Lecture Notes Differential Geometry 2

(Summer semester 2017)

Literature:

J. Milnor, J. Stasheff "Characteristic Classes"
H. Baum "Eichfeldtheorie"
W. Zhang "Lectures on Chern-Weil Theory and Witten Deformations"
D. McDuff, D. Salamon "Introduction to Symplectic Topology"

1. Manifolds and Vector Bundles

1.1 Smooth Manifolds

Recall:

Definition 1.1

A topological manifold of dimension $n \in \mathbb{N}$ is a second countable Hausdorff topological space that is locally Euclidean, i.e. for each $x \in M$ there exists an open subset $\mathcal{U} \subseteq M$, $x \in \mathcal{U}$, so that $\mathcal{U}$ is homeomorphic to an open subset of $\mathbb{R}^n$.

Remarks: 1) One can show that the locally euclidean condition implies that M is paracompact, i.e. every open cover of M has a locally finite refinement.

2) There are examples of locally Euclidean spaces which are neither Hausdorff, nor second countable (see problem sheet 1).

In order to introduce differentiation on M, an additional piece of structure is needed.

Recall that a coordinate chart on M is a pair (U, φ), where $U \subset M$ is an open subset and $\varphi : U \xrightarrow{\sim} W$ is a homeomorphism onto an open subset $W \subseteq \mathbb{R}^n$. An atlas is a family $\mathcal{A} = (U_i, \varphi_i)_{i \in I}$ of coordinate charts, s.t. $M = \bigcup_{i \in I} U_i$. An atlas is called differentiable (resp. smooth), if for all $i, j \in I$ the map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is continuously differentiable (resp. smooth).

A differentiable atlas is called <u>oriented</u>, if for all $i, j \in I$ $\varphi_j \circ \varphi_i^{-1}$ is orientation-preserving in the sense that $\det\left(J_{\varphi_j \circ \varphi_i^{-1}}\right)_x > 0$ for all $x \in \varphi_i(U_i \cap U_j)$. Here $J_{\varphi_j \circ \varphi_i^{-1}}$ is the Jacobi matrix of $\varphi_j \circ \varphi_i^{-1}$ (matrix of first partial derivatives).

Two differentiable (resp. smooth / oriented) atlases $\mathcal{A}, \mathcal{A}'$ on M are called <u>equivalent</u>, if $\mathcal{A} \cup \mathcal{A}'$ is a differentiable (resp. smooth / oriented) atlas.

<u>Definition 1.2</u> Let M be a topological manifold. A <u>differentiable structure</u> (resp. <u>smooth structure</u>) on M is an equivalence class $[\mathcal{A}]$ of differentiable (resp. smooth) atlases on M. An <u>orientation</u> of M is an equivalence class of oriented atlases on M.

<u>Remarks</u>: 1) Each equivalence class $[\mathcal{A}]$ of differentiable (resp. smooth or orientable) atlases contains a unique maximal atlas, given

by $\bigcup_{\mathcal{A}' \sim \mathcal{A}} \mathcal{A}'$.

2) There are examples of distinct differentiable structures on the same topological manifold (see problem sheet 1). There are examples of topological manifolds that do not admit a differentiable structure.

Unless specified otherwise, "manifold" will from now on always mean "topological manifold together with a choice of smooth structure".

Recall that given two smooth manifolds $(M, [\mathcal{A}_M])$ and $(N, [\mathcal{A}_N])$, a continuous map $f: M \to N$ is called smooth, if for all coordinate charts (U, φ) resp. (V, ψ) in $\mathcal{A}_M$ resp. in $\mathcal{A}_N$, the composition $\psi \circ f \circ \varphi^{-1}: \varphi(U \cap f^{-1}(V)) \to \psi(V)$ is smooth. f is called a <u>diffeomorphism</u>, if f is a homeomorphism and both f and f^{-1} are smooth.

Remarks: 1) Denoting by $F := C^\infty(M)$ the space of all smooth maps $M \to \mathbb{R}$ and by $\mathbb{R}^F$ the real vector space of all maps $F \to \mathbb{R}$, the map $i: M \to \mathbb{R}^F$, $p \mapsto (f \mapsto f(p))$ is smooth. Moreover, $i(M) \subset \mathbb{R}^F$ is a smooth submanifold and $i: M \to i(M)$ a diffeomorphism.

2) If M is a topological manifold of dimension ≤ 3, then for any two smooth structures $[A]$, $[A']$ on M, $(M, [A])$ and $(M, [A'])$ are diffeomorphic. There exist several non-diffeomorphic smooth structures on the 7-sphere $S^7 = \{ v \in \mathbb{R}^8 \mid \|v\| = 1 \}$. The proof of this fact relies on the theory of characteristic classes.

Finally, we recall

Proposition 1.3 (Partition of unity)
Let A be a smooth atlas on M. There exists a smooth partition of unity subordinate to A, i.e.

a sequence $(f_n)_{n\in\mathbb{N}}$ of non-negative smooth functions $f_n: M \to \mathbb{R}$, so that • For each $n \in \mathbb{N}$, $\operatorname{supp} f_n = \overline{\{x \in M \mid f(x) > 0\}}$ is compact and contained in a coordinate neighbourhood of $\mathcal{A}$,

• For each $x \in M$ there exists $\mathcal{U} \subseteq M$ open, $x \in \mathcal{U}$, so that $\mathcal{U} \cap \operatorname{supp} f_n \neq \emptyset$ only for finitely many $n \in \mathbb{N}$,

• $\sum_{n=1}^{\infty} f_n(x) = 1$ for all $x \in M$.

1.2 Vector Bundles

Definition 1.4

A fibre bundle is a tuple (E, π, M, F) consisting of

- Topological manifolds E, M and F, referred to as the total space, the base and the typical fibre respectively.
- A continuous map $\pi: E \to M$ satisfying the following property:

for every point $x \in M$ there exists an open neighbourhood $U \subseteq M$ of x and a homeomorphism $\varphi_U : \pi^{-1}(U) \to U \times F$, so that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\sim", "\varphi_U"'] \arrow[d, "\pi"'] & U \times F \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

commutes, i.e. $\pi = pr_1 \circ \varphi_U$. Here $pr_1 : U \times F \to U$ denotes the projection to the first factor.

The fibre bundle is called smooth, if E, M and F each carry a smooth structure, so that π is smooth and each φ_U is a diffeomorphism.

A map φ_U satisfying the conditions of Definition 1.4 is called a <u>bundle chart</u> or <u>local trivialization</u> of the fibre bundle. The fibre bundle is called <u>trivial</u> if a trivialization exists over $U = M$, i.e. if there is a diffeomorphism

$$E \underset{\varphi}{\overset{\sim}{\to}} M \times F$$

with $\pi = pr_1 \circ \varphi$.
For $x \in M$, $E_x := \pi^{-1}(x) \subset E$ is called the <u>fibre</u> over the point x.

<u>Definition 1.5</u> A real (resp. complex) <u>vector bundle</u> of rank $k \in \mathbb{N}$ is a fibre bundle $(E, \pi, M; F)$, so that $F \simeq \mathbb{R}^k$ (resp $F \simeq \mathbb{C}^k$), so that each fibre E_x is a real (resp. complex) vector space and for each point $x \in M$ there exists a local trivialization φ_U in a neighbourhood $U \subseteq M$ of x, so that the composition

$$E_x \xrightarrow{\varphi_U|_{E_x}} \{x\} \times \mathbb{R}^k \xrightarrow{pr_2} \mathbb{R}^k$$

(resp. $E_x \xrightarrow{\varphi_U|_{E_x}} \{x\} \times \mathbb{C}^k \xrightarrow{pr_2} \mathbb{C}^k$) is a vector space isomorphism.

<u>Examples</u>: 1) M - arbitrary manifold, $(E, \pi, M) = (M \times \mathbb{R}^k, pr_1, M)$.

2) For every differentiable manifold M of dimension $n \in \mathbb{N}$, the tangent bundle TM is a (topological) vector bundle of

rank n. If M is a smooth manifold, then TM is smooth.

3) With $\mathbb{RP}^n = \{\mathbb{R}x \mid x \in \mathbb{R}^{n+1} \setminus \{0\}\}$ consider $E := \{(\mathbb{R}x, v) : x \in \mathbb{R}^{n+1} \setminus \{0\}, v \in \mathbb{R}x\} \subset \mathbb{RP}^n \times \mathbb{R}^{n+1}$,
$\pi : E \to \mathbb{RP}^n$, $(\mathbb{R}x, v) \mapsto \mathbb{R}x$ and the natural vector space structure on each fibre $E_{\mathbb{R}x} = \mathbb{R}x \subset \mathbb{R}^{n+1}$.
Then $(E, \pi, \mathbb{RP}^n; \mathbb{R}^n)$ is a rank-one vector bundle over $\mathbb{RP}^n$ called the <u>canonical line bundle</u> over $\mathbb{RP}^n$.
The canonical line bundle over $\mathbb{CP}^n = \{\mathbb{C}x \mid x \in \mathbb{C}^{n+1} \setminus \{0\}\}$ is a rank-one complex line bundle defined analogously.

<u>Definition 1.6</u> A continuous (resp. smooth) <u>section</u> of a vector bundle (E, π, M) is a continuous (resp. smooth) map $s : M \to E$ s.t. $\pi \circ s = id_M$.
We write $\Gamma(E, \pi, M)$ (or $\Gamma(E)$) for the set of all sections. Note that $\Gamma(E, \pi, M)$ is a vector space whose zero element is the zero section

$$s(x) = 0_x = \text{zero element of the vector space } E_x = \pi^{-1}(x).$$

A vector bundle $(E, \pi, M; \mathbb{R}^k)$ is called *trivial* if there exists a diffeomorphism $E \xrightarrow[\varphi]{\sim} M \times \mathbb{R}^k$ s.t. $\pi = pr_2 \circ \varphi$ and for each $x \in M$ $\varphi|_{E_x} : E_x \to \{x\} \times \mathbb{R}^k \simeq \mathbb{R}^k$ is a vector space isomorphism.

<u>Proposition 1.7</u> A smooth vector bundle (E, π, M) of rank k is trivial if and only if it admits k smooth sections $s_1, \dots, s_k \in \Gamma(E)$, so that for every $x \in M$ the vectors $s_1(x), \dots, s_k(x) \in E_x$ are linearly independent.

For the proof of Proposition 1.7 we will need the following:

<u>Lemma 1.8</u> Let (E, π, M) and (E', π', M) be two smooth vector bundles over the same smooth manifold M and let $G: E \to E'$ be a smooth map so that for each $x \in M$, $G|_{E_x}$ is a vector space isomorphism $E_x \xrightarrow{\sim} E'_x$. Then G is a diffeomorphism.

<u>Proof</u>: It follows from the assumptions that G is a bijection. We show that $G^{-1}: E' \to E$ is smooth.

Given $x \in M$, fix an open neighbourhood $\mathcal{U} \subset M$ of x so that E and E' admit local trivialisations

$\varphi_{\mathcal{U}}: \pi^{-1}(\mathcal{U}) \xrightarrow{\sim} \mathcal{U} \times \mathbb{R}^k$ and $\varphi'_{\mathcal{U}}: (\pi')^{-1}(\mathcal{U}) \xrightarrow{\sim} \mathcal{U} \times \mathbb{R}^k$.

From the assumptions, for $(y, v) \in \mathcal{U} \times \mathbb{R}^k$

$$\left((\varphi'_{\mathcal{U}})^{-1} \circ G \circ \varphi_{\mathcal{U}} \right)(y, v) = (y, A_y \cdot v),$$

where $A_y \in \mathbb{R}^{k \times k}$ is invertible for each $y \in \mathcal{U}$ and the map $\begin{matrix} \mathcal{U} \to \mathbb{R}^{k \times k} \\ y \mapsto A_y \end{matrix}$ is smooth. It follows that

$\begin{matrix} \mathcal{U} \to \mathbb{R}^{k \times k} \\ y \mapsto (A_y)^{-1} \end{matrix}$ is smooth and thus

so is

$$\varphi_{\mathcal{U}}^{-1} \circ G^{-1} \circ \varphi'_{\mathcal{U}} : (\pi')^{-1}(\mathcal{U}) \to \pi^{-1}(\mathcal{U})$$
$$(y, w) \mapsto (y, A_y^{-1} \cdot w).$$

Since $\varphi_{\mathcal{U}}$ and $\varphi'_{\mathcal{U}}$ are diffeomorphisms and $x \in M$ was arbitrary, the claim follows. $\square$

Proof of Proposition 1.7

Given smooth sections $s_1, \dots, s_k$ as in the assumptions, define

$$\psi : M \times \mathbb{R}^k \to E$$
$$\psi(x, v) = v_1 \cdot s_1(x) + \dots + v_k s_k(x)$$

Since $s_1(x), \dots, s_k(x)$ form a basis of E_x for each $x \in M$,

$$\psi|_{\{x\}\times\mathbb{R}^k} : \{x\}\times\mathbb{R}^k \cong \mathbb{R}^k \to E_x$$

is a vector space isomorphism.

By Lemma 1.8, ψ is a diffeomorphism. Thus $\varphi := \psi^{-1}$ is a trivialization of E.

Conversely, assume that E is trivial and $\varphi : E \xrightarrow{\sim} M \times \mathbb{R}^k$ is a trivialization. Define

$$s_j(x) := \varphi^{-1}(x, e_j), \quad j = 1, \dots, k,$$

where $e_1, \dots, e_k \in \mathbb{R}^k$ are the standard basis vectors.

$\square$

<u>Examples</u>: 1) A smooth manifold M is called <u>parallelizable</u> if its tangent bundle is trivial.

For $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$,

$TS^n = \{(x,v) \in S^n \times \mathbb{R}^{n+1} \mid \langle x, v\rangle = 0\}$.

In the case $n = 1$,

$s_1(x_1, x_2) := \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$ for $x = (x_1, x_2) \in S^1$

is a nowhere vanishing section

$\Rightarrow TS^1$ is trivial.

2) Consider the canonical line bundle over $\mathbb{RP}^n$ (see above). We claim that every continuous section of this vector

bundle has a zero and hence the canonical line bundle is non-trivial.

Recall that the total space is $E = \{ (\mathbb{R}x, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in \mathbb{R}x \subset \mathbb{R}^{n+1} \}$.

Suppose that $s : \mathbb{RP}^n \to E$ is a continuous section with $\underset{\in \mathbb{R}x}{s(\mathbb{R}x)} \neq 0 \;\; \forall x$. Given $x \in S^n$, we have $s(\mathbb{R}x) = t(x) \cdot x$ for a unique $t(x) \in \mathbb{R} \setminus \{0\}$. The function $t : S^n \to \mathbb{R}$ is continuous and satisfies $t(-x) = -x$ for all $x \in S^n$. But since S^n is path connected, such a function cannot exist because of the intermediate value theorem.

<u>Definition 1.9</u> A <u>morphism</u> (or <u>bundle map</u>) between two smooth vector bundles (E, π, M) and (E', π', M') is a pair (h, g) of smooth maps $h : E \to E'$ and $g : M \to M'$, s.t. the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ M & \xrightarrow{g} & M' \end{array}$$

and so that for every $x \in M$,
$G|_{E_x} : E_x \to E_{g(x)}$ is a vector space isomorphism. (G, g) is called a <u>bundle isomorphism</u> if G and g are diffeomorphisms.

