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Problem Sheet 2

Differential Geometry II 2017

Problem 1
Recall that two vector bundles over the same base manifold M are isomorphic if there is a bundle isomorphism of the form $(\cdot, id_M)$ between them.
(i) Prove that every (real or complex) vector bundle over $M = (0,1)^n$, $n \in \mathbb{N}$ is trivial.
(ii) Let $(U_i)_{i\in I}$ be a cover of a manifold M by coordinate neighbourhoods so that $U_i \simeq (0,1)^n$ for all $i \in I$. A family of smooth maps $\mathcal{T} = (t_{ij} : U_i \cap U_j \to \mathrm{GL}(k,\mathbb{R}))_{i,j\in I}$ is called *transition data* for the cover $(U_i)_{i\in I}$ if $t_{ij} \circ t_{ji} = \mathrm{Id}_{\mathbb{R}^k}$ and $t_{jk} \circ t_{ij} = t_{ik}$ for all $i, j, k \in I$. We say that $\mathcal{T}$ and $\mathcal{T}'$are equivalent if there exists a fimily of smooth maps $(\tau_i : U_i \to \mathrm{GL}(k,\mathbb{R}))_{i\in I}$, so that $t'_{ij} = \tau_j^{-1} \circ t_{ij} \circ \tau_i$. Construct a bijection between the set of all real vector bundles of rank k over M up to isomorphism and the set of all transition data for the cover $(U_i)_{i\in I}$ up to equivalence.
(iii) Using (i) and (ii), classify all real vector bundles over S^1 and S^2 up to isomorphism.

Problem 2
(i) Let (E, π, M) be a (topological) vector bundle over $M = (0,1)$ and $r, s \in (0,1)$. Prove that the bundle $(\pi^{-1}(\{r\} \times S^1), \pi, S^1)$ is trivial if and only if the bundle $(\pi^{-1}(\{s\} \times S^1), \pi, S^1)$ is trivial.
(ii) Recall the definition of the first fundamental group $\pi_1(X)$ of a topological space X and the Hurewicz homomorphism $h_1 : \pi_1(X) \to H_1(X; \mathbb{Z}/2\mathbb{Z})$. Given a (topological) vector bundle $\xi = (E, \pi, M)$, consider the map $W : \pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$ for which $W([\gamma]) = 0$ if and only if the bundle $\gamma^*\xi$ is trivial. Show that composing W with h_1 and using the natural isomorphism $H_1(M; \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Hom}(H_1(M; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$ gives rise to a well-defined cohomology class $w_1(\xi) \in H^1(M; \mathbb{Z}/2\mathbb{Z})$. Prove that the map $\xi \mapsto w_1(\xi)$ is a characteristic class.
(iii) Show that w_1 is the unique non-trivial characteristic class with values in $H^1(\cdot; \mathbb{Z}/2\mathbb{Z})$, so that $w_1(\xi \oplus \xi') = w_1(\xi) \cup w_1(\xi')$ for all ξ, ξ'.

Problem 3

Recall that on orientation of a vector bundle $\xi = (E, \pi, M)$ consists of an orientation of each fibre as a real vector space, so that there exists a family $(U_i, \varphi_i)_{i\in I}$ of local trivializations of ξ, where $(U_i)_{i\in I}$ is a cover of M and each map $\varphi_i|_{E_x} : E_x \to \{x\} \times \mathbb{R}^k \simeq \mathbb{R}^k$ is orientation-preserving with respect to the standard orientation on $\mathbb{R}^k$.

(i) Show that orientations of ξ and M together naturally define an orientation of E.

(ii) Assume as known the fact that each vector bundle ξ admits a section s, so that $s(M) \subset E$ intersects the image $s_0(M) \subset E$ of the zero section transversely. Assuming that ξ and M are oriented and M is closed, define $e(\xi) := PD([s(M) \cap s_0(M)]) \in H^k(M; \mathbb{Z})$, where the intersection is oriented according to the orientation of E from (i) and $PD : H_{n-k}(M; \mathbb{Z}) \to H^k(M; \mathbb{Z})$ is the Poincaré duality isomorphism. Prove that $\xi \mapsto e(\xi)$ defines a characteristic class on oriented vector bundles with closed base manifold.

(iii) Prove that if the rank k of ξ is odd, then $2e(\xi) = 0$. *Hint:* Check that if $\overline{\xi}$ denotes the bundle ξ with its orientation reversed, then $e(\overline{\xi}) = -e(\xi)$. Consider the bundle map $\xi \to \xi$ given by multiplication with -1 in each fibre.