Let $F^A \in \Omega^2(P; \mathfrak{g})$ be the curvature form of A.

Denoting for $k \geq 1$

$$(F^A)^k := \underbrace{F_A \wedge \dots \wedge F_A}_{k \text{ times}} \in \Omega^{2k}(P; \mathfrak{g} \otimes \dots \otimes \mathfrak{g}),$$

we associate to every element $Q \in I^k(G)$ the $2k$-form

$$Q \circ (F^A)^k \in \Omega^{2k}(P; \mathbb{R}).$$

Since F^A is horizontal, so is $Q \circ (F^A)^k$. Moreover, since F^A is Ad-equivariant and Q is Ad-invariant, $Q \circ (F^A)^k$ is Ad-invariant. We conclude that $Q \circ (F^A)^k$ lies in the image of $\pi^*: \Omega^{2k}(M; \mathbb{R}) \to \Omega^{2k}(P; \mathbb{R})$.

By slight abuse of notation, we denote the element of $\Omega^{2k}(M; \mathbb{R})$ corresponding to $Q \circ (F^A)^k$ again by $Q \circ (F^A)^k$.

Theorem 2.37 (Main Theorem of Chern-Weil Theory).

1. For every connection form $A \in \Omega^1(P;\mathfrak{g})$ and every invariant polynomial $Q \in I^k(G)$, $Q \circ (F^A)^k \in \Omega^{2k}(M;\mathbb{R})$ is a closed two-form, i.e. $d(Q \circ (F^A)^k) = 0$. Moreover, the de Rham cohomology class $w(\xi, Q) := [Q \circ (F^A)^k] \in H^{2k}_{dR}(M)$ is independent of the choice of the connection A.
2. The map $w(\xi, \cdot) : (I^*(G), \circ) \to (H^*_{dR}(M), \wedge)$ is a ring homomorphism.
3. For every morphism $(F, f) : \eta \to \xi$ of G-principal bundles, we have
$$w(\eta, Q) = f^* w(\xi, Q).$$

<u>Proof</u>: 1. We first show that $Q(F^A)^k \in \Omega^{2k}(M;\mathbb{R})$ is closed. Since the map $\pi^*: \Omega^*(M;\mathbb{R}) \to \Omega^*(P;\mathbb{R})$ is injective, it suffices to prove that $Q(F^A)^k$ is closed as a form in $\Omega^{2k}(P;\mathbb{R})$.
Using the symmetry of Q, we have $d(Q(F^A)^k) = k\,Q(dF^A \wedge (F^A)^{k-1})$.
We have $dF^A = d(dA + \frac{1}{2}[A,A])$
$= \frac{1}{2}([dA,A] - [A,dA]) = [dA,A]$ and thus
$d(Q(F^A)^k) = k\,Q([dA,A] \wedge (F^A)^{k-1})$.
Note that $d(Q(F^A)^k)$ is horizontal (since $Q(F^A)^k$ lies in the image of $\pi^*: \Omega^*(M;\mathbb{R}) \to \Omega^*(P;\mathbb{R})$, so does $d(Q(F^A)^k)$). But on any tuple of horizontal vectors $[dA,A] \wedge (F^A)^{k-1}$ vanishes. We conclude $d(Q(F^A)^k) = 0$.
We now prove that the de Rham cohomology class $Q(F^A)^k \in H^{2k}_{dR}(M;\mathbb{R})$

is independent of the connection A. Let $\tilde{A} \in \Omega^1(P;\mathfrak{g})$ be a second connection form. We consider on $P \times \mathbb{R}$ the one-form

$B_{(p,s)} := (1-s) A_p + s \tilde{A}_p$. One readily checks that $B \in \Omega^1(P \times \mathbb{R}; \mathfrak{g})$ is a connection form on $(P \times \mathbb{R}, \pi \times id, M \times \mathbb{R}; G)$. Denoting by $i_0, i_1 : P \hookrightarrow P \times \mathbb{R}$ the inclusions $i_0 : p \mapsto (p,0)$ and $i_1 : p \mapsto (p,1)$, we have $i_0^* B = A$, $i_1^* B = \tilde{A}$, $i_0^* F^B = F^A$ and $i_1^* F^B = F^{\tilde{A}}$ (and thus $i_0^* Q((F^B)^k) = Q((F^A)^k)$, $i_1^* Q((F^B)^k) = Q((F^{\tilde{A}})^k)$). We want to use B to construct $h(Q(F^B)^k) \in \Omega^{2k-1}(M;\mathbb{R})$ with

$Q(F^A)^k - Q(F^{\tilde{A}})^k = d\, h(Q(F^B)^k))$.