<u>Remarks</u>: 1) If (E, π, M) is a trivial bundle with trivialization $\varphi : E \xrightarrow{\sim} M \times \mathbb{R}^k$ and $(G, g) : (E, \pi, M) \to (E', \pi', M')$ is a bundle isomorphism, then (E', π', M') is also trivial. Indeed, with
$$(\varphi \circ G^{-1})(e') =: (x(e'), v(e'))$$
for $e' \in E'$, a trivialization of (E', π', M') is given by the map
$$E' \xrightarrow{\sim} M \times \mathbb{R}^k$$
$$e' \longmapsto (g(x(e')), v(e')).$$

2) If $f : M \to M'$ is a diffeomorphism between two smooth manifolds, then (df, f) is an isomorphism between the tangent bundles TM and TM'. This observation can be used in order

to distinguish smooth structures: suppose that $[\mathcal{A}_M]$ and $[\mathcal{A}_M']$ are two smooth structures on the same topological manifold M, so that the tangent bundles $\tau = TM$ and $\tau' = (TM)'$ corresponding to $[\mathcal{A}_M]$ and $[\mathcal{A}_M']$ are non-isomorphic as topological vector bundles. Then it follows that $(M, [\mathcal{A}_M])$ and $(M, [\mathcal{A}_M'])$ are non-diffeomorphic.

1.3 Constructions of Vector Bundles

Proposition 1.10 Let $f: N \to M$ be a smooth map and (E, π, M) a smooth vector bundle over M. Then $f^*E := \{(x,e) : x \in N, e \in E, f(x) = \pi(e)\}$, together with the projection $\pi': f^*E \to N$, $(x,e) \mapsto x$, is a smooth vector bundle over N, called the pullback of E via f.

Proof: Let $(U_i, \varphi_i)_{i \in I}$ be a collection of bundle charts of

E, so that $\bigcup_{i\in I} \mathcal{U}_i = N$.
Then $\mathcal{U}_i' := f^{-1}(\mathcal{U}_i) \subset N$ is open and we define

$$\psi_i : (\pi')^{-1}(\mathcal{U}_i') \to \mathcal{U}_i' \times \mathbb{R}^k$$
$$(x,e) \mapsto (x, \mathrm{pr}_2\, \varphi_i(e)).$$

Note that ψ_i is a bijection and $\psi_i|_{(f^*E)_x} = \psi_i|_{E_{f(x)}} : E_{f(x)} \to \{x\}\times\mathbb{R}^k$ is a vector space isomorphism. Moreover, for $i,j \in I$

$$\psi_i \circ \psi_j^{-1} : (\mathcal{U}_i' \cap \mathcal{U}_j') \times \mathbb{R}^k \to (\mathcal{U}_i' \cap \mathcal{U}_j') \times \mathbb{R}^k$$
$$(x,v) \mapsto (x, \mathrm{pr}_2 \varphi_j \varphi_i^{-1}(f(x),v))$$

is smooth. Thus $\mathcal{A} = ((\pi')^{-1}(\mathcal{U}_i'), \psi_i)_{i\in I}$ defines a smooth structure on f^*E s.t. (f^*E, π, M) is a smooth vector bundle. $\square$

Recall that for two finite-dimensional vector spaces V, V' with bases $(e_1,\dots,e_k)$ and $(e_1',\dots,e_l')$ respectively, the direct sum $V\oplus V'$ resp. the tensor product $V\otimes V'$ is a vector space that has $(e_1,\dots,e_k, e_1',\dots,e_l')$ resp. $(e_i \otimes e_j')_{\substack{1\le i\le k\\ 1\le j\le l}}$ as a basis.

For every pair of linear maps $L: W \to V$, $L': W' \to V'$ there are natural corresponding maps
$L \oplus L' : W \oplus W' \to V \oplus V'$ and
$L \otimes L' : W \otimes W' \to V \otimes V'$. Moreover, if L, L' are isomorphisms, then so are $L \oplus L'$ and $L \otimes L'$.

We consider the following operations on the set of vector bundles over a manifold M:

(i) <u>The Whitney sum</u>

Given two vector bundles $\xi = (E, \pi, M)$ and $\xi' = (E', \pi', M)$, the space $E \oplus E' := \bigcup_{x \in M} (E_x \oplus E'_x)$, equipped with the projection $\pi_\oplus : E_x \oplus E'_x \ni v_x \mapsto x$, gives rise to a vector bundle $\xi \oplus \xi' := (E \oplus E', \pi_\oplus, M)$ called the Whitney sum of ξ and ξ'. The local trivializations of $E \oplus E'$ are given by the maps

$$\psi_u : \pi_\oplus^{-1}(U) \to U \times (\mathbb{R}^n \oplus \mathbb{R}^{n'})$$

determined by the condition

$$(pr_2 \circ \psi_u)|_{E_x \oplus E'_x} = \varphi_{u,x} \oplus \varphi'_{u,x}.$$ Here φ_u, φ'_u are local trivializations of ξ, ξ' and $\varphi_{u,x}$ is defined by $\varphi_u|_{E_x} =: (x, \varphi_{u,x})$.

k and k' denote the rank of ξ and of ξ' respectively.

The topology and smooth structure of $E \oplus E'$ are determined by requiring that each map $\psi_{\mathcal{U}}$ be a diffeomorphism. The Whitney sum of complex vector bundles is defined analogously.

(ii) <u>The tensor product</u>

Given vector bundles $\xi = (E, \pi, M)$ and $\xi' = (E', \pi', M)$, the tensor product of ξ and ξ' is defined as

$$\xi \otimes \xi' = (E \otimes E', \pi_{\otimes}, M), \quad \text{where}$$

$$E \otimes E' = \dot{\bigcup_{x \in M}} E_x \otimes E'_x, \quad \pi_{\otimes}: E_x \otimes E'_x \ni v_x \mapsto x$$

and the local trivializations are determined by

$$(pr_2 \circ \psi_{\mathcal{U}})|_{E_x \otimes E'_x} = \varphi_{\mathcal{U},x} \otimes \varphi'_{\mathcal{U},x}.$$

(iii) <u>The homomorphism bundle</u>

For $\xi = (E, \pi, M)$, $\xi' = (E', \pi', M)$, we define

$$\mathrm{Hom}(\xi, \xi') := (\mathrm{Hom}(E, E'), \pi_{\mathrm{Hom}}, M),$$

where

$$\mathrm{Hom}(E, E') = \bigcup_{x \in M} \mathrm{Hom}(E_x, E'_x) =$$

$$\dot{\bigcup_{x \in M}} \{ L_x : E_x \to E'_x \mid L_x \text{ is linear} \}$$

$\pi_{\mathrm{Hom}}: \quad \mathrm{Hom}(E_x, E'_x) \ni L \longmapsto x$

and local trivializations ψ_u given by

$$\psi_u(L_x) = (x, \varphi'_{u,x} \circ L_x \circ \varphi_{u,x}^{-1})$$

for $L_x \in \mathrm{Hom}(E, E')_x = \mathrm{Hom}(E_x, E'_x)$.
A special case of a homomorphism bundle is

(iv) <u>The dual bundle</u>

Denoting by $\xi_0 := (M \times \mathbb{C}, pr_1, M)$ the trivial rank one vector bundle over M, the dual of a complex vector bundle ξ over M is given by

$$\xi^* := \mathrm{Hom}(\xi, \xi_0).$$

Explicitly, if $\xi = (E, \pi, M)$, then $\xi^* = (E^*, \pi^*, M)$ where

$$E^* = \dot{\bigcup_{x \in M}} E_x^* = \dot{\bigcup_{x \in M}} \{ L_x : E_x \to \mathbb{R} \mid L_x \text{ linear} \},$$

$\pi^*: \quad E_x^* \ni L_x \mapsto x$ and local trivializations

$$\psi_u(L_x) = (x, L_x \circ \varphi_{u,x}^{-1}).$$

One can define the dual ξ^* of a <u>real</u> vector bundle ξ analogously, however it turns out that in this case ξ^* is always isomorphic to ξ. (see below).

(v) The conjugate vector bundle

For a complex vector bundle $\xi = (E, \pi, M)$ we define the conjugate bundle $\bar{\xi}$ by
$$\bar{\xi} := (\bar{E}, \bar{\pi}, M).$$
Here
$$\bar{E} = \dot{\bigcup_{x \in M}} \bar{E}_x,$$
where $\bar{E}_x$ is the conjugate complex vector space to E_x, i.e. $E_x \simeq \bar{E}_x$ as abelian groups, with the $\mathbb{C}$-action on $\bar{E}_x$ given by $\mathbb{C} \times E_x \ni (z, v_x) \mapsto \bar{z} \cdot v_x$. The projection $\bar{\pi}$ is given by
$\bar{\pi} : \bar{E}_x \ni v_x \mapsto x$ and after identifying the fibres $E_x \simeq \bar{E}_x$ as abelian groups, the local trivializations $\bar{\varphi}_u$ of $\bar{E}_x$ are given by
$$\bar{\varphi}_u : \bar{\pi}^{-1}(u) \to u \times \bar{\mathbb{C}}^k,$$
$$\bar{E}_x \ni v \mapsto (x, \bar{\varphi}_{u,x}(v)).$$
Here k is the rank of ξ, φ_u a local trivialization of ξ and the map $\bar{\varphi}_{u,x}$ is the composition of $\varphi_{u,x}$ with complex conjugation.

Finally, we recall the notion of a bundle metric on a vector bundle.

<u>Definition 1.11</u> Let $\xi = (E, \pi, M)$ be a real (resp. complex) vector bundle. A <u>bundle</u> <u>metric</u> on E is a section $\langle \cdot, \cdot \rangle$ of the bundle $\xi^* \otimes \xi^*$ (resp. $\xi^* \otimes \bar{\xi}^*$), so that for each $x \in M$ the symmetric bilinear form (resp. hermitian form) $\langle \cdot, \cdot \rangle_x : E_x \times E_x \to \mathbb{R}$ (resp. $\langle \cdot, \cdot \rangle_x : E_x \otimes E_x \to \mathbb{C}$) is nondegenerate.

<u>Proposition 1.12</u> Each real or complex vector bundle $\xi = (E, \pi, M)$ admits a bundle metric $\langle \cdot, \cdot \rangle$ s.t. $\langle \cdot, \cdot \rangle_x$ is positive definite for each $x \in M$.

<u>Proof</u>: Let $(\varphi_{U_i})_{i \in I}$ be a collection of local trivializations of, so that $(U_i)_{i \in I}$ is a cover of M. It follows from Proposition 1.3 that there exists a partition of unity $(f_i)_{i \in I}$ subordinate to $(U_i)_{i \in I}$ ($\operatorname{supp} f_i \subset U_i$ for $i \in I$).

Denote by $e_1, \dots, e_k$ the standard basis of $\mathbb{R}^k$ (resp. $\mathbb{C}^k$), where k is the rank of ξ.
Denote by $s_j^{(i)} : \mathcal{U}_i \to \pi^{-1}(\mathcal{U}_i)$ the local section of ξ given by $x \mapsto \varphi_{\mathcal{U}_i}^{-1}(x, e_j)$, $j = 1, \dots, k$.
We define for $x \in \mathcal{U}_i$

$$\langle \cdot, \cdot \rangle_x^{(i)} : E_x \times E_x \to \mathbb{R}$$

(resp. $\langle \cdot, \cdot \rangle^{(i)} : E_x \times \overline{E_x} \to \mathbb{C}$) as the bilinear (resp. hermitian) form with

$$\langle s_j^{(i)}(x), s_l^{(i)}(x) \rangle_x^{(i)} := \delta_{jl}$$

for $1 \leq j, l \leq k$. Then

$$\langle \cdot, \cdot \rangle_x := \sum_{i \in I} f_i(x) \langle \cdot, \cdot \rangle_x^{(i)}$$

is a positive definite bundle metric on ξ.

$\square$

<u>Corollary 1.13</u> (i) For every real vector bundle ξ, ξ^* is isomorphic to ξ.

(ii) For every complex vector bundle ξ, $\overline{\xi}^*$ is isomorphic to ξ.

<u>Proof</u>: Fix a bundle metric $\langle \cdot, \cdot \rangle$ on ξ and consider the map

$G: E \to E^*$, $E_x \ni v \mapsto (L_v(w) := \langle v, w \rangle_x)$

(resp. $E \to \bar{E}^*$, $E_x \ni v \mapsto (L_v(w) := \langle v, w \rangle_x)$).
Since $\langle \cdot, \cdot \rangle$ is nondegenerate, for each $x \in M$ the restriction of G to E_x is a vector space isomorphism $E_x \to E_x^*$ (resp. $E_x \to \bar{E}_x^*$). Using Lemma 1.8, we conclude that (G, id_M) is an isomorphism of vector bundles.

$\square$

2. Chern-Weil Theory

2.1 The Notion of a Characteristic Class

We first recall some basic facts from algebraic topology. Given a ring R and a topological space X, one defines for $k \geq 0$ $C^k(X; R)$ to be the space of maps which assign to every singular simplex in X, i.e. a continuous map $\sigma: \Delta^k \to X$, an element of R. Here

$\Delta^k = \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1}_{\geq 0} \mid \sum_{j=0}^{k} t_j = 1\}$ is the k-dimensional unit simplex. Denoting, for $j = 0, \dots, k+1$ by f_j the map $f_j : \Delta^k \to \Delta^{k+1}$, $(t_0, \dots, t_k) \mapsto (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_k)$, one defines

$$\delta^k : C^k(X; R) \to C^{k+1}(X; R) \text{ by}$$

$$(\delta\varphi)(\sigma) := \sum_{j=0}^{k+1} (-1)^j \varphi(\sigma \circ f_j).$$

Then $\delta^{k+1} \circ \delta^k = 0$ for all k and one defines the k-th cohomology group of X with coefficients in R as the quotient

$$H^k(X; R) = \ker \delta^k / \operatorname{im} \delta^{k-1}$$

$$= \{\varphi \in C^k(X; R) \mid \delta^k \varphi = 0\} / \{\delta^{k-1} \psi \mid \psi \in C^{k-1}(X; R)\}.$$

Every continuous map $f : X \to Y$ between topological spaces induces a pullback map

$$f^* : H^*(Y; R) \to H^*(X; R)$$
$$[\varphi] \mapsto [(\sigma \mapsto \varphi(f \circ \sigma))]$$

in cohomology. This is compatible with compositions: if $g : Y \to Z$ is also continuous, then $(g \circ f)^* = f^* \circ g^*$.

<u>Definition 2.1</u> A <u>characteristic class</u> over a ring R is a map which assigns to every (topological / smooth) vector bundle ξ over a manifold M a

cohomology class $c(\xi) \in H^*(M;R)$, so that for every continuous (resp. smooth) map $f: N \to M$, we have

$$c(f^*\xi) = f^*(c(\xi)).$$

Remarks: 1) Considering two vector bundles over the same manifold M as isomorphic if there is a bundle isomorphism of the form $(\cdot, id_M)$ between them, characteristic classes of isomorphic vector bundles coincide. This follows from the identity

$$(id_M)^* = id_{H^*(M;R)}.$$

2) One can adapt the above definition by assigning $c(\xi) \in H^*(M;R)$ to vector bundles ξ with some additional structure (e.g. orientation).

