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Problem Sheet 8

Differential Geometry II 2017

The following are meant to provide an overview of J. Milnor's construction of inequivalent differentiable structures on S^7 (see Milnor's paper 'On Manifolds Homeomorphic to the 7-Sphere' and the references therein for details and hints).

Problem 1
Let M be a closed oriented manifold of dimension 7 with $H^3(M;\mathbb{Z}) \simeq H^4(M;\mathbb{Z}) \simeq \{0\}$. Given a compact oriented manifold B with $M = \partial B$, define integers $\sigma(M,B)$ and $q(M,B)$ as follows: $\sigma(M,B)$ is the index of the quadratic form on $H^4(B,M;\mathbb{Z})$ (modulo torsion) given by $a \mapsto a^2([B])$, where $[B] \in H_8(B,M;\mathbb{Z})$ is the orientation class, while $q(M,B)$ is defined as $q(M,B) = (i^{-1}(p_1(TB))^2([B])$, where $p_1(TB)$ is the first Pontrjagin class of the tangent bundle of B and i the isomorphism $i : H^4(B,M;\mathbb{Z}) \to H^4(B;\mathbb{Z})$.

a) Let B' be a second compact oriented manifold with boundary $\partial B' = M$. Show that with $C := B \cup_M B'$, we have $\sigma(M,B) - \sigma(M,B') = \sigma(C)$ and $q(M,B) - q(M,B') = p_1^2(TC)([C])$.

b) Using the Hirzebruch Signature Theorem, conclude that $2q(M,B) - \sigma(M,B)$ is independent modulo 7 of the choice of B and in particular, if $\lambda(M) = [2q(M,B) - \sigma(M,B)] \in \mathbb{Z}/7\mathbb{Z}$ is non-zero, then M is not the (differentiable) boundary of a compact manifold B with $H_4(B;\mathbb{Z}) \simeq \{0\}$.

Problem 2
For $h, j \in \mathbb{Z}$, denote by $\xi_{h,j}$ the S^3-bundle over S^4 obtained by identifying two copies of $D^4 \times S^3$ along their boundary via the map $(u, v) \mapsto u^h \cdot v \cdot u^j$ for $u \in \partial D^4$, $v \in S^3$. Here the product stands for quaternion multiplication. For every odd integer k, let M_k be the total space of $\xi_{h,j}$ with h and j determined by $h + j = 1$, $h - j = k$.

a) Show that the first Pontrjagin class of $\xi_{h,j}$ is given (up to sign) by $2(h-j)$ times the generator of $H^4(S^4;\mathbb{Z})$.

b) By studying the unit disk bundle corresponding to $\xi_{h,j}$, conclude that $\lambda(M_k) = k^2-1$ modulo 7and thus for $k^2 \neq 1$ modulo 7, M_k is not diffeomorphic to S^7.

Problem 3

a) Construct a smooth function $f : M_k \to \mathbb{R}$ which has exactly two critical points, both of which are nondegenerate, i. e. the Hessian at the critical point does not have zero as an eigenvalue.

b) By studying the flow of the gradient of f with respect to a Riemannian metric, conclude that M_k is homeomorphic to S^7 for every k.