\mathrm{NC}}|||^2_{\mathrm{NC}}.$$

Lemma 2.2 and a Cauchy inequality yield

$$\int_\Omega (f-\lambda_{\mathrm{NC}})(1-I_{\mathrm{NC}})(u-v)\,\mathrm{d}x \leqslant \kappa_{\mathrm{NC}}\,\|h_{\mathcal{T}}(f-\lambda_{\mathrm{NC}})\|_{L^2(\Omega)}\,|||u-v|||.$$

The combination of the above-displayed estimates proves

$$\begin{aligned}2\Lambda(u-v)+|||e|||^2_{\mathrm{NC}}+|||u-v|||^2 \leqslant{}& 2\kappa_{\mathrm{NC}}\,\|h_{\mathcal{T}}(f-\lambda_{\mathrm{NC}})\|_{L^2(\Omega)}\,|||u-v|||\\ &+|||v-u_{\mathrm{NC}}|||^2_{\mathrm{NC}}+2\Lambda_{\mathrm{NC}}(u-v).\end{aligned} \tag{4.9}$$

Lemma 4.6 yields

$$\begin{aligned}\Lambda_{\mathrm{NC}}(u-v) ={}& \Lambda_{\mathrm{NC}}(u_{\mathrm{NC}}-v)+\int_{\mathcal{T}'}(\chi-u_{\mathrm{NC}})\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x-\int_{\mathcal{T}\setminus\mathcal{T}'}\lambda_{\mathrm{NC}}(\chi-u)\,\mathrm{d}x\\ &-\int_{\mathcal{T}'}(\chi-u)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x+\int_{\mathcal{T}'}(1-\Pi_0)\lambda_{\mathrm{NC}}(1-\Pi_0)(u-u_{\mathrm{NC}})\,\mathrm{d}x.\end{aligned} \tag{4.10}$$

The properties of the integral, the Poincaré inequality with the constant $h_T/j_{1,1}$ from Laugesen & Siudeja (2010) on each triangle $T \in \mathcal{T}'$, and the Cauchy inequality prove

$$\begin{aligned}\int_{\mathcal{T}'}(1-\Pi_0)\lambda_{\mathrm{NC}}(1-\Pi_0)(u-u_{\mathrm{NC}})\,\mathrm{d}x \leqslant{}& \operatorname{osc}(\lambda_{\mathrm{NC}},\mathcal{T}')|||u-v|||_{\mathrm{NC}(\mathcal{T}')}/j_{1,1}\\ &+\int_{\mathcal{T}'}(1-\Pi_0)\lambda_{\mathrm{NC}}(1-\Pi_0)(v-u_{\mathrm{NC}})\,\mathrm{d}x.\end{aligned}$$

It holds that

$$\Lambda_{\mathrm{NC}}(u_{\mathrm{NC}}-v)+\int_{\mathcal{T}'}(1-\Pi_0)\lambda_{\mathrm{NC}}(1-\Pi_0)(v-u_{\mathrm{NC}})\,\mathrm{d}x$$
$$=\int_{\mathcal{T}'}(u_{\mathrm{NC}}-v)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x+\int_{\mathcal{T}\setminus\mathcal{T}'}(u_{\mathrm{NC}}-v)\lambda_{\mathrm{NC}}\,\mathrm{d}x. \tag{4.11}$$

The first and second terms on the right-hand side of (4.11) satisfy

$$\int_{\mathcal{T}'}(u_{\mathrm{NC}}-v)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x+\int_{\mathcal{T}'}(\chi-u_{\mathrm{NC}})\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x=\int_{\mathcal{T}'}(\chi-v)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x$$

and

$$\int_{\mathcal{T}\setminus\mathcal{T}'}(u_{\mathrm{NC}}-v)\lambda_{\mathrm{NC}}\,\mathrm{d}x-\int_{\mathcal{T}\setminus\mathcal{T}'}\lambda_{\mathrm{NC}}(I_{\mathrm{NC}}\chi-u)\,\mathrm{d}x=\int_{\mathcal{T}\setminus\mathcal{T}'}(u-\chi)\lambda_{\mathrm{NC}}\,\mathrm{d}x+\int_{\mathcal{T}\setminus\mathcal{T}'}(\chi-v)\lambda_{\mathrm{NC}}\,\mathrm{d}x.$$

The combination with the above estimates and the absorption of $|||u-v|||$ conclude the proof of assertion ⓐ of Theorem 4.4.

The proof of ⓑ employs an auxiliary Laplace problem with right-hand side $f-\lambda_{\mathrm{NC}}\in L^2(\Omega)$. Following Braess (2005), let w denote the weak solution of the Laplace equation with Dirichlet boundary conditions u_{D} and right-hand side $f-\lambda_{\mathrm{NC}}\in L^2(\Omega)$, which corresponds to $F-\Lambda_{\mathrm{NC}}\in H^{-1}(\Omega)$. That is, $w\in H^1(\Omega)$ satisfies $w=u_{\mathrm{D}}$ on $\partial\Omega$ and

$$a(w,v)=(F-\Lambda_{\mathrm{NC}})(v),\qquad \text{for all } v\in V. \tag{4.12}$$

Since w is the solution to (4.12), any $v\in V$ satisfies $a(u-w,v)=(\Lambda-\Lambda_{\mathrm{NC}})(v)$.

The choice of $v:=u-w\in V$ leads to $|||u-w|||\leqslant|||\Lambda-\Lambda_{\mathrm{NC}}|||_*$. On the other hand, it holds that $(\Lambda-\Lambda_{\mathrm{NC}})(v)\leqslant|||u-w|||\,|||v|||$ for any $v\in V$.