3) Examples of characteristic classes are the so-called Stiefel-Whitney classes (over $\mathbb{Z}/2\mathbb{Z}$) or the Euler class (for oriented vector bundles, over $\mathbb{Z}$). These can be constructed using methods from algebraic topology (see problem sheet 2). From now on, we restrict attention to the case $R = \mathbb{R}$.

Recall that for a real vector space V, $\Lambda^k(V^*)$ is the space of all k-linear maps $L: V \times \ldots \times V \to \mathbb{R}$, so that $L(\ldots, v, \ldots, w, \ldots) = -L(\ldots, w, \ldots, v, \ldots)$ for all $v, w \in V$. If (E, π, M) is a smooth vector bundle over M, then $\Lambda^k E := \bigcup_{x \in M} \Lambda^k(E_x)$ is the total space of a smooth vector bundle $\Lambda^k \xi$ over M. In the special case when $\xi = TM$ is the tangent bundle of a differentiable manifold M, a (smooth) section of $\Lambda^k(TM)$ is called a (smooth) <u>k-form</u> on M and we write $\Omega^k(M;\mathbb{R}) = \Gamma(\Lambda^k(TM))$ for the space of k-form.

A k-form ω is uniquely determined by specifying $\omega(X_1, \ldots, X_k): M \to \mathbb{R}$ for all (smooth) vector fields $X_1, \ldots, X_k \in \Gamma(TM)$ on M and one defines

$$d: \Omega^k(M;\mathbb{R}) \to \Omega^{k+1}(M;\mathbb{R}) \text{ by}$$

$$(d\omega)(X_1, \ldots, X_{k+1}) := \sum_{j=1}^{k+1} (-1)^{j+1} X_j\, \omega(X_1, \ldots, \hat{X}_j, \ldots, X_{k+1})$$

$$+ \sum_{1 \le j < l \le k+1} (-1)^{j+l} \omega([X_j, X_l], \ldots, \hat{X}_j, \ldots, \hat{X}_l, \ldots, X_{k+1}).$$

Recall:

de Rham's Theorem:

We have $d^2=0$ and denoting $H^*_{dR}(M;\mathbb{R}) := \ker d / \operatorname{im} d$, there is an isomorphism

$$H^*_{dR}(M;\mathbb{R}) \cong H^*(M;\mathbb{R}),$$

where the RHS is singular cohomology with coefficients in $\mathbb{R}$.

In line with de Rham's Theorem, in order to obtain characteristic classes over $\mathbb{R}$, we should assign to vector bundles closed differential forms on their base manifold.

2.2 Principal Bundles

Recall that a <u>Lie group</u> is a group G that carries the structure of a smooth manifold, so that the map

$$G \times G \to G$$
$$(g,a) \mapsto g \cdot a^{-1}$$

is smooth.

Denoting for $g \in G$ by L_g the map $G \to G$, $h \mapsto g \cdot h$, a vector field $X \in \Gamma(TM)$ is called <u>left-invariant</u>, if $(dL_g)_{g^{-1}h} X(gh) = X$

for all $h \in G$. Setting

$$\mathfrak{g} := \{X \in \Gamma(TM) \mid X \text{ is left-invariant}\},$$

$\mathfrak{g}$, together with the Lie bracket $[\cdot,\cdot]$ on vector fields, is called the <u>Lie algebra</u> of G.

<u>Examples</u>: 1) $G = GL(n,\mathbb{R}) \cong \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\}$. We have $\mathfrak{g} \cong T_I GL(n,\mathbb{R}) \cong \mathbb{R}^{n\times n}$. Given $B, C \in \mathfrak{g}$, let $\gamma_B, \gamma_C : (-\varepsilon,\varepsilon) \to G$ be smooth curves with $\gamma_B(0) = \gamma_C(0) = I$ and $\gamma_B'(0) = B$, $\gamma_C'(0) = C$. The left-invariant vector field $X_B \in \Gamma(TG)$ with $X_B(I) = B$ is given by

$$X_B(A) = dL_A(B) = \frac{d}{dt}\Big|_{t=0} \left(L_A(\gamma_B(t))\right) = \frac{d}{dt}\Big|_{t=0} \left(A\cdot\gamma_B(t)\right) = A\cdot B.$$

It follows that for $B, C \in \mathfrak{g}$,

$$[B,C] = [X_B, X_C](I) = (X_B(X_C))(I) - (X_C(X_B))(I)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\gamma_B(t) \cdot C - \gamma_C(t) \cdot B \right)$$

$$= B \cdot C - C \cdot B$$

$GL_+(n, \mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} \mid \det A > 0 \} \subset GL(n, \mathbb{R})$ is a Lie group with the same Lie algebra.

2) $O(n) = \{ A \in \mathbb{R}^{n \times n} \mid A \cdot A^T = I \}$ is a Lie group whose Lie algebra is

$\mathfrak{o}(n) = \{ B \in \mathbb{R}^{n \times n} \mid X + X^T = 0 \}$, $[B, C] = B \cdot C - C \cdot B$.

$SO(n) = \{ A \in O(n) \mid \det A = 1 \} \subset O(n)$ is a Lie group which also has $\mathfrak{o}(n)$ as its Lie algebra.

3) $U(n) = \{ A \in \mathbb{C}^{n \times n} \mid A \cdot \bar{A}^T = I \}$ is a Lie group with Lie algebra

$\mathfrak{u}(n) = \{ B \in \mathbb{C}^{n \times n} \mid B + \bar{B}^T = 0 \}$.

(and $[B, C] = B \cdot C - C \cdot B$).

$SU(n) = \{ A \in U(n) \mid \det U = 1 \} \subset U(n)$ is a Lie group with Lie algebra

$\mathfrak{su}(n) = \{ B \in \mathbb{C}^{n \times n} \mid B + \bar{B}^T = 0$ and $\mathrm{Tr}(B) = 0 \}$

(and $[B, C] = B \cdot C - C \cdot B$).

A <u>right action</u> of a Lie group G on a smooth manifold M is a smooth map

$$G \times M \to M$$
$$(g, x) \mapsto x \cdot g,$$

so that $p \cdot e = p$ for all $p \in M$, where $e \in G$ is the identity element, and so that $(x \cdot g) \cdot h = x(g \cdot h)$ for all $g, h \in G$, $x \in M$. One defines left actions analogously, with the condition $h \cdot (g \cdot x) = (h \cdot g) \cdot x$.

<u>Definition 2.2</u> A (G-)<u>principal bundle</u> is a fibre bundle $(P, \pi, M; G)$, where G is a Lie group, together with a smooth right action $P \times G \to P$ of G on the total space P, so that:

1. The action is fibrewise, i.e. $\pi(p \cdot g) = \pi(p)$ for all $p \in P$, $g \in G$.
2. For every $x \in M$ there exists a local trivialization $\varphi_U : \pi^{-1}(U) \xrightarrow{\sim} U \times G$ of the fibre bundle over a neighbourhood $U \subset M$ of x, so that φ_U is <u>G-equivariant</u>, i.e.
$$\varphi(p \cdot g) = \varphi(p) \cdot g \quad \text{for all } p, g.$$

Here $(x, h) \cdot g = (y, h \cdot g)$ for $(y, h) \in U \times G$.

Remarks: 1) It follows from the first condition of the above definition that there is a well-defined map $P/G \to M$, $[p] \mapsto \pi(p)$. Using the second condition, the map is a homeomorphism with respect to the quotient topology on P/G (i.e. the topology for which $\mathcal{U} \subset P/G$ is open if and only if $\{p \mid [p] \in \mathcal{U}\} \subset P$ is open)

2) For each $x \in M$ the map $G \to E_x$, $g \mapsto p_0 \cdot g$, where $p_0 \in E_x$ is a fixed element, is a diffeomorphism.

The following Proposition establishes a criterion for a given fibre bundle to be a principal bundle:

Proposition 2.3

Let $(P, \pi, M; G)$ be a fibre bundle, where G is a Lie group.

Then $(P, \pi, M; G)$ is a principal bundle if and only if the following condition is satisfied: there exist local trivializations $(\varphi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times G)_{i \in I}$ of the bundle, where $\bigcup_{i \in I} U_i = M$, and smooth maps $(g_{ik} : U_i \cap U_k \to G)_{i,k \in I}$, so that $g_{ii} \equiv e$ and $g_{ij} \cdot g_{jk} = g_{ik}$ for all $i, j, k \in G$ and so that with $\varphi_i(p) =: (x, \varphi_{ix}(p))$ for $p \in P_x$, we have $\varphi_{ix} \circ \varphi_{kx}^{-1} = L_{g_{ik}(x)} : G \to G$.

Proof: Assume first that $(P, \pi, M; G)$ is a principal fibre bundle. Then there exists a cover $(U_i)_{i \in I}$ admitting local trivializations $\varphi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times G$ that are G-equivariant. Setting

$$g_{ik} : U_i \cap U_k \to G,$$
$$x \mapsto (\varphi_{ix} \circ \varphi_{kx}^{-1})(e),$$

we have $(\varphi_{ix} \circ \varphi_{kx}^{-1})(g) = g_{ik}(x) \cdot g$ due to the G-equivariance of $\varphi_{ix} \circ \varphi_{kx}^{-1}$. Conversely, assuming that local trivializations $(\varphi_i)_{i \in I}$ and maps $(g_{ik})_{i,k \in I}$ as in the assumptions of the Proposition are given, define a right action of G on P by

$$p \cdot g := \varphi_{ix}^{-1}(\varphi_{ix}(p) \cdot g).$$

for $x \in \mathcal{U}_i$, $p \in P_x$. It follows from the conditions on $(g_{ik})_{i,k \in I}$ that this is well-defined (independent of the choice of $\mathcal{U}_i$) and satisfies the required conditions.

$\square$

<u>Examples</u>: 1) The trivial bundle $(M \times G, pr_1, M; G)$.

2) Let G be a Lie group and $H \subset G$ a Lie subgroup. Denote by G/H the set of orbits of the right action of H on G, i.e. $G/H = \{gH \mid g \in G\}$ and by $\pi: G \to G/H$ the projection $g \mapsto gH$. Then G/H is a smooth manifold and $(G, \pi, G/H; H)$ a principal fibre bundle. Manifolds of the form $M = G/H$, with G and H as above, are called <u>homogeneous spaces</u>.

3) <u>The Hopf bundle</u> For $n \in \mathbb{N}$, $S^{2n+1} = \{(w_0, \dots, w_n) \in \mathbb{C}^{n+1} \mid |w_0|^2 + \dots + |w_n|^2 = 1\}$. $G = S^1$ acts on S^{2n+1} from the right by

$$S^{2n+1} \times S^1 \to S^{2n+1}$$

$$((w_0, \dots, w_n), z) \mapsto (w_0 \cdot z, \dots, w_n \cdot z).$$

With $\pi: S^{2n+1} \to S^{2n+1}/S^1 \cong \mathbb{C}P^n$

$$(w_0, \dots, w_n) \mapsto [w_0 : \dots : w_n],$$

$(S^{2n+1}, \pi, \mathbb{C}P^n; S^1)$ is a principal bundle.

A particularly important example is the <u>frame bundle</u> of a smooth manifold:

<u>Proposition 2.4</u> Let M be a smooth manifold.

(i) Denote for $x \in M$ by $GL(M)_x$ the set of all bases of the vector space T_xM. Define

$$GL(M) := \dot{\bigcup_{x \in M}} GL(M)_x.$$

Then with $\pi: GL(M) \to M$, $\pi(GL(M)_x) = \{x\}$, $(GL(M), \pi, M; GL(n,\mathbb{R}))$ is a principal bundle.

(ii) Suppose that M is oriented. Write $GL^+(M)_x$ for the set of all positive bases of T_xM, i.e.

all bases $(v_1, \dots, v_n)$, so that $(d\varphi_x(v_1), \dots, d\varphi_x(v_n))$ is a positive basis of $\mathbb{R}^n$ for each coordinate chart φ in a positively oriented atlas of M. Then $GL(M) = \dot{\bigcup_x} GL^+(M)_x$ gives rise to a $GL^+(n;\mathbb{R})$-principal bundle over M.

(iii) Let g be a Riemannian metric on M, i.e. a positive definite bundle metric on TM. Denoting by $O(M,g)_x$ the set of all g_x-orthonormal bases of T_xM,

$$O(M,g) := \dot{\bigcup_{x \in M}} O(M,g)_x$$

is the total space of an $O(n)$-principal bundle.

<u>Proof</u>: (i) $GL(n;\mathbb{R})$ acts from the right on $GL(M)$ by

$$(v_1, \dots, v_n) \cdot A := \left(\sum_i v_i A_{i1}, \dots, \sum_i v_i A_{in} \right),$$

$A = (A_{ij})$. This action is fibrewise. $GL(n;\mathbb{R})$-equivariant trivializations are obtained as follows: for each coordinate chart $\varphi: U \xrightarrow{\sim} W \subset \mathbb{R}^n$ on M, $V_x := ((d\varphi)^{-1}_{\varphi(x)}(e_1), \dots, (d\varphi)^{-1}_{\varphi(x)}(e_n))$, where

$(e_1, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$, is a basis of $T_x M$. Then

$$\psi_U : \mathcal{U} \to \bigcup_{x \in \mathcal{U}} T(M)_x$$

$$x \mapsto v_x$$ is a local section over $\mathcal{U}$. Define a G-equivariant local trivialisation φ_U over $\mathcal{U}$ by $\varphi_U(v_x \cdot g) = (x, g)$. There is a unique smooth structure on $GL(M)$, so that each map φ_U is a diffeomorphism.

(ii), (iii) are obtained analogously $\square$

<u>Definition 2.5</u> A <u>morphism</u> between two principal bundles $(P, \pi, M; G)$ and $(P', \pi', M'; G)$ is a pair (H, h) where $H : P \to P'$ and $h : M \to M'$, H is G-equivariant (i.e. $G(\cdot\, g) = G(\cdot) \cdot g$ for all $g \in G$ and so that the diagram

$$\begin{array}{ccc} P & \xrightarrow{H} & P' \\ \pi \downarrow & & \downarrow \pi' \\ M & & M' \end{array}$$

commutes. (G, g) is called an <u>isomorphism</u> if G and g are diffeomorphisms.

In the case $M = M'$, $(P, \pi, M; G)$ and $(P', \pi', M; G)$ are isomorphic if there exists an isomorphism of the form $(\cdot, \mathrm{id}_M)$ between them.

<u>Proposition</u> 2.6. A principal bundle $(P, \pi, M; G)$ is isomorphic to the trivial bundle $(M \times G, pr_1, M; G)$ iff it admits a global section, i.e. if there exists a smooth map $s: M \to P$ with $\pi \circ s = \mathrm{id}_M$.