To this end, use:

<u>Lemma 2.34</u> Define $h : \Omega^k(M \times \mathbb{R}) \to \Omega^{k-1}(M)$ by setting

$$h(\omega) := \int_0^1 \alpha_s \quad \text{for}$$

$\omega = ds \wedge \alpha + \beta$, where $\beta(\cdot, \ldots, \tfrac{\partial}{\partial s}) \equiv 0$

$(h(\omega) = 0$ for $\omega \in \Omega^0(M \times \mathbb{R}))$.

Then for all $\omega \in \Omega^k(M; \mathbb{R})$

$$d\, h(\omega) + h(d\omega) = i_1^* \omega - i_0^* \omega$$

Proof of Lemma: Apply the main Thm of calculus. $\square$

Using Lemma 2.34, we have

$$d h\left(Q\left(F^B\right)^k\right) = i_1^* Q\left(F^B\right)^k - i_0^* Q\left(F^B\right)^k = Q\left(F^A\right)^k - Q\left(F^{\tilde{A}}\right)^k.$$

2. For a permutation σ of $(1, .., k+l)$, write $S_\sigma : \mathfrak{g}^{\otimes(k+l)} \to \mathfrak{g}^{\otimes(k+l)}$ for the map

$$X_1 \otimes .. \otimes X_{k+l} \mapsto X_{\sigma(1)} \otimes .. \otimes X_{\sigma(k+l)}.$$

Using symmetry of Q and the fact that F^A has even rank,

$$(F^A)^{k+l} = T_\sigma \circ (F^A)^{k+l} = T_\sigma \circ \left((F^A)^k \wedge (F^A)^l\right)$$

Thus

$$(Q_1 \circ Q_2)\left((F^A)^{k+l}\right) =$$

$$\frac{1}{(k+l)!} \sum_\sigma (Q_1 \cdot Q_2) \circ T_\sigma \circ (F^A)^{k+l}$$

$$= \frac{1}{(k+l)!} \sum_\sigma Q_1\left((F^A)^k\right) \wedge Q_2\left((F^A)^l\right)$$

$= Q_1 (F^A)^k \wedge Q_2 (F^A)^\ell$.

3. Suppose that $\eta = (P', \pi', \sigma'; G)$.

$A' := F^* A$ is a connection form on η and we have

$$F^{A'} = F^* F^A \quad \text{and thus}$$

$Q(F^{A'})^k = F^*(Q(F^A)^k)$. Identifying with forms on the base, we have $Q(F^{A'})^k = f^*(Q(F^A)^k)$ and we conclude

$$w(\eta, Q) = f^* w(\xi, Q).$$

$\square$

By applying the Chern-Weil homomorphism to the cases $G = GL(k;\mathbb{C})$ or $G = GL(k;\mathbb{R})$ (and on suitable invariant polynomials Q), we can construct characteristic classes of complex or real vector bundles.

<u>Chern classes</u> Consider $G = GL(k,\mathbb{C})$, $\mathfrak{g} = \mathbb{C}^{k\times k}$. For $\ell = 0, 1, \ldots, k$, let $C_\ell : \mathfrak{g} \otimes \ldots \otimes \mathfrak{g} \to \mathbb{C}$ be

the symmetric form determined by

$$\det\left(\lambda I - \frac{1}{2\pi i} X\right) = \sum_{\ell=0}^{k} c_\ell(X,\dots,X)\lambda^{k-\ell}$$

for $X \in \mathbb{C}^{k\times k}$. We call $c_1, c_2, \dots$ the <u>Chern polynomials</u> and $c_\ell(\xi) := w(\xi, c_\ell) \in H^{2\ell}_{dR}(M;\mathbb{C})$ the <u>Chern classes</u> of the principal bundle ξ. The sum

$$c(\xi) := \sum_{\ell=0}^{\infty} c_\ell(\xi)$$

is called the total Chern class.