Altogether it follows that

$$|||\Lambda-\Lambda_{\mathrm{NC}}|||_*=|||u-w|||.$$

The triangle inequality leads to

$$|||\Lambda-\Lambda_{\mathrm{NC}}|||_*\leqslant|||u-u_{\mathrm{NC}}|||_{\mathrm{NC}}+|||w-u_{\mathrm{NC}}|||_{\mathrm{NC}}.$$

By definition of λ_{NC}, the discrete approximation u_{NC} equals the nonconforming finite element solution to the Poisson model problem (4.12), namely

$$a_{\mathrm{NC}}(u_{\mathrm{NC}},v_{\mathrm{NC}})=(F-\Lambda_{\mathrm{NC}})(v_{\mathrm{NC}}),\qquad \text{for all } v_{\mathrm{NC}}\in V_{\mathrm{NC}}$$

with right-hand side $f-\lambda_{\mathrm{NC}}\in L^2(\Omega)$ and exact solution w. Hence, the error of the continuous and discrete solution $|||w-u_{\mathrm{NC}}|||_{\mathrm{NC}}$ to the Poisson model problem and its error control may follow from the *a posteriori* error analysis of variational equations. For instance, Carstensen & Merdon (2013, Theorem 3.1) show that

$$|||w-u_{\mathrm{NC}}|||_{\mathrm{NC}} \leqslant |||u_{\mathrm{NC}}-v|||_{\mathrm{NC}} + \operatorname{osc}(f-\lambda_{\mathrm{NC}},\mathscr{T})/j_{1,1} \\ + \frac{1}{2}\left\|\Pi_0(f-\lambda_{\mathrm{NC}})(\bullet-\operatorname{mid}(\mathscr{T}))\right\|_{L^2(\Omega)} + |||w_{\mathrm{D}}|||.$$

This concludes the proof of ⓑ of Theorem 4.4. □

5. Efficiency

This section discusses the efficiency of the global upper bound (GUB) from Theorem 4.4 for piecewise affine obstacles $\chi \in V_1(\mathscr{T})$.

For any $v \in K$, the *a posteriori* error estimate from Theorem 4.4 leads to a computable global upper bound $\mathrm{GUB}(v)$ of the five non-negative error terms in $\mathrm{LHS}(v) \lesssim \mathrm{GUB}(v)$,

$$\mathrm{LHS}(v) := |||u-u_{\mathrm{NC}}|||_{\mathrm{NC}} + |||\Lambda-\Lambda_{\mathrm{NC}}|||_{*} + \Lambda(u-v)^{1/2} \\ + \left(\int_{\mathscr{T}'} \lambda_{\mathrm{NC}}\Pi_0(\chi-u)\,\mathrm{d}x\right)^{1/2} + \left(\int_{\mathscr{T}\setminus\mathscr{T}'} (\chi-u)\lambda_{\mathrm{NC}}\,\mathrm{d}x\right)^{1/2},$$
$$\mathrm{GUB}(v) := |||v-u_{\mathrm{NC}}|||_{\mathrm{NC}} + \|h_{\mathscr{T}}(f-\lambda_{\mathrm{NC}})\|_{L^2(\Omega)} + \operatorname{osc}(\lambda_{\mathrm{NC}},\mathscr{T}') + |||w_{\mathrm{D}}||| \\ + \left(\int_{\mathscr{T}'} \lambda_{\mathrm{NC}}\Pi_0(\chi-v)\,\mathrm{d}x\right)^{1/2} + \left(\int_{\mathscr{T}\setminus\mathscr{T}'} \lambda_{\mathrm{NC}}(\chi-v)\,\mathrm{d}x\right)^{1/2}.$$

For $\chi \in V_1(\mathscr{T})$, the *a posteriori* error estimate $\mathrm{LHS}(v) \lesssim \mathrm{RHS}(v)$ is efficient in the sense that the converse inequality holds up to some generic factor and oscillation terms. Recall the assumption $u_{\mathrm{D}} \in C(\partial\Omega)\cap H^2(\mathscr{E}(\partial\Omega))$ and suppose that v is postprocessed from u_{NC} such that it holds that

$$|||u-v||| \lesssim |||u-u_{\mathrm{NC}}|||_{\mathrm{NC}} + \left\|\frac{h_{\mathscr{E}}^{3/2}\partial^2 u_{\mathrm{D}}}{\partial s^2}\right\|_{L^2(\partial\Omega)}. \tag{5.1}$$

THEOREM 5.1 (Efficiency of GUB) Any function $v \in K$ with (5.1) satisfies

$$\mathrm{GUB}(v) \lesssim \mathrm{LHS}(v) + \operatorname{osc}(f,\mathscr{T}) + \operatorname{osc}(\lambda,\mathscr{T}) + \left\|\frac{h_{\mathscr{E}}^{3/2}\partial^2 u_{\mathrm{D}}}{\partial s^2}\right\|_{L^2(\partial\Omega)}.$$

REMARK 5.2 Given any postprocessing $v \in \mathscr{A}$ with (5.1), the function $\max\{v,\chi\}$ belongs to K and satisfies

$$|||u-\max\{v,\chi\}||| \lesssim |||u-u_{\mathrm{NC}}|||_{\mathrm{NC}} + \left\|\frac{h_{\mathscr{E}}^{3/2}\partial^2 u_{\mathrm{D}}}{\partial s^2}\right\|_{L^2(\partial\Omega)} + |||\min\{0,v-\chi\}|||.$$

The conforming companion u_2 from Section 2.4 satisfies this estimate.