<u>Proof</u>: If $s: M \to P$ is a global section, then
$$H: M \times G \to P, \quad (x, g) \mapsto s(x) g$$
defines a bundle isomorphism with the trivial bundle. Conversely, if $H: M \times G \to P$ is a bundle isomorphism, then $s(x) := H(x, e)$ is a global section. $\square$

2.3 Associated Vector Bundles

We saw above how to associate to the tangent bundle of a smooth manifold principal fibre bundles, namely the frame bundles.

Conversely, it is possible to construct vector bundles from principal bundles. Let $(P, \pi, M; G)$ be a principal bundle, V a (real or complex) vector space and ρ a representation of G, i.e. a smooth group homomorphism $\rho : G \to GL(V) = \{L : V \to V$ V is a vector space isomorphism$\}$. Define a right G-action on $P \times V$ by $(p, v) \cdot g := (p \cdot g, \rho(g^{-1}) \cdot v)$. Denote by

$$E := (P \times V)/G =: P \times_G V$$

the space of orbits and by $\hat{\pi} : E \to M$ the map with $\hat{\pi}((p, v) \cdot G) = \pi(p)$.

<u>Proposition 2.7</u> $\xi = (E, \hat{\pi}, M) =: P \times_{G,\rho} V$ is a smooth vector bundle, called the vector bundle <u>associated</u> to the principle bundle $(P, \pi, M; G)$ by the representation ρ.

<u>Proof</u> We construct local trivializations for ξ as follows.

Given $x \in M$, let $\varphi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times G$,
$p \mapsto (\pi(p), \psi_{\mathcal{U}}(p))$
be a G-equivariant local trivialization of $(P, \pi, M; G)$ in a neighbourhood $\mathcal{U} \subset M$ of x. Define

$$\hat{\varphi}_{\mathcal{U}} : \hat{\pi}^{-1}(\mathcal{U}) \to \mathcal{U} \times V$$
$$[p, v] \mapsto (\pi(p), \rho(\psi_{\mathcal{U}}(p)) \cdot v).$$

Since $\varphi_{\mathcal{U}}$ is G-equivariant, so is $\psi_{\mathcal{U}}$ and thus

$$\hat{\varphi}_{\mathcal{U}}([(p,v) \cdot g]) = \hat{\varphi}_{\mathcal{U}}([p \cdot g, \rho(g^{-1}) v])$$
$$= (\pi(p \cdot g), \rho(\psi_{\mathcal{U}}(p \cdot g)) \rho(g^{-1}) \cdot v) =$$
$$(\pi(p), \rho(\psi_{\mathcal{U}}(p) \cdot g \cdot g^{-1}) \cdot v) = (\pi(p), \rho(\psi_{\mathcal{U}}(p)) \cdot v).$$

Thus the map $\hat{\varphi}_{\mathcal{U}}$ is well-defined. The transition maps $\hat{\varphi}_{W} \circ \hat{\varphi}_{\mathcal{U}}^{-1}$ have the form

$$\hat{\varphi}_{W} \circ \hat{\varphi}_{\mathcal{U}}^{-1} : (\mathcal{U} \cap W) \times V \to (\mathcal{U} \cap W) \times V$$
$$(x, v) \mapsto (x, \rho(\psi_{W}(p) \cdot \psi_{\mathcal{U}}(p)^{-1}) \cdot v),$$

where $p \in \pi^{-1}(x)$. Thus the transition maps are smooth and act by vector space isomorphisms on the second factor. Hence there is a unique structure of a vector space on every fibre

$\hat{\pi}^{-1}(x)$, $x \in M$, so that ξ is a smooth vector bundle with $\hat{\varphi}_u$ as above as local trivializations.

$\square$

In Proposition 2.3, we saw how to describe a principle bundle by the transition functions between local trivializations. Analogously, we can describe the vector bundle $(E, \hat{\pi}, M) = \xi$ associated to a principal bundle $(P, \pi, M; G)$ via a representation ρ using transition functions.

Let $g_{ih} : U_i \cap U_h \to G$ be transition functions for local trivializations $\Psi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times G$ over the sets $(U_i)_{i \in I}$ of a cover of M (see Proposition 2.3). The local trivializations $\hat{\varphi}_i : \hat{\pi}^{-1}(U_i) \xrightarrow{\sim} U_i \times V$ of ξ defined in the proof of Proposition 2.7 then satisfy

$$(\hat{\varphi}_i \circ \hat{\varphi}_h^{-1})(x, v) = (x, \rho(\Psi_i(p) \cdot \Psi_h^{-1}(p)) v),$$

where $p \in P_x$. Fixing $p_0 \in P_x$ with $\Psi_h(p_0) = e$, we obtain

$$(\hat{\varphi}_i \circ \hat{\varphi}_h^{-1})(x, v) = (x, \rho(\Psi_i(p_0)) \cdot v)$$
$$= (x, \rho((\Psi_i \circ \Psi_h^{-1})(p_0)) \cdot v)$$

On the other hand, with $\pi(p) =: x$,

$$\varphi_i \circ \varphi_k^{-1}(p) = (\varphi_{ix} \circ \varphi_{kx}^{-1})(p) = g_{ik}(x) \cdot p; \text{ so}$$

$$\hat{\varphi}_i \circ \hat{\varphi}_k^{-1}(x, v) = (x, \rho(g_{ik}(x) v)).$$

<u>Proposition 2.8</u> Let $(U_i)_{i \in I}$ be an open cover of a manifold M, G a Lie group, $(g_{ik} : U_i \cap U_k \to G)_{ik \in I}$ a family of smooth maps with $g_{ii} \equiv e$ and $g_{ij} g_{jk} = g_{ik}$ for all $i, j, k \in I$ and $\rho : G \to \mathrm{Aut}(V)$ a representation of G. There exists a unique (up to isomorphism) vector bundle ξ over M with rank equal to $\dim V$, so that ξ admits trivializations over the sets U_i with transition functions given by $\rho(g_{ik})$. ξ has the form $\xi = P \times_{G,\rho} V$ for a G-principal bundle P.

<u>Proof</u>: Given a family $(g_{ik} : U_i \cap U_k \to G)_{ik \in I}$ as in the assumptions, construct a vector bundle ξ as follows:

$$E := \bigcup_{i \in I} U_i \times V / \sim,$$

where $(x_i, v_i) \sim (x_k, v_k)$ for $(x_i, v_i) \in U_i \times V$, $(x_k, v_k) \in U_k \times V$ if and only if $x_i = x_k \in U_i \cap U_k$ and $v_k = \rho(g_{ki}(x)) v_i$.

We set $\hat{\pi} : E \to M$,

$$[x, v] \mapsto x.$$

With $$\hat{\varphi}_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times V$$
$$[x_i, v] \mapsto (x_i, v),$$
$$(\hat{\varphi}_i \circ \hat{\varphi}_k^{-1})(x, v) = (x, \rho(g_{ik})v)$$
and hence $\hat{\varphi}_i \circ \hat{\varphi}_k^{-1}$ is smooth for all i, k. Thus there exists a unique structure of a manifold on E, so that $(E, \hat{\pi}, M)$ is a smooth vector bundle with local trivializations $\hat{\varphi}_i$.

With $P := \dot{\bigcup\limits_{i \in I}} \mathcal{U}_i \times G / \sim$, where $(x_i, h_i) \sim (x_k, h_k)$ iff $x_i = x_k$ and $h_k = g_{ki} \cdot h_i$, we obtain using Proposition 2.3 a principal bundle $(P, \pi, M; G)$. Then an isomorphism between $P \times_{G,\rho} V$ and $(E, \hat{\pi}, M)$ is given by
$$[[x_i, g], v] \mapsto [x_i, g \cdot v].$$
To show that ξ is unique up to isomorphism, suppose that ξ' is a second vector bundle satisfying the assumptions. Define $H : E \to E'$ by requiring that $H|_{\mathcal{U}_i} = \hat{\varphi}_i'^{-1} \circ \hat{\varphi}_i$ for all $i \in I$. The assumptions on the transition maps imply that H is well-defined and a diffeomorphism. $\square$

Examples: 1) Recall that for a smooth manifold M, $GL(M)$ is the frame bundle of M. This is a $GL(n;\mathbb{R})$-principal bundle, $n = \dim M$. Denoting by $\rho: GL(n,\mathbb{R}) \to GL(\mathbb{R}^n)$ the representation of $GL(n,\mathbb{R})$ by $n\times n$-matrices, we have

$$GL(M) \times_{GL(n,\mathbb{R}),\rho} \mathbb{R}^n \cong TM.$$

An explicit isomorphism is given by

$$[(s_1, \dots, s_n), (x_1, \dots, x_n)] \mapsto \sum_{i=1}^{n} x_i s_i.$$

2) Let $\xi = (H, \pi, \mathbb{C}P^n)$ be the canonical line bundle over $\mathbb{C}P^n$, i.e. $H = \{(\mathbb{R}z, v) \mid z \in \mathbb{C}^{n+1}\setminus\{0\}, v \in \mathbb{C}z\}$, $\pi(\mathbb{R}z, v) = \mathbb{R}z \in \mathbb{C}P^n$ and $(S^{2n+1}, \pi, \mathbb{C}P^n; S^1)$ be the Hopf bundle over $\mathbb{C}P^n$. Define for $k \in \mathbb{Z}$ $\rho_k : S^1 \to GL(\mathbb{C})$ by $\rho_k(z)\cdot w := z^k w$. Then

$$\xi \cong S^{2n+1} \times_{S^1,\rho} \mathbb{C},$$

$$\xi^* \cong S^{2n+1} \times_{S^1,\rho_{-1}} \mathbb{C} \text{ and}$$

$$\otimes^k \xi \cong S^{2n+1} \times_{S^1,\rho_k} \mathbb{C}.$$

There is a useful description for sections of associated vector bundles. Given a G-principal bundle P and a representation $\rho: G \to GL(V)$ of G, denote by $C^\infty(P, V)^G$ the space of all G-equivariant smooth maps $P \to V$, i.e.

$$C^\infty(P,V)^G := \{ \bar{s} \in C^\infty(P,V) \mid \bar{s}(p \cdot g) = \rho(g^{-1}) \cdot \bar{s}(p) \text{ for all } p \in P,\ g \in G \}.$$

<u>Proposition 2.9</u> Let $\xi = (P \times_{G,\rho} V, \hat{\pi}, M)$ be the vector bundle associated to P via ρ. There is a bijective correspondence between $C^\infty(P,V)^G$ and smooth sections of ξ.

<u>Proof</u>: Given $\bar{s} \in C^\infty(P,V)^G$, define $s: M \to E = P \times_{G,\rho} V$ by

$$s(x) := [p, \bar{s}(p)] \in E_x,$$ where $p \in P_x$. Since

$$[pg, \bar{s}(pg)] = [pg, \rho(g^{-1})\bar{s}(p)] = [p, \bar{s}(p)],$$

s is a well-defined section of ξ. Conversely, given a section s of ξ, define $\bar{s} : P \to V$ by $[(p, \bar{s}(p))] := s(x)$ for $p \in P_x$.
$\bar{s}$ is well-defined since $V \to E_x$, $v \mapsto [p, v]$ is a bijection. Moreover, from

$$[(p, \bar{s}(p))] = [(pg, \rho(g^{-1})\bar{s}(p))]$$

we conclude

$$\bar{s}(pg) = \rho(g^{-1})\bar{s}(p).$$

$\square$

2.4 Extension and Reduction of Principal Bundles

There are constructions that allow to change the underlying Lie group of a principal bundle.

Definition 2.10 Let $\xi = (P, \pi, M; G)$ be a principal bundle and $\lambda : H \to G$ a homomorphism of Lie groups (i.e. λ is a smooth map that is a group homomorphism).
A λ-reduction of ξ is

a pair (η, f) consisting of an H-principal bundle $\eta = (Q, \tilde{\pi}, M; H)$ and a smooth map $f : Q \to P$, so that the following hold:

- $\pi \circ f = \tilde{\pi}$
- $f(q \cdot h) = f(q) \cdot \lambda(h)$ for all $q \in Q$, $h \in H$.

There is also a notion of isomorphism for reductions:

Definition 2.11 Two λ-reductions (η, f) and $(\tilde{\eta}, \check{f})$ are called isomorphic if there exists an isomorphism $\Phi : Q \xrightarrow{\sim} \tilde{Q}$ of H-principal bundles (over M) with $\Phi \circ \tilde{f} = f$.

In the special case when $\lambda = i : H \hookrightarrow G$ is the inclusion of a Lie subgroup, a λ-reduction is also referred to as a reduction from G to H.

Example: Recall that for a smooth manifold M, the frame bundle $GL(M)$ is a $GL(n;\mathbb{R})$-principal bundle, where $n = \dim M$. Every Riemannian metric g on M defines the reduction $O(M,g)$ of $GL(M)$ to $O(n) \subset GL(n;\mathbb{R})$.

Conversely, suppose that (η, f), where $\eta = (Q, \tilde{\pi}, M; O(n))$ and $f : Q \to GL(M)$ as in Definition 2.10, is a reduction of $GL(M)$ to $O(n)$. For $x \in M$, $q \in Q_x$, $f(q) = (v_1, \dots, v_n)$ is a basis of $T_x M$. There is a unique Riemannian metric g on M s.t.

$g(v_i, v_j) = \delta_{ij}$ for $i,j = 1, \dots, n$.

Since $f(q \cdot A) = f(q) \cdot A$ for every $A \in O(n)$, g is well-defined (independent of the choice of $q \in Q_x$ for each $x \in M$).

Then $\eta \cong O(M, g)$.

<u>Proposition 2.12</u> Let $\xi = (P, \pi, M; G)$ be a principal bundle and $\lambda : H \to G$ a Lie group homomorphism. There exists a λ-reduction of ξ

if and only if the following condition is satisfied:
there exist trivializations $\varphi_i : \pi^{-1}(\mathcal{U}_i) \xrightarrow{\sim} \mathcal{U}_i \times G$ of η over the sets $(\mathcal{U}_i)_{i\in I}$ in a cover of M and smooth maps $h_{ik} : \mathcal{U}_i \cap \mathcal{U}_k \to H$, $i,k \in I$, so that

$$(\varphi_i \circ \varphi_k^{-1})(x,e) = (x, \lambda(h_{ik}(x)))$$

for all $i,k \in I$, $x \in \mathcal{U}_i \cap \mathcal{U}_k$.