Recall that for a complex vector bundle (E,π,M) of rank k the frame bundle $GL(E)$ is a $GL(k;\mathbb{C})$-principal bundle. We define $c_\ell(E) := c_\ell(GL(E))$ and $c(E) := c(GL(E))$.

<u>Proposition 2.35</u>

1. For every $\ell \geq 0$, the map $E \mapsto c_\ell(E)$ is a characteristic class on complex vector bundles.
2. We have $c_\ell(E) \in H^{2\ell}_{dR}(M;\mathbb{R})$ for all ℓ.

3. If $\overline{E}$ denotes the conjugate to the complex vector bundle E, then

$$c_\ell(\overline{E}) = (-1)^\ell c_\ell(E).$$

4. The total Chern class of a Whitney sum $E_1 \oplus E_2$ is given by

$$c(E_1 \oplus E_2) = c(E_1) \cup c(E_2).$$

<u>Proof</u>: To prove 1., note that a morphism $(F, f) : E' \to E$ of vector bundles gives rise to a morphism $(GL(F), f) : GL(E) \to GL(E')$. For every connection form A on $GL(E')$ the pullback $(GL(F))^* A =: A'$ is a connection form on $GL(E)$ and we have $F^{A'} = f^* F^A$.

For the second claim, note that since E admits a bundle metric, there is a $\mathcal{U}(n)$-principal reduction of $GL(E)$ to a bundle $\mathcal{U}(E)$. It follows from Proposition 2.18 that for every connection form A on $\mathcal{U}(E)$, there exists a unique connection form $\tilde{A}$ on $GL(E)$ so that $\tilde{A}^s|_{T\mathcal{U}(E)} = A^s$ for all local sections s of $\mathcal{U}(E)$. Since for $X \in \mathfrak{u}(n)$, $\det\left(x I - \frac{1}{2\pi i} X\right) = \det\left(x I + \frac{1}{2\pi i} \overline{X}^T\right)$

$= \det(xI - \frac{1}{2\pi i}X)$, we have $c_\ell(X,\dots,X) \in \mathbb{R}$ for all ℓ and thus $c_\ell(F^A)^\ell \in \Omega^{2\ell}(M;\mathbb{R})$. The third claim follows analogously, using the fact that the total spaces of $GL(E)$ and $GL(\bar{E})$ are diffeomorphic and for every connection form A on $GL(E)$, $\bar{A}$ is a connection form on $GL(\bar{E})$.

For 3: the frame bundle $GL(E \oplus E')$ of a Whitney sum is an extension of the $GL(k,\mathbb{C}) \times GL(k';\mathbb{C})$ - principal bundle $GL(E) \times_M GL(E') = \{(p,p') \mid p \in GL(E), p' \in GL(E'), \pi(p) = \pi(p')\}$. Using Proposition 2.18, every connection form B on $GL(E) \times_M GL(E')$ gives rise to a unique connection form $\tilde{B}$ on $GL(E \oplus E')$ and moreover, $Q(F^B)^\ell = Q(F^{\tilde{B}})^\ell$ for every $Q \in I^\ell(GL(k+k'))$. Now given A and A' on connection forms on $GL(E)$ and $GL(E')$ respectively, a connection form on $GL(E) \times_M GL(E')$ is given by $A \oplus A'$. The embedding $\mathfrak{gl}(k,\mathbb{C}) \oplus \mathfrak{gl}(k';\mathbb{C}) \hookrightarrow \mathfrak{gl}(k+k';\mathbb{C})$ induced by $GL(k,\mathbb{C}) \times GL(k';\mathbb{C}) \hookrightarrow GL(k+k';\mathbb{C})$

is given by $X \oplus X' \mapsto \operatorname{diag}(X, X') = \begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix}$.
The claim now follows from the identity

$$\det\left(\lambda I_{k+k'} - \operatorname{diag}(X, X')\right) =$$

$$\det(\lambda I_k - X) \cdot \det(\lambda I_{k'} - X')$$ and the second part of Theorem 2.33.