Proof of Theorem 5.1. **Step 1** is the proof of

$$\|h_{\mathcal{T}}(f - \lambda_{\mathrm{NC}})\|_{L^2(\Omega)} \lesssim \operatorname{osc}(f, \mathcal{T}) + |||\Lambda - \Lambda_{\mathrm{NC}}|||_* + |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}. \tag{5.2}$$

Proof. The triangle inequality and $f_{\mathcal{T}} := \Pi_0 f$ with $f_T := f_{\mathcal{T}}|_T$ for $T \in \mathcal{T}$ imply

$$\|h_{\mathcal{T}}(f - \lambda_{\mathrm{NC}})\|_{L^2(\Omega)} \leqslant \operatorname{osc}(f, \mathcal{T}) + \|h_{\mathcal{T}}(f_{\mathcal{T}} - \lambda_{\mathrm{NC}})\|_{L^2(\Omega)}.$$

The efficiency of $\|h_T(f_T - \lambda_{\mathrm{NC}})\|_{L^2(T)}$ is shown as for boundary value problems in Verfürth (1996). Consider the cubic bubble function $b_T := 60\Pi_{z\in\mathcal{N}(T)}\varphi_z$ on the triangle $T \in \mathcal{T}$ and set $w_T := (f_T - \lambda_{\mathrm{NC}})b_T$ which satisfies

$$\|h_T(f_T - \lambda_{\mathrm{NC}})\|^2_{L^2(T)} \lesssim h_T^2 \int_T w_T(f_T - \lambda_{\mathrm{NC}})\,\mathrm{d}x. \tag{5.3}$$

The Cauchy and Friedrichs inequalities lead to

$$\int_T w_T(f_T - f)\,\mathrm{d}x \lesssim |||w_T|||_T \operatorname{osc}(f, T).$$

An integration by parts yields

$$\int_T (f - \lambda_{\mathrm{NC}})w_T\,\mathrm{d}x = (\Lambda - \Lambda_{\mathrm{NC}})(w_T) - \int_T \nabla u \cdot \nabla w_T\,\mathrm{d}x.$$

Since $\int_T \nabla w_T\,\mathrm{d}x = 0$, it follows that

$$\int_T \nabla u \cdot \nabla w_T\,\mathrm{d}x = \int_T (\nabla u - \nabla_{\mathrm{NC}} u_{\mathrm{NC}}) \cdot \nabla w_T\,\mathrm{d}x \lesssim |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}(T)} |||w_T|||_T.$$

The combination of the previous estimates shows

$$\|h_T(f_T - \lambda_{\mathrm{NC}})\|^2_{L^2(T)} \leqslant h_T^2 |||w_T|||_T \Big(|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}(T)} + |||\Lambda - \Lambda_{\mathrm{NC}}|||_{*,T} + \operatorname{osc}(f, T) \Big). \tag{5.4}$$

The triangle inequality implies

$$|||w_T|||_T \leqslant \|(f_T - \lambda_{\mathrm{NC}})\nabla b_T\|_{L^2(T)} + \|b_T \nabla (f_T - \lambda_{\mathrm{NC}})\|_{L^2(T)}.$$

The properties of b_T (Verfürth, 1996, Lemma 1.3) result in

$$\|(f_T - \lambda_{\mathrm{NC}})\nabla b_T\|_{L^2(T)} \lesssim h_T^{-1} \left\| b_T^{1/2}(f_T - \lambda_{\mathrm{NC}}) \right\|_{L^2(T)}.$$

An inverse estimate shows

$$\|b_T \nabla (f_T - \lambda_{\mathrm{NC}})\|_{L^2(T)} \lesssim h_T^{-1} \left\| b_T^{1/2}(f_T - \lambda_{\mathrm{NC}}) \right\|_{L^2(T)}.$$

Therefore,

$$|||w_T|||_T \lesssim h_T^{-1} \left\| b_T^{1/2}(f_T - \lambda_{\mathrm{NC}}) \right\|_{L^2(T)}. \tag{5.5}$$

The combination of (5.4) and (5.5) implies

$$\|h_T(f_T - \lambda_{\mathrm{NC}})\|_{L^2(T)} \lesssim \operatorname{osc}(f, T) + |||\Lambda - \Lambda_{\mathrm{NC}}|||_{*,T} + |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}(T)}.$$

This is already a local version of the assertion for some fixed triangle T. To prove the global form, let $w_{\mathscr{T}}$ be defined as above on each triangle. Then $w_{\mathscr{T}} \in V$ and $h_{\mathscr{T}}^2 w_{\mathscr{T}} \in V$ and (5.4) is replaced by

$$\|h_{\mathscr{T}}(f_{\mathscr{T}} - \lambda_{\mathrm{NC}})\|_{L^2(\Omega)}^2 \lesssim \sum_{T\in\mathscr{T}} h_T^2 |||w_{\mathscr{T}}|||_T \Big(\operatorname{osc}(f, \mathscr{T}) + |||\lambda - \lambda_{\mathrm{NC}}|||_{*,T} + |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}(T)} \Big).$$

Since (5.5) holds, this leads to

$$\|h_{\mathscr{T}}(f - \lambda_{\mathrm{NC}})\|_{L^2(\Omega)} \lesssim |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}} + |||\Lambda - \Lambda_{\mathrm{NC}}|||_{*} + \operatorname{osc}(f, \mathscr{T}).$$

