Dr. Viktor Fromm
Institut für Mathematik
Rudower Chaussee 25
Haus 1 Raum 306

# Problem Sheet 5

## Differential Geometry II 2017

**Problem 1**
Let $G$ be a Lie group acting from the right on a manifold $M$ and let $x : (-\varepsilon, \varepsilon) \to M$ and $g : (-\varepsilon, \varepsilon) \to G$ be smooth paths in $M$ and $G$ respectively, where $x(0) = x_0$ and $g(0) = g_0$. Prove:

$$\frac{d}{dt}|_{t=0} (x(t) \cdot g(t)) = dR_{g_0}(x'(0)) + (d\widetilde{L_{g_0^{-1}}(g'(0))})(x_0 \cdot g_0).$$

**Problem 2**
Let $GL(M)$ denote the frame bundle of a smooth $n$-dimensional manifold $M$. Prove that the following maps $A \mapsto \nabla^A$ and $\nabla \mapsto A^\nabla$ establish a bijection between the set of all connection forms $A \in \Omega^1(GL(M); \mathfrak{gl}(n, \mathbb{R}))$ on $GL(M)$ and the set of all connections $\nabla$ on the tangent bundle $TM$:

- Given a connection form $A \in \Omega^1(GL(M); \mathfrak{gl}(n, \mathbb{R}))$, define $\omega_{ij} \in \Omega^1(GL(M); \mathbb{R})$ by

  $$A =: \sum_{i,j=1}^{n} \omega_{ij} B_{ij},$$

  where $B_{ij} \in \mathbb{R}^{n \times n}$ denotes the matrix with entry 1 on the intersection of the $i$-th row and the $j$-th column, and all other entries equal to zero. Then $\nabla^A$ is the uniquely determined connection on $TM$ so that for every local section $s = (s_1, \ldots, s_n)$ of $GL(M)$,

  $$\nabla^A_{\cdot} s_k := \sum_{i=1}^{n} \omega_{ik}(ds(\cdot)) s_i.$$

- Given a connection $\nabla$ on $TM$, define for every local section $s = (s_1, \ldots, s_n)$ of $GL(M)$ over $U \subset M$ one-forms $\omega_{ij} \in \Omega^1(U; \mathfrak{gl}(n, \mathbb{R}))$ for $i, j = 1, \ldots, n$ by

  $$\nabla_{\cdot} s_i =: \sum_{i,j=1}^{n} \omega_{ij}(\cdot) s_j$$

  and let $A^s \in \Omega^1(U, \mathfrak{gl}(n, \mathbb{R}))$ be given by

  $$A^s := \sum_{i,j=1}^{n} \omega_{ij} B_{ij}.$$

  Then $A^\nabla$ is the unique connection form on $GL(M)$ whose local connection form with respect to each local section $s$ is given by $A^s$.

**Problem 3**
Let $(M, g)$ be a Riemannian manifold and $O(M, g)$ the pricipal bundle of $g$-orthonormal frames in $TM$.

a) Let $\nabla$ be a connection on $TM$ which is metric with respect to $g$. For every local section $s : U \to O(M, g)$ define $A^s \in \Omega^1(U; \mathfrak{o}(n))$ by

$$A^s(\cdot) := \sum_{i,j=1}^{n} g(\nabla_{\cdot} s_i, s_j) E_{ij},$$

where $E_{ij} = B_{ij} - B_{ji} \in \mathfrak{o}(n)$. Show that there is a unique connection form $A^\nabla$ on $O(M, g)$ whose local connection form with respect to each $s$ is given by $A^s$.

b) Prove that the map $\nabla \mapsto A^\nabla$ establishes a bijection between the set of all $g$-metric connections on $TM$ and the set of all connection forms on $O(M, g)$.