<u>Proof</u>: Suppose first that (η, f), where $\eta = (Q, \tilde{\pi}, M; H)$ and $f : Q \to P$, is a λ-reduction of ξ. Let $\psi_i : \tilde{\pi}^{-1}(\mathcal{U}_i) \xrightarrow{\sim} \mathcal{U}_i \times H$ be trivializations of η over the sets in a cover $(\mathcal{U}_i)_{i\in I}$ of M. Using Proposition 2.3, there are smooth functions $h_{ik} : \mathcal{U}_i \cap \mathcal{U}_k \to H$, $i,k \in I$, so that

$$(\psi_i \circ \psi_k^{-1})(x, h) = (x, h_{ik}(x) \cdot h).$$

Explicitly, $h_{ik}(x)$ is determined by the condition $(\psi_i \circ \psi_k^{-1})(x,e) = (x, h_{ik}(x))$.
Define $\varphi_i : \pi^{-1}(\mathcal{U}_i) \longrightarrow \mathcal{U}_i \times G$ as follows: given $x \in \mathcal{U}_i$,

fix $q \in Q_x$ and put

$$\varphi_i(f(q)\cdot g) := (x, \lambda(\psi_{ix}(q))\cdot g),$$

where $\psi_{ix}(q) \in H$ is determined by $\psi_i(q) = (x, \psi_{ix}(q))$. Then φ_i is a well-defined G-equivariant diffeomorphism $\pi^{-1}(\mathcal{U}_i) \xrightarrow{\sim} \mathcal{U}_i \times G$ and we have

$$\varphi_i \circ \varphi_k^{-1}(x,e) = (x, \lambda(\psi_{ix}(\psi_{kx}^{-1}(e))))$$
$$= (x, \lambda(h_{ik}(x))).$$

Conversely, suppose that $(\varphi_i)_{i\in I}$ are local trivializations of $\mathcal{E}$ with $\varphi_{ix}\circ\varphi_{kx}^{-1}(e) = \lambda(h_{ik}(x))$, where $h_{ik}: \mathcal{U}_i \cap \mathcal{U}_k \to H$ are smooth. Using Proposition 2.3, there is a principal bundle $(Q, \tilde{\pi}, M; H)$ which admits local trivializations $\psi_i : \tilde{\pi}^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times H$ so that

$$\psi_{ix}\circ\psi_{kx}^{-1}(e) = h_{ik}(x) \quad \text{for all}$$

$i,k \in I$, $x \in \mathcal{U}_i \cap \mathcal{U}_k$. Define $f_i: \tilde{\pi}^{-1}(\mathcal{U}_i) \to \pi^{-1}(\mathcal{U}_i)$ by requiring commutativity of the diagram

$$\begin{array}{ccc} \tilde{\pi}^{-1}(U_i) & \xrightarrow{f_i} & \pi^{-1}(U_i) \\ \downarrow \psi_i & & \downarrow \varphi_i \\ U_i \times H & \xrightarrow{id \times \lambda} & U_i \times G \end{array}$$

(i.e. $f_i = \varphi_i^{-1} \circ (id_{U_i} \times \lambda) \circ \psi_i$).

Then by construction of $(Q, \tilde{\pi}, M; H)$, $f_i = f_k$ on $\tilde{\pi}^{-1}(U_i \cap U_k)$ for all $i, k \in I$.

Then (η, f), where $\eta = (Q, \tilde{\pi}, M; H)$ is a reduction of ξ.

$\square$

Finally, we show that reductions are compatible with passing to associated vector bundles:

<u>Proposition 2.13</u> Let $\lambda: H \to G$ be a Lie group homomorphism, $\xi = (P, \pi, M; G)$ a principal bundle and (η, f) a λ-reduction of ξ, where $\eta = (Q, \tilde{\pi}, M; H)$. Then for every representation

$\rho : G \to GL(V)$ of G, the associated vector bundles

$$P \times_{(G,\rho)} V \quad \text{and} \quad Q \times_{(H, \rho\circ\lambda)} V$$

are isomorphic.

<u>Proof</u>: Define
$$\Psi : Q \times_{(H,\rho\circ\lambda)} V \to P \times_{(G,\rho)} V$$
$$[q, v] \mapsto [f(q), v].$$

Since
$$\Psi([qh, (\rho\circ\lambda)(h^{-1})v]) = [f(q)\lambda(h), (\rho\circ\lambda)(h^{-1})v]$$
$$= [f(q), v],$$

Ψ is well-defined. Moreover, Ψ is a fibrewise map and linear on the fibres. If $\Psi([q,v]) = \Psi([\tilde{q}, \tilde{v}])$, $q, \tilde{q} \in Q_x$, then there exists $h \in H$ with $\tilde{q} = q \cdot h$, thus $f(\tilde{q}) = f(q) \cdot \lambda(h)$ and
$$[f(q), v] = [f(q)\lambda(h), (\rho\circ\lambda)(h^{-1})v]$$
$$= [f(\tilde{q}), \tilde{v}] \quad \text{implies} \quad \tilde{v} = (\rho\circ\lambda)(h^{-1})v$$
$$\Rightarrow [q, v] = [qh, (\rho\circ\lambda)(h^{-1})v] = [\tilde{q}, \tilde{v}].$$

Thus Ψ is injective. For surjectivity, let $[p, v] \in P \times_{(G,\rho)} V$,

$p \in P_x$. Fix $q \in Q_x$ arbitrary.
There exists $g \in G$ with $f(q) = p \cdot g$.
Then

$$\bar{\Psi}\left(\left[q, \rho(g^{-1})v\right]\right) = \left[f(q), \rho(g^{-1})v\right]$$

$= [p, v]$ Thus $\bar{\Psi}$ is also surjective.

$\square$

2.5 Connections on Principal Bundles

For a principal bundle $(P, \pi, M; G)$ and $p \in P$,

$$Tv_p P := T_p(P_{\pi(p)}) \subset T_p P$$

is called the vertical tangent space to p at P.
Recall that for a Lie group G, the exponential map $\exp: \mathfrak{g} \to G$ is given by $\exp(X) = \varphi_X(1)$, where $\varphi_X: \mathbb{R} \to G$ is the maximal integral curve of X (viewed as a left-invariant vector field on G) with $\varphi_X(0) = e$. Given a right action of G

on a manifold M,

$$\tilde{X}(x) := \frac{d}{dt}\Big|_{t=0} (x \cdot \exp(tX)) \in T_x M, x \in M$$

is called the <u>fundamental vector field</u> generated by $X \in \mathfrak{g}$.

<u>Lemma 2.14</u> Let $(P, \pi, M; G)$ be a principal bundle and $p \in P$. Then

1. $Tv_p P = \ker d\pi_p$.
2. The map $X \mapsto \tilde{X}(p)$ defines an isomorphism $\mathfrak{g} \xrightarrow{\sim} Tv_p P$.

<u>Proof</u>: Let $\varphi_U : \pi^{-1}(U) \xrightarrow{\sim} U \times G$ be a trivialization of $(P, \pi, M; G)$ in a neighbourhood $U \subset M$ of $x = \pi(p)$. Let $(x, g) = \varphi_U(p)$. Since $\pi = pr_1 \circ \varphi_U$, a vector $v \in T_p P$ lies in the kernel of $d\pi_p$ if and only if $d\varphi_U(v) = (0, w)$ for some $w \in T_g G$. This in turn is the case if and only if $d\varphi_p(v) = \dot{\gamma}(0)$ for some smooth curve $\gamma : (-\varepsilon, \varepsilon) \to U \times G$ of the form $\gamma(t) = (x, g(t))$. But this is equivalent to $v = \frac{d}{dt}\big|_{t=0} (\varphi_U^{-1}(\gamma(t)))$,

where $t \mapsto \varphi_x^{-1}(\gamma(t))$ is a curve in P_x, i.e. with the condition $v \in T_p P_x = Tv_p P$.
To show 2., note first that the map
$$\begin{aligned} \mathfrak{g} &\to Tv_p P \\ X &\mapsto \tilde{X}(p) \end{aligned}$$
is linear. Comparing dimensions, it suffices to check that the map is injective. Suppose that $\tilde{X}(p) = 0$. Using linearity of the map $X \to \tilde{X}(p)$, it follows that $p = p \cdot \exp(tX)$ for all $t \in \mathbb{R}$. Since the G-action on M is free, we conclude $\exp(tX) = e$ for all $t \in \mathbb{R}$ and hence $X = 0$.

$\square$

We say that a subspace $Th_p P \subset T_p P$ is a <u>horizontal tangent space</u> if $Th_p P$ is complementary to the vertical tangent space $Tv_p P$, i.e. if $T_p P = Tv_p P \oplus Th_p P$. While the vertical tangent spaces are naturally determined, horizontal subspaces are not. A connection on $(P, \pi, M; G)$ can be defined as a choice of horizontal tangent spaces at all points $p \in P$, which is compatible with the G-action.

<u>Definition 2.15</u> A <u>connection</u> on a principal bundle $(P, \pi, M; G)$ is an assignment

$$Th : P \ni p \longmapsto Th_p P \subset T_p P,$$

so that the following conditions are satisfied:

1. $$T_p P = Tv_p P \oplus Th_p P$$
   for all $p \in P$

2. Denoting for $g \in G$ $\quad R_g : \begin{array}{l} P \to P \\ p \mapsto p \cdot g, \end{array}$

   we have
   $$dR_g \, (Th_p P) = Th_{p \cdot g} P$$
   for all $g \in G$, $p \in P$.

3. For every $p \in P$ there exists a neighbourhood $W \subset P$ of p and smooth vector fields $X_1, \dots, X_m$ on W, so that
   $$Th_q P = \operatorname{span} (X_1(q), \dots, X_m(q))$$

for all $g \in W$.
One can refer to the above conditions 1, 2 and 3 as "complementarity", "right invariance" and "smoothness" respectively.
Lemma 2.14 and the complementarity condition imply that $d\pi_p$ defines an isomorphism
$Th_p P \xrightarrow{\sim} T_{\pi(p)} M$ for all $p \in P$.
We can characterize connections on $(P, \pi, M; G)$ using suitable one-forms on the total space P. To this end, recall that for a Lie group G the <u>adjoint representation</u> of G is defined as follows: for $g \in G$, denote by α_g the map $G \to G$, $h \mapsto ghg^{-1}$.
Then identifying $\mathfrak{g}$ with the space of left-invariant vector fields on G, we define $Ad(g)X := (\alpha_g)_* X$.

Definition 2.16 A *connection one-form* on a principal bundle $(P, \pi, M; G)$ is an element $A \in \Omega^1(P; \mathfrak{g})$, so that the following conditions are satisfied:

1. $A(\tilde{X}) = X$ for all $X \in \mathfrak{g}$
2. $R_g^* A = Ad(g^{-1}) \circ A$ for all $g \in G$.

Proposition 2.17 The following maps establish a bijection between the set of all connections and the set of all connection one-forms on a principal bundle $(P, \pi, M; G)$:

1. Given a connection $Th : P \ni p \mapsto Th_p P$ on $(P, \pi, M; G)$,
$$A_p(\tilde{X}(p) \oplus Y_h) := X$$
for all $X \in \mathfrak{g}$, $Y_h \in Th_p P$, $p \in P$ is a connection one-form.
2. Given a connection one-form A on $(P, \pi, M; G)$,
$$Th : P \ni p \mapsto Th_p P := \ker A_p$$
is a connection.

Proof: 1. We use the following relation between fundamental vector fields and the adjoint representation (proof: exercise)

$$dR_g(\tilde{X}(p)) = \widetilde{(Ad(g^{-1})X)}(p\cdot g).$$

Using the right invariance of the connection, $Y_h \in T_h p$ implies $dR_g(Y_h) \in T_h{}_{p\cdot g}$. We conclude

$$(R_g^* A)_p(\tilde{X}(p) + Y_h) = A_{p\cdot g}(dR_g(\tilde{X}(p)) + dR_g Y_h) = A_{p\cdot g}(\widetilde{Ad(g^{-1})X}(p\cdot g) + dR_g Y_h)$$
$$= Ad(g^{-1})X = Ad(g^{-1}) \circ A_p(\tilde{X}(p) + Y_h).$$

It follows that $R_g^* A = Ad(g^{-1}) \circ A$.

2. We show:

- Complementarity

To check that $\operatorname{Ker} A_p \cap T_{v}{}_p P = \{0\}$ for all $p \in P$, let $Y \in \operatorname{Ker} A_p \cap T_{v}{}_p P$. We have $Y = \tilde{X}(p)$ for some $X \in \mathfrak{g}$, then $0 = A(Y) = X$, thus $Y = 0$. Moreover, since A_p is surjective,

$$\dim \operatorname{Ker} A_p = \dim T_p P - \dim \mathfrak{g} = \dim T_p P - \dim T_{v}{}_p P$$ and we

Conclude $\quad T_pP = \ker A_p \oplus Tv_p P.$

- Right invariance

Let $Y \in T_pP$ be a vector with $A_p(Y) = 0$. Then

$$A_{p\cdot g}(dR_g Y) = (R_g^* A)_p (Y) =$$

$$Ad(g^{-1})(A_p(Y)) = 0.$$

- Smoothness

Let $(x_1, \dots, x_n)$ be local coordinates on $\mathcal{U} \subset P$, $p \in P$ and $(a_1, \dots, a_\ell)$ a basis of $\mathfrak{g}$. Let $Y = \sum_i y_i \frac{\partial}{\partial x_i}(p) \in T_pP$. Since A is smooth,

$$A\left(\frac{\partial}{\partial x_i}\right) = \sum_j A_{ij} a_j, \text{ where}$$

A_{ij} are smooth functions on $\mathcal{U}$. Y lies in the kernel of A if and only if

$$(*) \qquad \sum_i y_i A_{ij} = 0 \quad \text{for } j = 1, \dots, \ell.$$

Since the solutions of $(*)$ are smooth on a neighbourhood of p, it follows that $\ker A$ is locally spanned by smooth vector fields. $\square$

A third way to characterize connections on principal bundles is via locally defined one-forms on the base. Given a connection form $A \in \Omega^1(P; \mathfrak{g})$ and $s : \mathcal{U} \to P$ a local section over $\mathcal{U} \subset M$,

$$A^s := A \circ ds \in \Omega^1(\mathcal{U}; \mathfrak{g})$$

is called the <u>local connection form</u> determined by s.

For any two local sections $s_i : \mathcal{U}_i \to P$, $s_j : \mathcal{U}_j \to P$, we have $s_i(x) = s_j(x) \cdot g_{ij}(x)$ for $x \in \mathcal{U}_i \cap \mathcal{U}_j$, where $g_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \to G$ are smooth functions. Denoting by $\theta_G \in \Omega^1(G; \mathfrak{g})$ the form with

$$\theta_G(X_g) = dL_{g^{-1}}(X_g) \in T_eG \cong \mathfrak{g}$$

for all $X \in \Gamma(TG)$ (the so-called canonical one-form of G), we put

$$\theta_{ij}(X) := dL_{g_{ij}^{-1}(x)}(dg_{ij}(X)) = g_{ij}^* \theta_G$$

for $X \in T_x(\mathcal{U}_i \cap \mathcal{U}_j)$.

Proposition 2.18 Let $(P, \pi, M; G)$ be a principal bundle.