This concludes the proof of (5.2). □

Step 2 verifies

$$\operatorname{osc}(\lambda_{\mathrm{NC}}, \mathscr{T}) \lesssim |||\Lambda - \Lambda_{\mathrm{NC}}|||_{*} + \operatorname{osc}(\lambda, \mathscr{T}). \tag{5.6}$$

Proof of Step 2. Let $b_T := 60\Pi_{z\in\mathscr{N}(T)}\varphi_z$ be the cubic bubble function on the triangle $T \in \mathscr{T}$ and set $w_T := b_T(\lambda_{\mathrm{NC}} - \Pi_0\lambda_{\mathrm{NC}})$. This gives

$$\|h_T(\lambda_{\mathrm{NC}} - \Pi_0\lambda_{\mathrm{NC}})\|_{L^2(T)}^2 \lesssim h_T^2 \int_T w_T(\lambda_{\mathrm{NC}} - \Pi_0\lambda_{\mathrm{NC}})\,\mathrm{d}x.$$

Since $\int_T w_T\,\mathrm{d}x = 0$, it follows that

$$\int_T w_T(\lambda_{\mathrm{NC}} - \Pi_0\lambda_{\mathrm{NC}})\,\mathrm{d}x = \int_T w_T(\lambda_{\mathrm{NC}} - \lambda)\,\mathrm{d}x + \int_T w_T(\lambda - \Pi_0\lambda)\,\mathrm{d}x.$$

With $w_{\mathscr{T}}|_T := w_T$ on each triangle $T \in \mathscr{T}$, the summation over all triangles results in

$$\begin{aligned}\|h_{\mathscr{T}}(\lambda_{\mathrm{NC}} - \Pi_0\lambda_{\mathrm{NC}})\|_{L^2(\Omega)}^2 &\lesssim \int_\Omega h_{\mathscr{T}}^2 w_T(\lambda_{\mathrm{NC}} - \lambda)\,\mathrm{d}x + \int_\Omega h_{\mathscr{T}}^2 w_T(\lambda - \Pi_0\lambda)\,\mathrm{d}x \\ &\leqslant |||\Lambda - \Lambda_{\mathrm{NC}}|||_{*} |||h_{\mathscr{T}}^2 w_{\mathscr{T}}||| + \operatorname{osc}(\lambda, \mathscr{T})\, \|h_{\mathscr{T}} w_{\mathscr{T}}\|_{L^2(\Omega)}.\end{aligned}$$

An inverse estimate as in Step 1 and a Friedrichs inequality prove

$$|||h_{\mathscr{T}}^2 w_{\mathscr{T}}||| \approx \|h_{\mathscr{T}} w_{\mathscr{T}}\|_{L^2(\Omega)} \approx \operatorname{osc}(\lambda_{\mathrm{NC}}, \mathscr{T}).$$

The division by $\operatorname{osc}(\lambda_{\mathrm{NC}}, \mathscr{T})$ concludes the proof of (5.6).

Step 3 is the proof of

$$\left(\int_{\mathcal{T}'}(\chi - v)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x\right)^{1/2} + \left(\int_{\mathcal{T}\setminus\mathcal{T}'}(\chi - v)\lambda_{\mathrm{NC}}\,\mathrm{d}x\right)^{1/2} \lesssim \mathrm{LHS}(v). \tag{5.7}$$

Proof of Step 3. The proof starts with the split of the first term:

$$\begin{aligned}\int_{\mathcal{T}'}(\chi - v)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x &= \int_{\mathcal{T}'}(\chi - u)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x + \int_{\mathcal{T}'}\lambda(u - v)\,\mathrm{d}x \\ &\quad + \int_{\mathcal{T}'}(\lambda - \lambda_{\mathrm{NC}})(v - u)\,\mathrm{d}x + \int_{\mathcal{T}'}(\lambda_{\mathrm{NC}} - \Pi_0\lambda_{\mathrm{NC}})(1 - \Pi_0)(v - u)\,\mathrm{d}x.\end{aligned}$$

The Cauchy, Poincaré and Young inequalities and (5.1) show

$$\int_{\mathcal{T}'}(\lambda_{\mathrm{NC}} - \Pi_0\lambda_{\mathrm{NC}})(1 - \Pi_0)(v - u)\,\mathrm{d}x \lesssim |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}^2 + \mathrm{osc}(\lambda_{\mathrm{NC}}, \mathcal{T}')^2 + \left\| \frac{h_{\mathcal{E}}^{3/2}\partial^2 u_{\mathrm{D}}}{\partial s^2} \right\|_{L^2(\partial\Omega)}.$$

The second term satisfies

$$\int_{\mathcal{T}\setminus\mathcal{T}'}(\chi - v)\lambda_{\mathrm{NC}}\,\mathrm{d}x = \int_{\mathcal{T}\setminus\mathcal{T}'}(\chi - u)\lambda_{\mathrm{NC}}\,\mathrm{d}x + \int_{\mathcal{T}\setminus\mathcal{T}'}(u - v)(\lambda_{\mathrm{NC}} - \lambda)\,\mathrm{d}x + \int_{\mathcal{T}\setminus\mathcal{T}'}(u - v)\lambda\,\mathrm{d}x.$$