□

Pontrjagin classes

Define for $X \in gl(k, \mathbb{R})$

$$\det\left(\lambda I - \frac{1}{2\pi} X\right) =: \sum_{\ell=0}^{k} P_{\ell/2}(X, .., X)\, \lambda^{k-\ell}.$$

Then $P_{\ell/2}$ is an Ad-invariant polynomial of degree k on $G = GL(k, \mathbb{R})$. For a $GL(k, \mathbb{R})$-principal bundle ξ over M, we call $p_{\ell/2}(\xi) := w(\xi, P_{\ell/2}) \in H^{2\ell}_{dR}(M; \mathbb{R})$ the <u>Pontrjagin classes</u> of ξ. For a vector bundle E,

we define $p_{l/2}(E) := p_{l/2}(GL(E))$.
We have $p_{l/2} = 0$ for l odd: the frame bundle of every vector bundle can be reduced to $O(n) \subset GL(n, \mathbb{R})$ and on the other hand, for $X \in \mathfrak{o}(n) = \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid X^T = -X\}$ we have

$$\det\left(\lambda I - \frac{1}{2\pi}X\right) = \det\left(\lambda I - \frac{1}{2\pi}X\right)^T$$

$$= \det\left(\lambda I + \frac{1}{2\pi}X\right) \Rightarrow p_{l/2}(X, \dots, X) \equiv 0$$

for l odd. It follows from the definitions that $P_{l/2}(E) = C_l(E \otimes \mathbb{C})$ for every real vector bundle E. We state without proof two additional important properties of the Pontrjagin classes:

Cobordism invariance

Recall that a cobordism between to closed oriented manifolds M_1 and M_2 is a compact oriented manifold N with boundary $\partial N = M_1 \cup (-M_2)$.
If $\dim M_1 = \dim M_2 = 4n$ and if there exists a cobordism between

M_1 and M_2, then for all $i_1, \dots, i_r \geq 0$ with $i_1 + \dots + i_r = n$,

$$(p_{i_1}(TM_1) \cup \dots \cup p_{i_r}(TM_1))[M_1] = (p_{i_1}(TM_2) \cup \dots \cup p_{i_r}(TM_2))[M_2],$$

where $[M_i]$ denotes the fundamental class of M_i.

<u>Hirzebruch signature Theorem</u>

For a closed oriented manifold M of dimension $4n$, define the <u>signature</u> $\sigma(M)$ as the difference between the number of positive and the number of negative entries of a diagonalization of the symmetric form

$$H^{2n}(M;\mathbb{R}) \times H^{2n}(M;\mathbb{R}) \to \mathbb{R}$$
$$(a,b) \mapsto (a \cup b)[M].$$

There exists a sequence of polynomials, where $L_1, L_2, \dots$ $L_n \in \mathbb{Q}[p_1, \dots, p_n]$, so that for every closed oriented manifold M,

$$\sigma(M) = L(p_1(TM), \dots, p_n(TM))[M].$$

We have

$$L_1 = \frac{1}{3} p_1$$

$$L_2 = \frac{1}{45} (7p_2 - p_1^2)$$

$$L_3 = \frac{1}{945} (62 p_3 - 13 p_2 p_1 + 2 p_1^3).$$

As an application, it is possible to use the properties of the Pontrjagin classes in order to distinguish different differentiable structures on the seven-sphere S^7 (see problem sheet 8).

3. De Rham Hodge Theory

3.1 The Hodge Decomposition

In the last section we saw how to systematically construct characteristic classes of vector bundles using the notions of connections and curvature. In turn, characteristic classes prominently play a role in the index theory of elliptic differential operators over manifolds. While this general theory is out of reach for the purposes of this course, the goal of this section is to introduce some of the underlying ideas by looking at the simplest case of de Rham Hodge Theory.