1. Let $A \in \Omega^1(P, \mathfrak{g})$ be a connection form. Then for all local sections $s_i : U_i \to P$, $s_j : U_j \to P$, the local connection forms corresponding to s_i and s_j are related by

$$A^{s_i} = Ad(g_{ij}^{-1}) \circ A^{s_j} + \theta_{ij}$$

2. Conversely, suppose that $(s_i : U_i \to P)_{i \in I}$ are local sections, where $(U_i)_{i \in I}$ is a cover of M, and $(A_i \in \Omega^1(U_i; \mathfrak{g}))_{i \in I}$ a family of one-form so that

$$A_i = Ad(g_{ij}^{-1}) \circ A_j + \theta_{ij}$$

for all $i, j \in I$. Then there exists a unique connection form A on $(P, \pi, M; G)$, so that

$$A^{s_i} = A_i \quad \text{for all } i \in I.$$

Examples: 1. If $G \in GL(k,\mathbb{R})$, then identifying $GL(k,\mathbb{R})$ with the space of non-degenerate $k \times k$-matrices,

$$dl_g X = g \cdot X \quad \text{and}$$
$$Ad(g)X = g \cdot X \cdot g^{-1}$$

for $g \in G$, $X \in \mathfrak{g}$. Thus the transformation formula from 2. in the above Proposition becomes

$$A_i = g_{ij}^{-1} \circ A_j \circ g_{ij} + g_{ij}^{-1} dg_{ij}.$$

2. If $(P, \pi, M; G) = (M \times G, pr_1, M; G)$ is the trivial G-principal bundle over M, then there is a section of the bundle over $U = M$ and hence by Proposition 2.18, a connection is uniquely determined by a one-form $A \in \Omega^1(M; \mathfrak{g})$.

Proof of Proposition 2.18

We will use the following statement whose proof is left as an exercise: if G is a Lie group acting from the right on a smooth manifold M, $x_0 \in M$ and $g_0 \in G$, then for all smooth curves $x : (-\varepsilon, \varepsilon) \to M$, $g : (-\varepsilon, \varepsilon) \to G$, $x(0) = x_0$, $g(0) = g_0$,

$$\frac{d}{dt}\Big|_{t=0} (x(t) \cdot g(t)) = d\widetilde{R_{g_0}} (x'(0)) + (dL_{g_0^{-1}}(g'(0)))\,\widetilde{}\,(x_0 \cdot g_0).$$

1. Let $x \in U_i \cap U_j$ and let $\gamma : (-\varepsilon, \varepsilon) \to U_i \cap U_j$ be a smooth curve with $\gamma(0) = x$ and $\gamma'(0) = X \in T_x M$. Then

$$ds_i(X) = \frac{d}{dt}\Big|_{t=0} s_i(\gamma(t))$$

$$= \frac{d}{dt}\Big|_{t=0} (s_j(\gamma(t)) \cdot g_{ij}(\gamma(t)))$$

$$= dR_{g_{ij}(x)} (ds_j(X)) + \widetilde{\theta_{ij}(X)}(s_j(x))$$

and thus

$$A^{s_i}(X) = A(ds_i(X)) = A(dR_{g_{ij}(x)} \cdot ds_j(X)) + \theta_{ij}(X) = \mathrm{Ad}(g_{ij}(x)^{-1}) A^{s_j}(X) + \theta_{ij}(X).$$

2. Let $s_i : U_i \to P$ be local sections and $A_i \in \Omega^1(U_i; \mathfrak{g})$

one-forms satisfying the above condition.
For $x \in \mathcal{U}_i$, $p := s_i(x) \in P$

$$T_pP = T_{V_p}P \oplus ds_i(T_x\mathcal{U}_i)$$

Define $A_p : T_pP \to \mathfrak{g}$ by

$$A_p(\tilde{Y}(p) \oplus ds_i(X)) := A_i(X) + Y$$

for $Y \in \mathfrak{g}$, $X \in T_x\mathcal{U}_i$.
Moreover, for $g \in G$ put

$$A_{p\cdot g} := Ad(g^{-1}) \cdot A_p(dR_{g^{-1}}(\cdot))$$

Then for all $Y \in \mathfrak{g}$,

$$A_{p\cdot g}(\tilde{Y}(p\cdot g)) = Ad(g^{-1}) A_p(dR_{g^{-1}}(\tilde{Y}(p\cdot g)))$$
$$= Ad(g^{-1}) A_p(Ad(g) Y(p)) =$$
$$Ad(g^{-1}) Ad(g) Y = Y.$$

Moreover, for $h \in G$

$$(R_h^* A)_{pg}(V) = A_{pgh}(dR_h(V))$$
$$= Ad(h^{-1}) Ad(g^{-1}) A_p(dR_{hg^{-1}}^{-1}(dR_h(V)))$$
$$= Ad(h^{-1}) A_p(V).$$

It follows that A is a connection

form on $(\pi^{-1}(U_i), \pi|_{\pi^{-1}(U_i)}, U_i; G)$.

Next, we show that for all i, j, the one-forms A and $\tilde{A}$ obtained from (A_i, s_i) and (A_j, s_j) coincide on $\pi^{-1}(U_i \cap U_j)$.

A and $\tilde{A}$ coincide by construction on the vertical tangent spaces and are uniquely determined by the values $(A_{s_i(x)})_{x \in U_i \cap U_j}$ and $(\tilde{A}_{s_j(x)})_{x \in U_i \cap U_j}$. Thus it suffices to check that

$$\tilde{A}_p(ds_i(X)) = A_p(ds_i(X)) = A_i(X).$$

We have $s_i(x) = s_j(x) \cdot g_{ij}(x)$.

Given $X \in T_pP$, $p \in U_i \cap U_j$, let $\gamma : (-\varepsilon, \varepsilon) \to U_i \cap U_j$ be a smooth curve with $\gamma'(0) = X$.

Then

$$ds_i(X) = \frac{d}{dt}\Big|_{t=0} \left(s_j(\gamma(t)) \cdot g_{ij}(\gamma(t)) \right)$$

$$= dR_{g_{ij}(x)}(ds_j(X)) + \vartheta_{ij}(X)\left(s_j(x) \cdot g_{ij}(x)\right))$$

and thus using the assumptions on $(A_i)_{i\in I}$,

$$\tilde{A}(ds_i(x)) = \tilde{A}\left(dR_{g_{ij}(x)}(ds_j(x)) + \widetilde{\theta_{ij}(x)}(s_j(x)\cdot g_{ij}(x))\right) =$$

$$Ad(g_{ij}(x)^{-1})\,\tilde{A}(ds_j(x)) + \theta_{ij}(X)$$

$$= Ad(g_{ij}(x)^{-1})\,A_j(x) + \theta_{ij}(x) = A_i(x).$$

$\square$

Examples: 1. (The flat connection):
If $(P = M\times G,\ pr_1,\ M; G)$ is the trivial G-principal bundle over M, then $TV_{(x,g)}P \cong T_gG$ for all $x\in M$, $g\in G$ and we have for $X\in\mathfrak{g}$

$$\tilde{X}(x,g) = \frac{d}{dt}\Big|_{t=0}\left((x,g)\cdot \exp(tX)\right)$$

$$= \frac{d}{dt}\Big|_{t=0}(x, g\cdot\exp(tX)) = dl_g(X).$$

Define $Th_{(x,g)}P := T_{(x,g)}(M\times\{g\}) \cong T_xM$.

This is a connection whose corresponding connection form is given by

$$A : T_{(x,g)}(M\times G) \simeq T_xM \oplus T_gG \to \mathfrak{g}$$
$$X+Y \mapsto dL_{g^{-1}}Y.$$

Recall that a connection (or "covariant derivative") on a vector bundle (E,π,M) is bilinear map

$$D: \Gamma(TM)\times\Gamma(E) \to \Gamma(E)$$
$$(X,s)\mapsto D_X s,$$

so that for every smooth function $f: M\to\mathbb{R}$, $D_X(fs) = fD_Xs + X(f)\, s$ and $D_{(fX)}s = f\cdot D_X s$.

<u>Example 2</u> (Connections on frame bundles) For a smooth manifold M, there is a bijective correspondence between connections on the frame bundle $GL(M)$, and connections on the tangent bundle TM. The correspondence can be obtained as follows:

- Suppose that $A\in\Omega^1(GL(M), \mathfrak{gl}(n,\mathbb{R}))$ is a connection form on the frame bundle. Denote for $i,j\in\{1,..,n\}$ by $B_{ij}\in\mathbb{R}^{n\times n}$ the matrix whose entry in the i-th row and the j-th column is 1, and all other entries are zero. Then A can be written as

$$A = \sum_{i,j=1}^{n} \omega_{ij} B_{ij},$$

where $\omega_{ij} \in \Omega^1(GL(M); \mathbb{R})$. Given a local section $s = (s_1, .., s_n) : \mathcal{U} \to GL(M)$ of the frame bundle, define for $k = 1, .., n$

$$\overset{A}{\nabla}_X s_k := \sum_{i=1}^{n} \omega_{ik}\,(ds(X))\, s_i .$$

- Conversely, let ∇ be a connection on TM. Given a local section $s = (s_1, ..., s_n) : \mathcal{U} \to GL(M)$ of the frame bundle, we have

$$\nabla s_i = \sum_{i,j=1}^{n} \omega_{ij} \otimes s_j$$

with $\omega_{ij} \in \Omega^1(\mathcal{U}; \mathbb{R})$. Define a local connection form $A^s \in \Omega^1(\mathcal{U}, \mathfrak{gl}(n, \mathbb{R}))$ by

$$A^s := \sum_{i,j=1}^{n} \omega_{ij} B_{ij} .$$

One can show that the local connection forms satisfy the transformation rule of Proposition 2.18 and thus give rise to a connection form A^∇ on $GL(M)$ (see Problem Sheet 5 for the details).

3. Recall that if g is a Riemannian metric on M, then a connection D on TM is called metric w.r.t. g if

$$X g(Y,Z) = g(D_X Y, Z) + g(Y, D_X Z)$$

for all $X, Y, Z \in \Gamma(TM)$. Every metric connection on TM defines a connection form on the bundle $O(M,g)$ of g-orthonormal frames as follows: identifying the Lie algebra $\mathfrak{o}(n)$ of the orthogonal group $O(n)$ as the space of all antisymmetric $n \times n$-matrices and denoting for $i, j \in 1, \dots, n$

$$E_{ij} := B_{ij} - B_{ji},$$

we define for a local section $s = (s_1, \dots, s_n) : \mathcal{U} \to O(M,g)$ of $O(M,g)$ $A^s \in \Omega^1(\mathcal{U}; \mathfrak{o}(n))$ by

$$A^s(X) := \sum_{i,j=1}^{n} g(D_X s_i, s_j) E_{ij} \in \mathfrak{o}(n).$$

One can show that the conditions of Proposition 2.18 are satisfied and thus the local connection forms A^s give rise to a connection form

A^D on $O(M,g)$. Moreover $D \mapsto A^D$ is a bijection between the set of all connections on TM that are metric w.r.t. g, and the set of all connections on $O(M,g)$ (see problem sheet 5).

As another application of Proposition 2.18, one can show that every principal bundle admits a connection:

<u>Proposition 2.19</u> Every principal bundle $\xi = (P, \pi, M; G)$ admits a connection form $A \in \Omega^1(P; \mathfrak{g})$.

<u>Proof</u>: Let $(U_i)_{i \in I}$ be an open cover of M, so that ξ admits a G-equivariant trivialization $\varphi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times G$ over each U_i. Let $(f_i)_{i \in I}$ be a partition of unity subordinate to $(U_i)_{i \in I}$ (see Proposition 1.3). For every

$i \in I$, denote by $A_i \in \Omega^1(\mathcal{U}_i \times G; \mathfrak{g})$ the natural connection on the trivial principal bundle $\mathcal{U}_i \times G$ from example 1 above (i.e. $A_i(X+Y) = dL_g^{-1}(Y)$ for $X+Y \in T_{(x,g)}(\mathcal{U}_i \times G) = T_x\mathcal{U}_i \oplus T_g G$, $x \in \mathcal{U}_i$, $g \in G$). We define

$$A := \sum_{i \in I} (f_i \circ \pi)\, \varphi_i^* A_i .$$

Using G-equivariance of φ_i, we have $(\varphi_i^* A_i)(\tilde{X}) = X$ for every $X \in \mathfrak{g}$ and thus

$$A(\tilde{X}) = \Big(\sum_{i \in I} f_i \circ \pi\Big) \cdot X = X.$$

Moreover, for $g \in G$

$$\begin{aligned}
R_g^* A &= \sum_{i \in I} (f_i \circ \pi)\, R_g^*(\varphi_i^* A_i) \\
&= \sum_{i \in I} (f_i \circ \pi)\, Ad(g^{-1})(\varphi_i^* A_i) \\
&= Ad(g^{-1}) \sum_{i \in I} (f_i \circ \pi) \cdot (\varphi_i^* A_i) \\
&= Ad(g^{-1})\, A .
\end{aligned}$$

Thus A is a connection form on S.

$\square$

Finally, let us show that the space of all connections on a principal bundle is an affine vector space.

For a vector bundle (E, π, M) and $k \geq 0$, denote by

$$\Omega^k(M;E) = \Gamma(\Lambda^k M \otimes E)$$

the space of all k-forms on M with values in E, i.e. maps of the form

$$\omega : x \in M \mapsto \omega_x : T_x \times \ldots \times T_x M \to E_x,$$

where each ω_x is multilinear antisymmetric and depends smoothly on $x \in M$ in the sense that for every k-tuple of smooth vector fields $X_1, \ldots, X_k$ on M,

$s : x \in M \mapsto \omega_x(X_1, \ldots, X_k) \in E_x$ is a smooth section of E.

Consider now the special case where $E \times_{(G,\rho)} V$ is the vector bundle associated to the principal bundle $(P, \hat{\pi}, M; G)$ via the representation $\rho : G \to GL(V)$. We will see that in this case $\Omega^k(M;E)$ can be identified with a certain subset of $\Omega^k(P;V)$.

<u>Definition 2.20</u> Let $(P, \hat{\pi}, M; G)$ be a principal bundle and $\rho: G \to GL(V)$ a representation of G. A k-form $\omega \in \Omega^k(P, V)$ is called

- <u>horizontal</u>, if $\omega_p(X_1, .., X_k) = 0$ whenever $X_i \in T_{V,p}P$ for some $i = 1, .., k$

- <u>ρ-equivariant</u>, if $R_g^* \omega = \rho(g^{-1}) \circ \omega$ for all $g \in G$.

For example, suppose that $A, A' \in \Omega^1(P; \mathfrak{g})$ are two connection forms on the principal bundle. Then it follows from the definition of a connection form that $A - A' \in \Omega^1(P; \mathfrak{g})$ is horizontal and Ad-equivariant, where $Ad: G \to GL(\mathfrak{g})$ is the adjoint representation of G. Conversely if $\omega \in \Omega^1(P; \mathfrak{g})$ is horizontal and Ad-equivariant, then for every connection form $A \in \Omega^1(P; \mathfrak{g})$

$A' := A + \omega$ is also a connection form.

<u>Proposition 2.21</u> The vector space of all horizontal ρ-equivariant k-forms in $\Omega^k(P;V)$ is isomorphic to the vector space $\Omega^k(M;E)$ of all k-forms on M with values in $E = P \times_\rho V$.