The square roots of the terms $\int_{\mathcal{T}'}\lambda_{\mathrm{NC}}\Pi_0(\chi - u)\,\mathrm{d}x$, $\mathrm{osc}(\lambda_{\mathrm{NC}}, \mathcal{T}')^2$, $|||u - u_{\mathrm{NC}}|||^2$ and $\int_{\mathcal{T}\setminus\mathcal{T}'}(I_{\mathrm{NC}}\chi - u)\lambda_{\mathrm{NC}}\,\mathrm{d}x$ are part of LHS(v). The combination of the remaining terms with (5.1) leads to

$$\int_{\mathcal{T}'}\lambda(u - v)\,\mathrm{d}x + \int_{\mathcal{T}\setminus\mathcal{T}'}\lambda(u - v)\,\mathrm{d}x = \Lambda(u - v)$$

and

$$\int_{\mathcal{T}'}(\lambda - \lambda_{\mathrm{NC}})(v - u)\,\mathrm{d}x + \int_{\mathcal{T}\setminus\mathcal{T}'}(\lambda - \lambda_{\mathrm{NC}})(v - u)\,\mathrm{d}x \lesssim |||\Lambda - \Lambda_{\mathrm{NC}}|||_*^2 + |||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}^2.$$

The square roots of these terms are part of LHS(v). This concludes the proof of (5.7).

Step 4. The efficiency of $\left\|\fint_{\mathcal{T}}(f - \lambda_{\mathrm{NC}})\,\mathrm{d}x(\bullet - \mathrm{mid}(\mathcal{T}))\right\|_{L^2(\Omega)} + |||w_{\mathrm{D}}|||$ follows from Carstensen & Merdon (2013, Theorem 3.1) and Lemma 2.6.

The combination of Steps 1–4 and the triangle inequality conclude the proof of Theorem 5.1.

6. Computational benchmarks

This section is devoted to the performance of the *a priori* and *a posteriori* error estimates in three benchmark examples implemented as in Alberty *et al.* (1999) and Bahriawati & Carstensen (2005).

6.1 *Numerical realization*

The output of the following algorithm are the two lower energy bounds, the value of two error estimators and the corresponding efficiency indices. The lower bounds $\mu_1,\ \mu_2$ of $E(u)$ from Theorem 4.1 are given by

$$\begin{aligned}\mu_1 :=& E_{\mathrm{NC}}(u_{\mathrm{NC}}) - \kappa_{\mathrm{NC}}^2/2\,\|h_{\mathscr{T}} f\|_{L^2(\Omega)}^2,\\ \mu_2 :=& E_{\mathrm{NC}}(u_{\mathrm{NC}}) - \frac{1}{2}\Big(\kappa_{\mathrm{NC}}\,\|h_{\mathscr{T}}(f-\lambda_{\mathrm{NC}})\|_{L^2(\Omega)} + \mathrm{osc}(\lambda_{\mathrm{NC}},\mathscr{T}')\Big)^2\\ &- \int_{\mathscr{T}'}(\chi - u_{\mathrm{NC}})\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x + \int_{\mathscr{T}\setminus\mathscr{T}'}(I_{\mathrm{NC}}\chi-\chi)\lambda_{\mathrm{NC}}\,\mathrm{d}x.\end{aligned}$$

With $\mathscr{T}'$ from Section 4 and $v := \max\{\chi, w_{\mathrm{D}} + u_{\mathrm{D}2} + J_2(u_{\mathrm{NC}} - I_{\mathrm{NC}}u_{\mathrm{D}2})\} \in K$ for $w_{\mathrm{D}} \in H^1(\Omega)$ as in Lemma 2.6 with $w_{\mathrm{D}}|_{\partial\Omega} = u_{\mathrm{D}} - u_{\mathrm{D}2}$ and J_2 from (2.7), the estimator η_1 and η_2 are given by

$$\begin{aligned}\eta_1 :=& |||u_{\mathrm{NC}} - v|||^2 + 2\Big(E(v) - E_{\mathrm{NC}}(u_{\mathrm{NC}})\Big) + \Big(\kappa_{\mathrm{NC}}\,\|h_{\mathscr{T}}(f-\lambda_{\mathrm{NC}})\|_{L^2(\Omega)} + \mathrm{osc}(f,\mathscr{T}')\Big)^2\\ &+ 2\int_{\mathscr{T}'}(\chi - u_{\mathrm{NC}})\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x + 2\int_{\mathscr{T}\setminus\mathscr{T}'}(\chi - I_{\mathrm{NC}}\chi)\lambda_{\mathrm{NC}}\,\mathrm{d}x,\\ \eta_2 :=& |||v - u_{\mathrm{NC}}|||_{\mathrm{NC}}^2 + \Big(\kappa_{\mathrm{NC}}\,\|h_{\mathscr{T}}(f-\lambda_{\mathrm{NC}})\|_{L^2(\Omega)} + \mathrm{osc}(f,\mathscr{T}')\Big)^2\\ &+ 2\int_{\mathscr{T}'}(\chi - v)\Pi_0\lambda_{\mathrm{NC}}\,\mathrm{d}x + 2\int_{\mathscr{T}\setminus\mathscr{T}'}(\chi - v)\lambda_{\mathrm{NC}}\,\mathrm{d}x.\end{aligned}$$

Algorithm. INPUT an initial triangulation $\mathscr{T}_0$.

LOOP for all $\ell = 0, 1, 2, \ldots$ until termination repeating the three steps ①–③

① COMPUTE the discrete solution u_{NC} on $\mathscr{T}_\ell$ with `ndof` unknowns.
② ESTIMATE the error $|||u - v|||$ for $v := \max\{\chi, w_{\mathrm{D}} + u_{\mathrm{D}2} + J_2(u_{\mathrm{NC}} - I_{\mathrm{NC}}u_{\mathrm{D}2})\}$ with η_1 and η_2 and compute the lower energy bounds μ_2 and μ_2.