Let (M,g) be a closed oriented Riemannian manifold. Let $\omega \in \Omega^n(M;\mathbb{R})$, $n = \dim M$ be the volume form determined by the condition that $\omega_x(v_1,\dots,v_n) = 1$ for every positive g_x-orthonormal basis $(v_1,\dots,v_n)$ of T_xM, where $x \in M$.

<u>Definition 3.1</u> We define for $k = 0,\dots,n$ a $C^\infty(M;\mathbb{R})$-linear map $*: \Omega^k(M;\mathbb{R}) \to \Omega^{n-k}(M;\mathbb{R})$

as follows: if $e_1, .., e_n$ are local vector fields on M that are orthonormal and so that $(e_1, .., e_n)_x$ is a positive basis of T_xM for each x, then

$$*(e_1^* \wedge .. \wedge e_k^*) := e_{k+1}^* \wedge .. \wedge e_n^* \quad (i),$$

$*(1) := e_1^* \wedge .. \wedge e_n^*$ and
$*(e_1^* \wedge .. \wedge e_n^*) := 1$. Here $e_j^* \in \Omega^1(M, \mathbb{R})$ is the dual 1-form to e_j, i.e. $e_j^*(e_l) = \delta_{jl}$ for $j, l = 1, .., n$.

We call $*$ the "Hodge-Star operator". Note that $*$ is well-defined since every every k-form α can be uniquely written (locally) as

$$\alpha = \sum_{1 \le j_1 < j_2 < .. < j_k \le n} \alpha_{j_1 .. j_k} \, e_{j_1}^* \wedge .. \wedge e_{j_k}^*,$$

where $\alpha_{j_1 .. j_k}$ are real valued functions on M and from (i), we have

$$*(e_{j_1}^* \wedge .. \wedge e_{j_k}^*) = \operatorname{sign}(\sigma) \cdot e_{i_1}^* \wedge .. \wedge e_{i_{n-k}}^*$$

where $1 \le i_1 < i_2 < .. < i_{n-k} \le n$ are determined by $\{i_1, .., i_{n-k}\} = \{1, .., n\} \setminus \{j_1, .., j_k\}$ and σ is the permutation of $(1,..,n)$ with $\sigma(s) = j_s$ for $s = 1,..,k$ and $\sigma(s) = i_s$ for $s = k+1,..,n$.

The Riemannian metric g gives rise to a $C^\infty(M;\mathbb{R})$-bilinear inner product $\langle \cdot,\cdot \rangle$ on $\Omega^k(M;\mathbb{R})$, determined by

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge (*\beta)$$

for $\alpha, \beta \in \Omega^k(M;\mathbb{R})$.

<u>Definition 3.2</u> We define the <u>Laplace-Beltrami</u> operator $\Delta : \Omega^k(M;\mathbb{R}) \to \Omega^k(M;\mathbb{R})$ as

$$\Delta := \delta d + d\delta,$$

where $d : \Omega^l(M;\mathbb{R}) \to \Omega^{l+1}(M;\mathbb{R})$ is the exterior differential and where $\delta : \Omega^l(M;\mathbb{R}) \to \Omega^{l-1}(M;\mathbb{R})$ is given by

$$\delta = (-1)^{n(l+1)+1} * d *$$

The following Proposition states some basic properties of the Laplace-Beltrami operator:

Proposition 3.3

1. Δ commutes with d, δ and $*$.
2. Δ is <u>self-adjoint</u> with respect to $\langle\cdot,\cdot\rangle$, i.e.

$$\langle \Delta\alpha, \tilde{\alpha}\rangle = \langle \alpha, \Delta\tilde{\alpha}\rangle$$

for all $\alpha, \tilde{\alpha} \in \Omega^k(M;\mathbb{R})$, $0 \le k \le n$.

<u>Proof</u>: The fact that Δ commutes with d and δ is immediate from the definition. For commutativity with $*$, we note that for all $\alpha \in \Omega^k(M;\mathbb{R})$,

$$**\alpha = (-1)^{k(n-k)}\alpha,$$

as follows from definition 3.1 using the fact that the sign of the permutation $(e_1,..,e_k,e_{k+1},..,e_n) \mapsto (e_{k+1},..,e_n,e_1,..,e_k)$ is $(-1)^{k(n-k)}$.