<u>Proof:</u> For $p \in P_x$, $x \in M$, denote by $[p] : V \to E_x$ the vector space isomorphism

$$[p]\ V \ni v \mapsto [p, v] \in E_x.$$

Given a horizontal, ρ-equivariant k-form $\bar{\omega} \in \Omega^k(P;V)$ define $\omega \in \Omega^k(M;E)$ as follows: for $x \in M$, $t_1, \dots, t_k \in T_x M$,

$$\omega_x(t_1, \dots, t_k) := [p, \bar{\omega}_p(X_1, \dots, X_k)],$$

where $p \in P_x$ and $X_1, \dots, X_k \in T_p P$ are tangent vectors with $d\pi_p(X_i) = t_i$, $i = 1, \dots, k$. We claim that ω is well-defined, i.e. independent of the choice of $p \in P_x$ and $X_1, \dots, X_k \in T_p P$.

a) If $\tilde{X}_1, \dots, \tilde{X}_k \in T_pP$ also satisfy $d\hat{\pi}_p(\tilde{X}_j) = t_j$, $j = 1, \dots, k$, then

$$d\hat{\pi}_p(X_j - \tilde{X}_j) = 0$$ and thus

$$X_j - \tilde{X}_j \in T_{Vp}P$$ for all j.

It follows that $\bar{\omega}_p(\dots, X_j - \tilde{X}_j \dots) = 0$ and thus $\bar{\omega}_p(X_1, \dots, X_k) = \omega_p(\tilde{X}_1, \dots, \tilde{X}_k)$.

b) Suppose that $\tilde{p} = p \cdot g$, $g \in G$ is a second element of the fibre P_x and $Y_1, \dots, Y_k \in T_{\tilde{p}}P$ tangent vectors with $d\hat{\pi}_{\tilde{p}}(Y_j) = t_j$ for $j = 1, \dots, k$. Then

$$[\tilde{p}, \bar{\omega}_{\tilde{p}}(Y_1, \dots, Y_k)] = [p \cdot g, \bar{\omega}_{p \cdot g}(Y_1, \dots, Y_k)]$$

$$= [p, \rho(g)\bar{\omega}_{pg}(Y_1, \dots, Y_k)] =$$

$$= [p, (R^*_{g^{-1}}\bar{\omega})_{pg}(Y_1, \dots, Y_k)] =$$

$$= [p, \bar{\omega}_p(dR_{g^{-1}}Y_1, \dots, dR_{g^{-1}}Y_k]$$

$$= [p, \bar{\omega}_p(X_1, \dots, X_k)],$$

where in the last equality we used the fact that $\tilde{X}_j := dR_{g^{-1}}Y_j$ satisfy $d\hat{\pi}_p(\tilde{X}_j) = t_j$ and a).

c) To show that ω is smooth, note that if $s : \mathcal{U} \to P$, $\mathcal{U} \subset M$ is a (smooth) local section of P and $T_1, .., T_k$ smooth vector fields on $\mathcal{U}$, then

$$\omega(T_1, .., T_k) = [s, \bar{\omega}_{s(\cdot)}(ds(T_1), .., ds(T_k))]$$

d) The map $\bar{\omega} \mapsto \omega$ is bijective, with the inverse map given by

$$\bar{\omega}_p(X_1, \dots, X_k) := [p]^{-1} \omega_{\pi(p)}(d\pi_p(X_1), .., d\pi_p(X_k))$$

for $p \in P$, $X_1, .., X_k \in T_p P$.

$\square$

<u>Corollary 2.22</u> The set of all connections on a principal bundle carries the structure of an affine space over the vector space $\Omega^1(M; Ad(P))$ of all one-forms on M with values in

$$Ad(P) := P \times_{(G, Ad)} \mathfrak{g} .$$

As another application of Proposition 2.21, consider the case where $\rho : G \to GL(V)$ is the trivial representation (i.e. $\rho(g) = e$ for all $g \in G$). Then $\omega \in \Omega^1(P; V)$ is ρ-equivariant if and only if ω is G-invariant in the sense that $R_g^* \omega = \omega$ for all $g \in G$.

<u>Definition 2.23</u> Let $(P, \pi, M; G)$ be a principal bundle and V a vector space. $\omega \in \Omega^1(P; V)$ is called <u>basic</u> if ω is horizontal and G-invariant.

From Proposition 2.21 we obtain:

<u>Lemma 2.24</u> Let $(P, \pi, M; G)$ be a principal bundle and V a vector space. Then $\pi^* : \Omega^k(M; V) \to \Omega^k(P; V)$ induces an isomorphism between the space of all k-forms

on M with values in V, and the space of all horizontal G-invariant k-forms on P.

Proof: If $\rho: G \to GL(V)$ denotes the trivial representation, then $(x,v) \mapsto [x,v]$ defines an isomorphism between the trivial bundle $M \times V$ and $E = P \times_{G,\rho} V$. Composing with this isomorphism, the map $[p]: V \xrightarrow{\sim} E_{\pi(p)}$ from the proof of Proposition 2.21. is the identity map on V. Thus the map $\omega \mapsto \bar{\omega}$ from part d) of the proof of the Proposition is given by $\omega \mapsto \pi^* \omega$. $\square$

2.5 Curvature of a Connection on a Principal Bundle

In the previous section we saw that there is a correspondence between connections on the tangent bundle TM of a smooth manifold M, and connections on the principal bundle $GL(M)$ of frames in TM. In this section, we generalize this correspondence to the case of arbitrary vector bundles associated to principal bundles and introduce a notion of curvature of a connection on a principal bundle that corresponds to curvature of a connection on a vector bundle.

A connection on a vector bundle E can be interpreted as a map

$$\begin{aligned} D: \Gamma(E) = \Omega^0(M;E) &\to \Gamma(T^*M \otimes E) \\ &\cong \Omega^1(M;E), \\ Y &\mapsto (X \mapsto D_X Y). \end{aligned}$$

On the other hand, as we saw in the previous section, if $E = P \times_{(G,\rho)} V$ for a G-principal bundle P and a representation $\rho : G \to GL(V)$ of G, then $\Omega^k(M;E)$ is isomorphic to the space of all horizontal ρ-equivariant forms in $\Omega^k(P;V)$. On P, a natural map $\Omega^k(P;V) \to \Omega^{k+1}(P;V)$ is given by exterior the exterior derivative $\omega \mapsto d\omega$.
However, for a horizontal form ω the differential $d\omega$ is in general no longer horizontal.
As a simple example, take $M = \mathbb{R}$, $G = \mathbb{R}$ and the 'trivial' G-principal bundle $P = \mathbb{R} \times \mathbb{R}$ over $\mathbb{R}$. The one-form $\omega(t,s) := s \cdot dt$ (where $t \in M$, $s \in P_t$) is horizontal but $d\omega = ds \wedge dt$ is not.
It turns out that a connection form on P defines a deformation

D_A of $d: \Omega^k(P;V) \to \Omega^{k+1}(P;V)$ which sends horizontal forms to horizontal forms.

<u>Definition 2.25</u> Let A be a connection form on the principal bundle $(P,\pi,M;G)$. For $p \in P$, denote by $\mathrm{pr}_h : T_pP \to Th_pP$ the projection to the first summand in the direct sum decomposition $T_pP = Th_pP \oplus Tv_pP$ defined by A. We define for a vector space V

$$D_A : \Omega^k(P;V) \to \Omega^{k+1}(P;V)$$

$$(D_A\omega)_p(t_1, \dots, t_{k+1}) := (d\omega)(\mathrm{pr}_h t_1, \dots, \mathrm{pr}_h t_{k+1}).$$

<u>Proposition 2.26</u> Let $\rho: G \to GL(V)$ be a representation of G and $\omega \in \Omega^k(P;V)$ a ρ-equivariant horizontal k-form. Then

1. $D_A\omega \in \Omega^{k+1}(P;V)$ is again horizontal ρ-equivariant and
2. We have

$$D_A\omega = d\omega + \rho_*(A) \wedge \omega,$$

where

$$(\rho_*(A) \wedge \omega)(t_1, .., t_{k+1}) := \sum_{j=1}^{k+1} (-1)^{j+1} \rho_*(A(t_j))(\omega(t_1, .., \hat{t_j}, .., t_{k+1}))$$

$$\rho_*(X)(v) := \frac{d}{dt}\Big|_{t=0} (\rho(\exp tX) v).$$

Proof: 1. It is immediate from the definition that $D_A \omega$ is horizontal for every k-form $\omega \in \Omega^k(P;V)$. For ω ρ-equivariant, we have

$(R_g^* D_A \omega)(t_1, .., t_{k+1}) =$

$(D_A\omega)(dR_g t_1, .., dR_g t_{k+1}) =$

$(d\omega)(\mathrm{pr}_h\, dR_g t_1, .., \mathrm{pr}_h\, dR_g t_{k+1})$

$= (d\omega)(dR_g\, \mathrm{pr}_h t_1, .., dR_g\, \mathrm{pr}_h t_{k+1})$

$= (R_g^* d\omega)(\mathrm{pr}_h t_1, .., \mathrm{pr}_h t_{k+1})$

$= d(R_g^* \omega)(\mathrm{pr}_h t_1, .., \mathrm{pr}_h t_{k+1})$

$= d(\rho(g^{-1}) \circ \omega)(\mathrm{pr}_h t_1, .., \mathrm{pr}_h t_{k+1})$

$= (\rho(g^{-1}) \circ d\omega)(\mathrm{pr}_h t_1, .., \mathrm{pr}_h t_{k+1})$

$= \rho(g^{-1})(D_A\omega(t_1, .., t_{k+1})).$

Thus $D_A \omega$ is also ρ-equivariant.

2. It suffices to check the desired formula for vectors $t_1, .., t_{k+1}$ each of whom is either horizontal, or vertical.
Assume first that $t_1, .., t_{k+1}$ are all horizontal. Then
$(D_A\omega)(pr_h t_1, .., pr_h t_{k+1}) =$
$(d\omega)(pr_h t_1, .., pr_h t_{k+1}) = d\omega(t_1, .., t_{k+1})$
and since $A(t_j)=0$ for all j, the claim holds true.
Assume now that at least one of the vectors $t_1, .., t_{k+1}$ is vertical.
If two of the vectors, say t_i and t_j are vertical, then we have $(D_A\omega)(t_1, \dots, t_{k+1}) = 0$ and $(\rho_*(A) \wedge \omega)(t_1, .., t_{k+1}) = 0$. On the other hand, we have
$$(d\omega)(t_1, .., t_{k+1}) = \sum_{j=1} (-1)^j t_i(\omega(t_1, .., \hat{t}_j, .., t_{k+1}))$$
$$+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([t_i, t_j], t_1, .., \hat{t}_i, .., \hat{t}_j, .., t_{k+1}))$$
Since $T_{\gamma p} P = T_p P_{\pi(p)})$, for every two horizontal vector fields $\tilde{X}, \tilde{Y}$, the commutator $[\tilde{X}, \tilde{Y}]$ is again

horizontal. Thus in this case all summands on the right hand side of the above formula for $d\omega$ vanish.

Finally, consider the case when exactly one of $t_1, .., t_{k+1}$ is vertical, and the other vectors are all horizontal. We may assume w. l. o. g. that t_1 is vertical. Fix $X \in \mathfrak{g}$ with $\tilde{X}(p) = t_1$ and vector fields $V_2, .., V_k$ on M with $V_j(\pi(p)) = d\pi_p(t_j)$. Denoting $T_j := (d\pi|_{Th})^{-1}(V_j)$, $T_2, .., T_k$ are vector fields on P with $T_j(p) = t_j$. We have

$$(D_A\omega)(t_1, .., t_{k+1}) = (d\omega)(pr_h t_1, t_2, .., t_{k+1}) = 0$$

and

$$(\rho_* A \wedge \omega)(t_1, .., t_{k+1}) = \rho_*(A(t_1))\omega(t_2, .., t_{k+1})$$

$= \rho_*(X)(\omega(t_2, .., t_{k+1}))$. On the other hand,

$$(d\omega)(t_1, .., t_{k+1}) = \tilde{X}(\omega(T_2, .., T_k))(p)$$
$$+ \sum_{j=2}^{k+1} (-1)^{j+1} \omega([\tilde{X}, T_j], T_1, .., \hat{T}_j, .., T_{k+1})(p).$$

since the flow of $\tilde{X}$ is given by

$$\varphi_t : P \to P$$
$$p \mapsto p \cdot \exp(tX)$$
$$= R_{\exp(tX)},$$

$$[\tilde{X}, T_j](p) = \frac{d}{dt}\Big|_{t=0} \left(d\varphi_{-t} T_j(\varphi_t(p)) \right)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(dR_{\exp(-tX)} \left(T_j(p \cdot \exp tX) \right) \right).$$

It follows from the definition of T_j that $R_g^* T_j = T_j$ for all $g \in G$, i.e. T_j is G-invariant and we conclude that $[\tilde{X}, T_j] = 0$.

Thus we obtain

$$d\omega(t_1, \dots, t_{k+1}) = \tilde{X}(\omega(T_2, \dots, T_{k+1}))(p)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\omega(T_2(p \cdot \exp tX)), \dots, T_{k+1}(p \cdot \exp(tX)) \right).$$

Due to the G-invariance of T_j,

$$T_j(p \cdot \exp tX) = dR_{\exp(tX)}(T_j(p))$$
$$= dR_{\exp tX}(t_j)$$ and thus

$$d\omega(t_1, \dots, t_{k+1}) = \frac{d}{dt}\Big|_{t=0} \left((R^*_{\exp tX}\, \omega)(t_2, \dots, t_{k+1}) \right)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\rho(\exp(-tX))\, \omega(t_2, \dots, t_{k+1}) \right)$$

$$= -\frac{d}{dt}\Big|_{t=0} \left(\rho(\exp tX)\, \omega(t_2, \dots, t_{k+1}) \right)$$

$$= - \rho_*(X)\, \omega(t_2, \dots, t_{k+1}).$$

Thus

$$d\omega(t_1, \dots t_{k+1}) + \rho_*(A) \wedge \omega(t_1, \dots, t_{k+1})$$
$$= \rho_*(X)\, \omega(t_2, \dots, t_{k+1}) - \rho_*(X)\, \omega(t_2, \dots, t_{k+1})$$
$$= 0.$$

□

Using Proposition 2.26, we may define for $k \geq 0$

$$d_A : \Omega^k(M;E) \to \Omega^{k+1}(M;E),$$
$$\omega \longmapsto d_A\omega, \text{ where } d_A\omega := D_A\bar{\omega}.$$

Here for a k-form $\omega \in \Omega^k(M;E)$ the term $\bar{\omega}$ denotes the ρ-equivariant horizontal form $\bar{\omega} \in \Omega^k(P;V)$ corresponding to ω under the isomorphism of Proposition 2.21.