The related efficiency indices $\mathrm{Eff}(\eta_1)$ and $\mathrm{Eff}(\eta_2)$ read

$$\mathrm{Eff}(\eta_j) := \sqrt{\sum_{T\in\mathscr{T}} \eta_j^2(v)}/|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}} \text{ for } j = 1, 2.$$

③ REFINE $\mathscr{T}_\ell$ by red refinement of all triangles and compute $\mathscr{T}_{\ell+1}$.

OUTPUT efficiency indices $\mathrm{Eff}(\eta_1)$, $\mathrm{Eff}(\eta_2)$, and the lower energy bounds $\mu_1,\ \mu_2$ on each level ℓ.

6.2 *Square domain*

The function $u(r,\varphi) := \max\{r^2 - 0.49, 0\}^2$ solves the obstacle problem from Nochetto *et al.* (2003) with the constant obstacle $\chi \equiv 0$, the nonhomogeneous Dirichlet data $u_{\mathrm{D}} = u|_{\partial\Omega}$ and the right-hand side $f(r,\varphi)$ which equals $-16r^2 + 3.92$ for $r > 0.7$ and $-5.8408 + 3.92r^2$ for $r \leqslant 0.7$ on the square

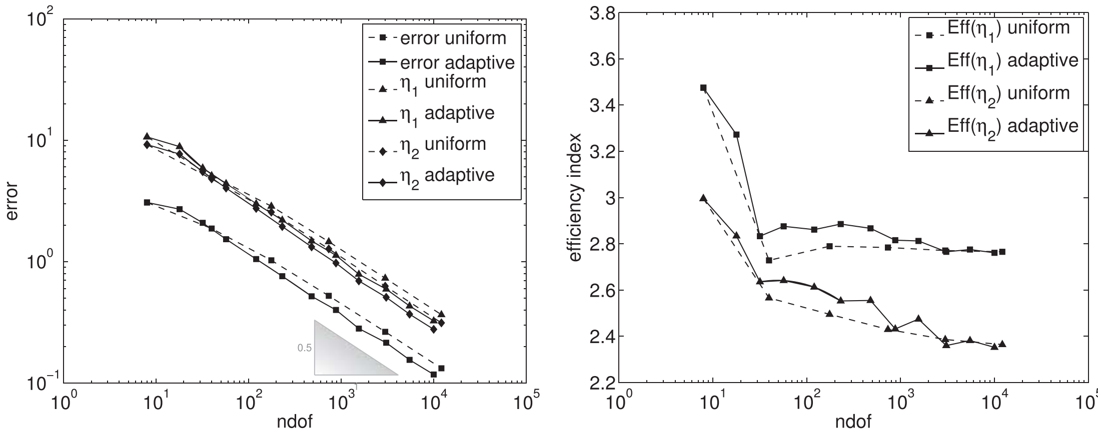

FIG. 3. Convergence history plot of the error $|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}$ and the two error estimators (left) and efficiency indices of the two error estimators (right) for the NCFEM as functions of `ndof` in Section 6.2.

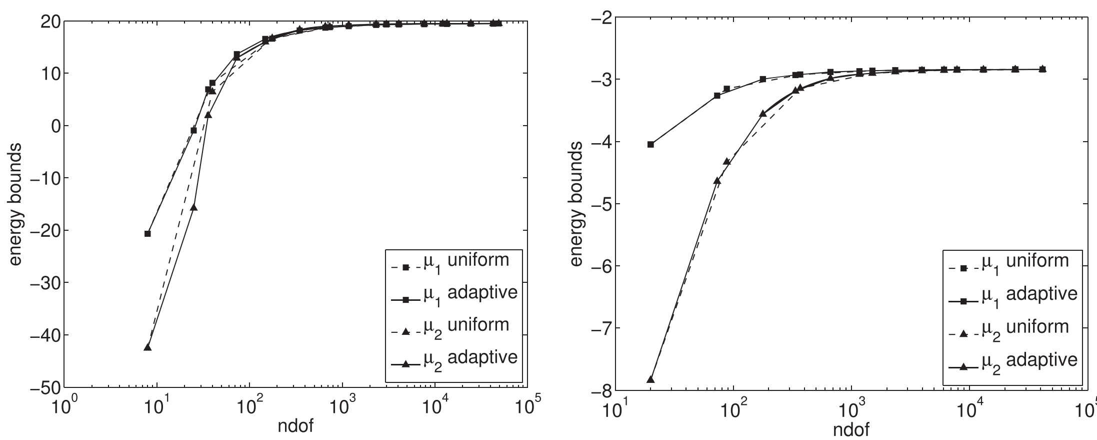

FIG. 4. Upper and lower energy bounds in Sections 6.2 (left) and 6.3 (right).

domain $(-1,1)^2$ in polar coordinates (r,φ) at the origin. Figure 3 displays the error estimators η_1 and η_2 of $|||u - u_{\mathrm{NC}}|||$. On the left, the error estimator and the corresponding exact error converge with a convergence rate -0.5 with respect to `ndof` as anticipated by Theorem 3.1 both for the uniform algorithm (described above) and an adaptive algorithm based on the error estimator η_2 as a refinement indicator with Dörfler marking and a bulk parameter $\Theta = 0.5$. On the right, Fig. 3 shows that the *a posteriori* error estimators are efficient with efficiency indices between 2 and 3. Figure 4 shows the lower energy bounds μ_1 and μ_2 for adaptive and uniform mesh refinement and their convergence towards the exact energy on the left. Both lower bounds converge and they exhibit the same overall behaviour.