We obtain for $\alpha \in \Omega^k(M;\mathbb{R})$

$$\begin{aligned}
*\Delta\alpha &= *\delta d\alpha + *d\delta\alpha \\
&= (-1)^{nk+1} *(*d*)d\alpha + (-1)^{n(k+1)+1} *d(*d*)\alpha \\
&= (-1)^{nk+1}\cdot(-1)^{(n-k)k} d*d\alpha + (-1)^{n(k+1)+1} *d*d*\alpha \\
&= \left((-1)^{k+1} d*d + (-1)^{n(k+1)+1} *d*d*\right)\alpha \quad \text{and}
\end{aligned}$$

$$\Delta * \alpha = \delta d * \alpha + d\delta * \alpha$$

$$= (-1)^{n(n-k)+1} * d * d * \alpha + (-1)^{n(n-k+1)+1} d(*d*) * \alpha$$

$$= (-1)^{n(n-k)+1} * d * d * \alpha + (-1)^{n(n-k+1)+1} \cdot (-1)^{k(n-k)} d * d\alpha$$

$$= (-1)^{nk+n+1} * d * d * \alpha + (-1)^{k+1} d * d\alpha = * \Delta \alpha$$

For the second claim, let $\alpha \in \Omega^{k-1}(M;\mathbb{R})$ and $\beta \in \Omega^k(M;\mathbb{R})$. We have

$$* \delta \beta = (-1)^{n(k+1)+1} * * d * \beta = (-1)^{n(k+1)+1} (-1)^{(n-k+1)(k+1)} d * \beta$$

$$= (-1)^{(k+1)^2+1} d * \beta = (-1)^k d * \beta$$ and

thus

$$d(\alpha \wedge * \beta) = d\alpha \wedge * \beta + (-1)^{k-1} \alpha \wedge d * \beta$$

$$= d\alpha \wedge * \beta - \alpha \wedge * \delta \beta$$

Applying Stokes' Theorem, we conclude

$$0 = \int_M d\alpha \wedge * \beta - \int_M \alpha \wedge * \delta \beta$$

$$= \langle d\alpha, \beta \rangle - \langle \alpha, \delta \beta \rangle,$$ i.e.

$\langle d\alpha, \beta \rangle = \langle \alpha, \delta \beta \rangle$. It follows that

$$\langle \Delta \alpha, \tilde{\alpha} \rangle = \langle (d\delta + \delta d)\alpha, \tilde{\alpha} \rangle$$

$$= \langle \delta \alpha, \delta \tilde{\alpha} \rangle + \langle d\alpha, d\tilde{\alpha} \rangle = \langle \alpha, \Delta \tilde{\alpha} \rangle.$$

$\square$

We say that $\alpha \in \Omega^k(M;\mathbb{R})$ is <u>harmonic</u> if $\Delta\alpha = 0$.
Since by the proof of Proposition 3.3,

$$\langle \Delta\alpha, \alpha\rangle = \langle \delta\alpha, \delta\alpha\rangle + \langle d\alpha, d\alpha\rangle$$

for all $\alpha \in \Omega^k(M;\mathbb{R})$, we conclude:

<u>Corollary 3.4</u> $\alpha \in \Omega^k(M;\mathbb{R})$ is harmonic if and only if $d\alpha = 0$ and $\delta\alpha = 0$. In particular, every harmonic function on a closed connected manifold is constant.

In order to understand harmonic forms of rank $k > 0$, consider more generally for a given $\omega \in \Omega^k(M;\mathbb{R})$ the equation

$$\Delta\alpha = \omega, \quad \alpha \in \Omega^k(M;\mathbb{R}).$$

For every solution α, we have

$$\langle \alpha, \Delta\beta\rangle = \langle \Delta\alpha, \beta\rangle = \langle \omega, \beta\rangle$$

for all $\beta \in \Omega^k(M;\mathbb{R})$. Thus if we denote by $\ell : \Omega^k(M;\mathbb{R}) \to \Omega^k(M;\mathbb{R})$ the linear functional

$$\ell(\beta) = \langle \alpha, \beta\rangle,$$

then we have $\ell(\Delta\beta) = \langle \omega, \beta\rangle$ for all $\beta \in \Omega^k(M;\mathbb{R})$.