<u>Proposition 2.28</u>

1. Denoting by ∇^A the map

$$d_A : \Omega^0(M;E) = \Gamma(E) \to \Omega^1(M;E) = \Gamma(T^*M \otimes E),$$

∇^A is a connection on $E = P \times_{(G,\rho)} V$

2. Let s be a local section of P over $U \subset M$ and $v : U \to V$ a smooth map. Denoting by e the local section of E defined by

$$e(x) := [(s(x), v(x))],$$

we have for $y \in T_x M$

$$(\nabla^A e)(x) = [(s(x), dv_x + \rho_*(A^s) v(x)]$$

<u>Proof</u>:

1. Let e be a section of E.
Denote by $\bar{e} \in \Omega^0(P; V) = C^\infty(P, V)$ the corresponding ρ-equivariant map $P \to V$. Then for $f \in C^\infty(M; \mathbb{R})$, $\bar{f} := f \circ \hat{\pi} : P \to \mathbb{R}$, $x \in M$ and $p \in P_x$, identifying $H_p P \cong T_{\pi(p)} M$,

$$(\nabla^A (f \cdot e))(x) = [(p, (D_A(\bar{f} \cdot \bar{e}))(p)]$$

$$= [(p, (d(\bar{f}\bar{e}) \circ pr_h)(p))] =$$

$$[(p, (d\bar{f} \circ pr_h)(p) \cdot \bar{e}(p))]$$

$$+ [(p, \bar{f}(p) \cdot (d\bar{e} \circ pr_h)(p))].$$

For $X \in T_x M$, denote by $X^* \in T_p P$ the unique vector with $d\pi_p X^* = X$.
Then

$$\left(D^A_X (f \cdot e)\right)(x) = [(p, \overbrace{(d\bar{f} \circ pr_h)(X^*)}^{= df(X)} \cdot \bar{e}(p))] + [(p, \underbrace{\bar{f}(p)}_{= f(x)} \cdot (d\bar{e} \circ pr_h)(X^*))]$$

$$= df(X) \cdot [(p, \bar{e}(p))] + f(x)\, [(p, d\bar{e} \circ pr_h(X^*))]$$

$$= df(X) \cdot e(x) + f(x)\,(D^A_X e)(x).$$

2. Suppose that $e(x) = [s(x), v(x)]$ is a local section of E. Using the second part of Proposition 2.26, we have with $v(x) =: \bar{e}(s(x))$:

$$D^A_X e = [s(x), (D_A \bar{e})(ds(X))]$$

$$= [s(x), d\bar{e}(ds(X)) + \rho_*(A(ds(X)))\,\bar{e}(s(x))]$$

$$= [s(x), d(\bar{e} \circ s)(X) + \rho_*(A^s(X))(\bar{e} \circ s)(x)]$$

$$= [s(x), dv(X) + \rho_*(A^s(X))\, v(x)]$$

$\square$

<u>Definition 2.27</u> Let $A \in \Omega^1(P; \mathfrak{g})$ be a connection form on a principal bundle $(P, \pi, M; G)$. We define the <u>curvature form</u> of A

by

$$F^A := D_A A \in \Omega^2(P;\mathfrak{g}).$$

It follows from Proposition 2.26 that F^A is a horizontal Ad-equivariant two-form.
Given a local section s of the bundle, we define the <u>local</u> <u>curvature</u> <u>form</u> corresponding to s by

$$F^s := s^* F^A = F^A(ds(\cdot), ds(\cdot)).$$

To determine the behaviour of the local curvature forms under passing from s to a different local section $\tau = s \cdot g$, recall from the proof of Proposition 2.18 the formula

$$d\tau(\cdot) = dR_g(ds(\cdot)) + \widetilde{dL_{g^{-1}}\, dg(\cdot)}.$$

Using the fact that F^A is Ad-equivariant and horizontal, we obtain

$$F^\tau = \tau^* F^A = Ad(g^{-1}) \circ F^s.$$

The proof of the following proposition is left as an exercise (see Problem Sheet 6):

<u>Proposition 2.29</u> Let $E = P \times_{(G,\rho)} V$ for a representation $\rho: G \to GL(V)$ of G and let D^A denote the connection on E from Proposition 2.26. For $p \in P_x$, $x \in M$, denote by $[p]: V \xrightarrow{\sim} E_x$ the isomorphism given by $v \mapsto [p, v]$. Then for $X, Y \in T_x M$,

$$R_x^{D^A}(X, Y) = [p] \circ \rho_*\left(F_p^A(X^*, Y^*)\right) \circ [p]^{-1},$$

where $X^*, Y^* \in T_p P$ are uniquely determined by $d\pi_p(X^*) = X$, $d\pi_p(Y^*) = Y$).
Here

$$R^D \in \Gamma\left(\Lambda^2(T^*M) \otimes \mathrm{End}(E, E)\right)$$

is the curvature endomorphism

$$R^D(X, Y)\, e = \left(D_X D_Y - D_Y D_X - D_{[X,Y]}\right) e$$

for $e \in \Gamma(E)$.

Next, we study the basic properties of the curvature form of a connection. To this end, we introduce some additional notation. For $\omega \in \Omega^k(P;\mathfrak{g})$ and $\tau \in \Omega^e(P;\mathfrak{g})$, define $[\omega,\tau] \in \Omega^{k+e}(P;\mathfrak{g})$ as follows: after fixing a basis $(a_1,..,a_m)$ of $\mathfrak{g}$ and writing $\omega = \sum_{i=1}^m \omega^i a_i$, $\tau = \sum_{i=1}^m \tau^i a_i$, where $\omega^j, \tau^j \in \Omega^*(P;\mathbb{R})$,

$$[\omega,\tau] := \sum_{i,j} (\omega^i \wedge \tau^j) \cdot [a_i,a_j].$$

It follows from the fact that the Lie bracket $[\cdot,\cdot]$ on $\mathfrak{g}$ is bilinear that $[\omega,\tau]$ is well-defined, i.e. independent of the choice of the basis $(a_1,..,a_m)$. Moreover, we have for all $\omega \in \Omega^k(P;\mathfrak{g})$, $\tau \in \Omega^e(P;\mathfrak{g})$:

- $[\omega,\tau] = (-1)^{k\cdot e+1}[\tau,\omega]$

- $d[\omega, \tau] = [d\omega, \tau] + (-1)^k [\omega, d\tau]$
- if $k=1$, i.e. $\omega \in \Omega^1(P; \mathfrak{g})$, then $[\omega, \omega](X, Y) = 2[\omega(X), \omega(Y)]$ for all $X, Y \in \Gamma(TP)$.

<u>Proposition 2.30</u> The curvature form F^A satisfies the following:

1. $F^A = dA + \frac{1}{2}[A, A]$
2. $D_A F^A = 0$ "Bianchi - Identity"
3. If $\omega \in \Omega^k(P, V)$ is horizontal and ρ-equivariant for a representation $\rho: G \to GL(V)$, then
$$D_A D_A \omega = \rho_*(F^A) \wedge \omega.$$

<u>Proof</u>: 1. It suffices to check the desired identity for pairs (X, Y) of vectors where each of X, Y is either horizontal, or vertical. If both X and Y are horizontal, then $A(X) = A(Y) = 0$ and
$$F^A(X, Y) = (D_A A)(X, Y) = dA(X, Y).$$

Thus the identity is satisfied.
Assume now that X is horizontal and Y is vertical. As in the proof of Proposition 2.26, we may assume that X, Y have the form $X = V^*$, where $dR_g V^* = V^* \circ R_g$ for $g \in G$ and $Y = \tilde{T}$ for some $T \in \mathfrak{g}$. Using again the proof of Proposition 2.26, $[X, Y] = [V^*, \tilde{T}] = 0$. Since moreover $A(V^*) = 0$ and $A(\tilde{T}) = T = \text{const}$, we have $dA(X, Y) = 0$. Thus

$$F^A(X,Y) = 0 = \underset{\overset{\|}{0}}{dA(X,Y)} + \underset{\overset{\|}{0}}{[A(X), A(Y)]}$$

$$= dA(X,Y) + \frac{1}{2}[A, A](X,Y).$$

Finally, consider the case where X, Y are both vertical. We may assume $X = \tilde{T}$, $Y = \tilde{S}$, $T, S \in \mathfrak{g}$. We have $F^A(X,Y) = 0$ and

$$dA(X,Y) = X(A(Y)) - Y(A(X)) - A([X,Y]) = -A[\tilde{T}, \tilde{S}] = -A(\widetilde{[T,S]}) =$$

$$= -[T,S] = -[A(\tilde{T}), A(\tilde{S})]$$

$$= -\frac{1}{2}[A,A](X,Y).$$

2. We have

$$dF^A = d(dA) + \frac{1}{2} d([A,A])$$

$$= \frac{1}{2}([dA,A] - [A,dA]) =$$

$[dA,A]$. Thus

$$D_A F^A = dF^A(\rho^h(\cdot), \rho^h(\cdot), \rho^h(\cdot))$$

$$= [dA(\rho^h(\cdot), \rho^h(\cdot)), \underset{0}{\underset{\|}{A(\rho^h(\cdot))}}] = 0.$$

3. Using Proposition 2.26,

$$D_A(D_A w) = d(dw + \rho_*(A)\wedge w)$$

$$+ \rho_*(A)\wedge(dw + \rho_*(A)\wedge w) =$$

$$ddw + d(\rho_*(A))\wedge w - \rho_*(A)\wedge dw$$

$$+ \rho_*(A)\wedge dw + \rho_*(A)\wedge\rho_*(A)\wedge w$$

$$= \rho_*(dA)\wedge w + \rho_*(A)\wedge\rho_*(A)\wedge w$$

We have

$$(\rho_*(A)\wedge\rho_*(A))(X,Y) = \rho_*(A(X))\circ$$

$$\rho_*(A(Y)) - \rho_*(A(Y))\circ\rho_*(A(X)) =$$

$$= [\rho_*(A(x)), \rho_*(A(y))]_{\mathfrak{gl}(V)}$$

$$= \rho_*([A(x), A(y)])_{\mathfrak{g}}$$

$$= \frac{1}{2}\rho_*([A,A](x,y)).$$ Thus

we obtain

$$D_A D_A \omega = \rho_*(dA + \tfrac{1}{2}[A,A]) \wedge \omega$$

$$= \rho_*(F^A) \wedge \omega.$$

$\square$

<u>Example</u>: consider the special case when $G = S^1$. We identify $\mathfrak{g} \cong i\mathbb{R}$. Since G is abelian, the adjoint representation is trivial. It follows that every Ad-equivariant horizontal form $\bar{\omega} \in \Omega^k(P; i\mathbb{R})$ has the form $\bar{\omega} = \pi^*\omega$, $\omega \in \Omega^k(M; i\mathbb{R})$. In particular, for any two connection forms $A, \tilde{A} \in \Omega^1(P; i\mathbb{R})$ the difference $A - \tilde{A}$ has the form $A - \tilde{A} = \pi^*\alpha$, $\alpha \in \Omega^1(M; i\mathbb{R})$.

We have

$$F^A = D_A A = dA + \frac{1}{2}[A,A] = dA$$

since $G = S^1$ is abelian. In particular, $F^A \in \Omega^2(P; i\mathbb{R})$ is closed. Since F^A is horizontal (and G-invariant), we may identify F^A with a closed two-form in $\Omega^2(M; i\mathbb{R})$. Moreover, since $F^A - F^{\tilde{A}} = d(\pi^*\alpha) = \pi^*(d\alpha)$, the difference $F^A - F^{\tilde{A}}$ in $\Omega^2(M; i\mathbb{R})$ is exact. It follows that

$$c_1(P) := \left[-\frac{1}{2\pi i} F^A \right] \in H^2_{dR}(M; \mathbb{R})$$

is independent of the choice of the connection form A. $c_1(P)$ is called the <u>first</u> <u>Chern class</u> of P.

2.6 The Chern-Weil Homomorphism

We saw in the last example of section 2.5 that in the case of S^1-principal bundles the curvature form of a connection gives rise to a de Rham cohomology class on the base manifold and that class is independent of the choice of connection. We want to generalize this to the case of arbitrary principal bundles and show that characteristic classes can be obtained as the result of the construction.

For a real vector space V, we denote by $S^k(V^*)$, $k \geq 1$, the set of all multilinear symmetric maps $Q : \underbrace{V \times \ldots \times V}_{k \text{ times}} \to \mathbb{R}$

There is a product

$$\circ : S^k(V^*) \times S^l(V^*) \to S^{k+l}(V^*)$$

$$(Q_1 \circ Q_2)(v_1, \ldots, v_{k+l}) =$$

$$\frac{1}{(k+l)!}\sum_{\sigma} Q_1(v_{\sigma(1)},\dots,v_{\sigma(k)})\,Q_2(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)}),$$

where the sum is over all permutations σ of $(1,\dots,k+l)$.

Let $(e_1,\dots,e_n)$ be a basis of V and let $\mathbb{R}[x_1,\dots,x_n]^k$ denote the set of homogeneous polynomials of degree k in $x_1,\dots,x_n$.

We define a map

$$S^k(V^*) \to \mathbb{R}[x_1,\dots,x_n]^k$$
$$Q \mapsto \tilde{Q}$$

by $\tilde{Q}(x_1,\dots,x_n) := Q(v,\dots,v)$ with $v := \sum_{j=1}^{n} x_j e_j$.

<u>Proposition 2.31</u>

1. We have $\widetilde{Q_1 \circ Q_2} = \tilde{Q}_1 \cdot \tilde{Q}_2$ for all $Q_1 \in S^k(V^*)$, $Q_2 \in S^l(V^*)$.
2. The map $S^k(V^*) \to \mathbb{R}[x_1,\dots,x_n]^k$, $Q \mapsto \tilde{Q}$ is a vector space isomorphism.

Proof: 1. is immediate from the definition of $\tilde{Q}$. For 2, note that the coefficient of $x_1^{j_1} \cdots x_n^{j_n}$ in $\tilde{Q}$ is

$$Q(e_1, .., e_1, e_2, .., e_2, \dots, e_n, .., e_n),$$

where e_m repeats j_m times.

If $\tilde{Q} = 0$, then we must have $Q(e_1, .., e_1, e_2, .., e_2, \dots, e_n, .., e_n) = 0$ for all $(j_1, .., j_n)$, but then using multilinearity and symmetry of P, we conclude $Q = 0$.

Thus the map $Q \mapsto \tilde{Q}$ is injective. Surjectivity of the map is immediate in the case $k = 1$ as

$$S^1(V^*) = \mathrm{Hom}(V; \mathbb{R}) \longrightarrow \mathbb{R}[x_1, .., x_n]^1$$
$$\sum_{j=1}^n a_j e_j^* \mapsto \sum_{j=1}^n a_j x_j$$

is an isomorphism. For $k > 1$, surjectivity follows from 1. and the fact that the map

$$\left(\mathbb{R}[x_1, .., x_n]^1\right)^{\times k} \longrightarrow \mathbb{R}[x_1, .., x_n]^k$$
$$(q_1, .., q_k) \mapsto q_1 \cdots q_k$$

is surjective. $\square$

Consider now the special case when $V = \mathfrak{g}$ is the Lie algebra of a Lie group G.

<u>Definition 2.32</u> We denote by $I^k(G) \subset S^k(\mathfrak{g}^*)$ the subspace of all $Q \in S^k(\mathfrak{g}^*)$, so that $Q(Ad(g^{-1})X_1, .., Ad(g^{-1})X_k) = Q(X_1, .., X_k)$ for all $X_1, .., X_k \in \mathfrak{g}$, $g \in G$.

Using Proposition 2.31, $I^k(G)$ is in bijective correspondence to a subspace of $\mathbb{R}[X_1, .. X_n]^k$, $n = \dim G$, and one sometimes refers to the elements of $I^*(G)$ as <u>invariant polynomials</u> on $\mathfrak{g}$.

Suppose now that $\xi = (P, \pi, M; G)$ is a principal bundle with connection $A \in \Omega^1(P; \mathfrak{g})$.