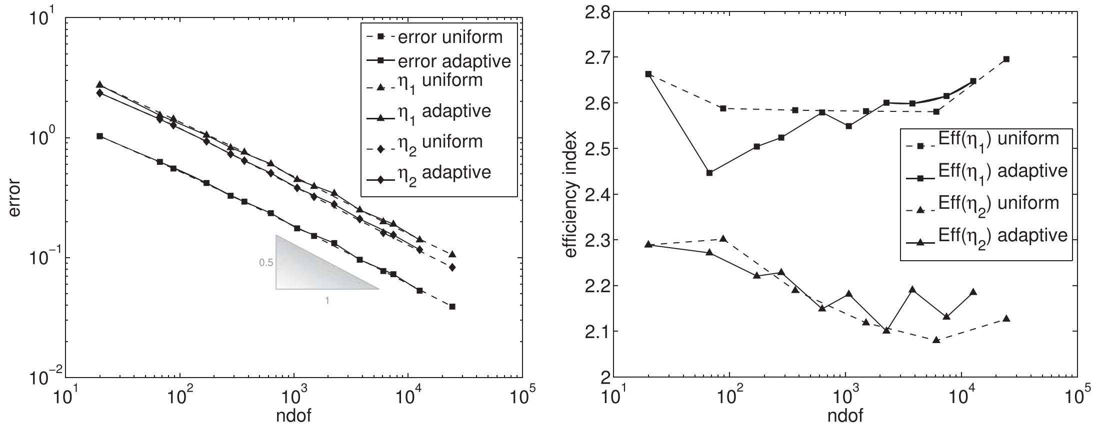

FIG. 5. Convergence history plot of the error $|||u - u_{NC}|||_{NC}$ and the two error estimators (left) and efficiency indices of the two error estimators (right) for the NCFEM as functions of `ndof` in Section 6.3.

6.3 *Smooth obstacle*

The function $u \equiv \chi \in K$ from Gräser & Kornhuber (2009) solves the obstacle problem on the square domain with the smooth obstacle $\chi(x, y) := -(x^2-1)(y^2-1)$, the homogeneous Dirichlet data $u_D|_{\partial\Omega} := 0$ and the source term $f := -\Delta\chi$. Figure 5 investigates the quality of the error estimators for $|||u - u_{NC}|||$ on the left and confirms that the error estimator and the corresponding error converge with a convergence rate -0.5 as anticipated by Theorem 3.1, for the adaptive and uniform mesh refinements. Figure 5 reveals on the right that all three error estimators are efficient with efficiency indices between 2 and 2.8. Figure 4 shows the lower energy bounds μ_1 and μ_2 on adaptive and uniform meshes and their convergence towards $E(u)$ on the right.

6.4 *L-shaped domain*

The example from Bartels & Carstensen (2004) considers a zero obstacle and Dirichlet data $u_D \equiv \chi \equiv 0$ on the L-shaped domain $\Omega := (-2, 2)^2 \backslash [0, 2] \times [-2, 0]$ with the source term

$$f(r, \varphi) := -r^{2/3} \sin(2\varphi/3) \left(\frac{7}{3r} \frac{\partial g(r)}{\partial r} + \frac{\partial^2 g(r)}{\partial r^2} \right) - H(r - 5/4),$$
$$g(r) := \max\{0, \min\{1, -6s^5 + 15s^4 - 10s^3 + 1\}\} \quad \text{for } s := 2(r - 1/4)$$

with the Heaviside function H. The exact solution

$$u(r, \varphi) = r^{2/3} g(r) \sin(2\varphi/3)$$

has a typical corner singularity at the reentrant corner and illustrates the superiority of an adaptive mesh-refinement strategy that accompanies Theorem 4.4. Figure 6 (left) displays a significantly improved convergence rate for the adaptive algorithm compared with uniform mesh refinement. The efficiency indices for guaranteed error control in Fig. 6 on the right range between 2 and 3 for uniform and adaptive

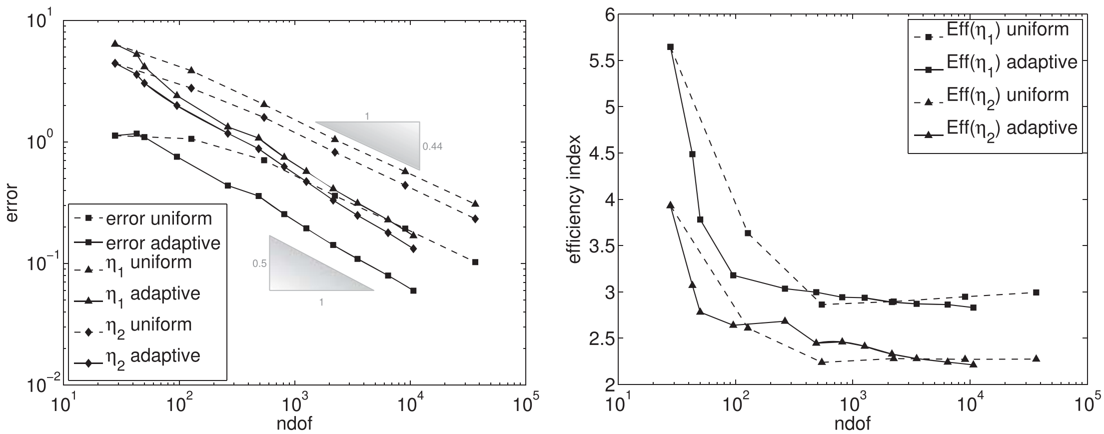

FIG. 6. Convergence history plot of the error $|||u - u_{\mathrm{NC}}|||_{\mathrm{NC}}$ and the two error estimators (left) and efficiency indices of the two error estimators (right) for the NCFEM as functions of `ndof` in Section 6.4.

mesh refinement. The errors in Fig. 6 on the left converge with the same rate as the estimators, which also follows from the efficiency indices displayed in Fig. 6 (right).

6.5 *Comments*

All numerical experiments confirm the *a priori* convergence rates anticipated by Theorem 3.1 even in Section 6.4 with a singular solution on a polygon; the theoretical result in Wang (2003) does not cover this situation. The guaranteed error estimates lead to upper error bounds confirmed in all numerical examples. Additional undisplayed numerical experiments with nonconforming and conforming finite element methods show comparable accuracies even in the presence of singular solutions. The lower energy bound μ_2 leads to a better approximation of the exact energy $E(u)$ on coarse grids. This behaviour also holds true for the experiment in Section 6.4 (undisplayed). On fine grids, the two lower energy bounds μ_1 and μ_2 lead to comparable bounds. Adaptive mesh refinement leads to optimal convergence rates in all considered experiments. In all numerical examples, the error estimator η_2 leads to slightly better efficiency indices with less over-estimation of the true error. Overall efficiency indices between 2 and 3.5 are obtained for the estimators η_1 and η_2.

Funding

This research is carried out in the framework of Matheon supported by Einstein Foundation Berlin. The author K. K. was supported by the Berlin Mathematical School.

REFERENCES

ALBERTY, J., CARSTENSEN, C. & FUNKEN, S. (1999) Remarks around 50 lines of Matlab: short finite element implementation. *Numer. Algorithms*, **20**, 117–137.

BAHRIAWATI, C. & CARSTENSEN, C. (2005) Three Matlab implementations of the lowest-order Raviart–Thomas MFEM with a posteriori error control. *CMAM*, **5**, 333–361.

BARTELS, S. & CARSTENSEN, C. (2004) Averaging techniques yield reliable a posteriori finite element error control for obstacle problems. *Numer. Math.*, **99**, 225–249.

BARTELS, S., CARSTENSEN, C. & DOLZMANN, G. (2004) Inhomogeneous Dirichlet conditions in *a priori* and *a posteriori* finite element error analysis. *Numer. Math.*, **99**, 1–24.

BRAESS, D. (2005) *A posteriori* error estimators for obstacle problems—another look. *Numer. Math.*, **101**, 415–421

BRAESS, D., HOPPE, R. & SCHÖBERL, J. (2008) *A posteriori* estimators for obstacle problems by the hypercircle method. *Comput. Vis. Sci.*, **11**, 351–362.

BRENNER, S. C. (2003) Poincaré–Friedrichs inequalities for piecewise H^1 functions. *SIAM J. Numer. Anal.*, **41**, 306–324.

CARSTENSEN, C. & GALLISTL, D. (2014) Guaranteed lower eigenvalue bounds for the biharmonic equation. *Numer. Math.*, **126**, 33–51.

CARSTENSEN, C., GALLISTL, D. & SCHEDENSACK, M. (2015) Adaptive nonconforming Crouzeix–Raviart FEM for eigenvalue problems. *Math. Comp.*, **84**, 1061–1087.

CARSTENSEN, C., GEDICKE, J. & RIM, D. (2012) Explicit error estimates for Courant, Crouzeix–Raviart and Raviart–Thomas finite element methods. *J. Comput. Math.*, **30**, 337–353.

CARSTENSEN, C. & MERDON, C. (2013) Computational survey on *a posteriori* error estimators for nonconforming finite element methods for the Poisson problem. *J. Comput. Appl. Math.*, **249**, 74–94.

CIARLET, P. (1978) *The Finite Element Method for Elliptic Problems*, Amsterdam: North-Holland Publishing pp. xix+530. Studies in Mathematics and its Applications, vol. 4.

FALK, R. (1974) Error estimates for the approximation of a class of variational inequalities. *Math. Comput.*, **28**, 963–971.

GRÄSER, C. & KORNHUBER, R. (2009) Multigrid methods for obstacle problems. *J. Comput. Math.*, **27**, 1–44.

GUDI, T. & PORWAL, K. (2014) A posteriori error control of discontinuous Galerkin methods for elliptic obstacle problems. *Math. Comput.*, **83**, 579–602.

KINDERLEHRER, D. & STAMPACCHIA, G. (1980) *An Introduction to Variational Inequalities and Their Applications*. New York, London: Harcourt Brace Jovanovich.

LAUGESEN, R. & SIUDEJA, B. (2010) Minimizing Neumann fundamental tones of triangles: an optimal Poincaré inequality. *J. Differ. Equ.*, **249**, 118–135.

NOCHETTO, R., SIEBERT, K. & VEESER, A. (2003) Pointwise *a posteriori* error control for elliptic obstacle problems. *Numer. Math.*, **95**, 163–195.

RODRIGUES, J. (1987) *Obstacle Problems in Mathematical Physics*. Amsterdam: Elsevier Science Publishers B.V.

VERFÜRTH, R. (1996) *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*. Chichester, Stuttgart: Wiley-Teubner.

WANG, L. (2003) On the error estimate of nonconforming finite element approximation to the obstacle problem. *J. Comput. Math.*, **21**, 481–490.