We say that a linear functional $l: \Omega^k(M;\mathbb{R}) \to \Omega^k(M;\mathbb{R})$ which is bounded with respect to $\langle \cdot, \cdot \rangle$ is a **weak solution** of $\Delta\alpha = \omega$, if $l(\Delta\beta) = \langle \omega, \beta \rangle$ for all $\beta \in \Omega^k(M;\mathbb{R})$.

It turns out that weak solutions can be constructed using input from functional analysis (see below). On the other hand, a fundamental property of the Laplace-Beltrami operator is regularity of all weak solutions:

Proposition 3.5 Let $\omega \in \Omega^k(M;\mathbb{R})$ and let $l: \Omega^k(M;\mathbb{R}) \to \Omega^k(M;\mathbb{R})$ be a weak solution of $\Delta\alpha = \omega$. Then there exists $\alpha \in \Omega^k(M;\mathbb{R})$, so that

$$l(\beta) = \langle \alpha, \beta \rangle$$

for all $\beta \in \Omega^k(M;\mathbb{R})$.

Note that since $\langle \Delta\alpha, \beta \rangle = \langle \omega, \beta \rangle$ for all $\beta \in \Omega^k(M;\mathbb{R})$ implies $\Delta\alpha = \omega$ due to non-degeneracy of the inner product, in the situation of Proposition 3.5, $l(\beta) = \langle \alpha, \beta \rangle$ implies $\Delta\alpha = \omega$, i.e. α is automatically a (strong) solution of the equation.

The proof of Proposition 3.5 will be given further below.

Denote by $H^k := \{\alpha \in \Omega^k(M;\mathbb{R}) \mid \Delta\alpha = 0\} \subset \Omega^k(M;\mathbb{R})$ the subspace of harmonic k-forms. It turns out that H^k is finite dimensional. This can be concluded from the following:

<u>Proposition 3.6</u> Let $(\alpha_j)_{j\in\mathbb{N}} \subset \Omega^k(M;\mathbb{R})$ be a sequence so that $\|\alpha_j\| \leq C$ and $\|\Delta\alpha_j\| \leq C$ for all $j \in \mathbb{N}$, where $C \in \mathbb{R}$. Then $\exists$ a subsequence of $(\alpha_j)_{j\in\mathbb{N}}$ a Cauchy sequence.

Finally, we recall from functional analysis the Hahn-Banach theorem for linear functionals.

<u>Proposition 3.7</u> (Hahn-Banach Theorem) Let $(V, \|\cdot\|)$ be a vector space with a norm, $W \subset V$ a subspace and $\ell: W \to \mathbb{R}$ a bounded linear functional. Then there exists a linear functional $\tilde{\ell}: V \to \mathbb{R}$ so that $\tilde{\ell}|_W = \ell$ and $\|\tilde{\ell}\| = \|\ell\|$.

The proofs of Propositions 3.6 and 3.7 are also deferred to a later point.

Assuming Propositions 3.5 - 3.7, we can now state and prove the main Theorem of this section:

<u>Theorem 3.8</u> (The Hodge Decomposition Theorem) For every $k \in \{0, .., n\}$, $n = \dim M$, the space $\mathcal{H}^k \subset \Omega^k(M;\mathbb{R})$ of harmonic k-forms on M is finite-dimensional. Moreover, we have the following orthogonal sum decompositions of $\Omega^k(M;\mathbb{R})$:

$$\Omega^k(M;\mathbb{R}) = \Delta\left(\Omega^k(M;\mathbb{R})\right) \oplus \mathcal{H}^k$$
$$= d\left(\Omega^{k-1}(M;\mathbb{R})\right) \oplus \delta\left(\Omega^{k+1}(M;\mathbb{R})\right) \oplus \mathcal{H}^k.$$

In particular, for $\omega \in \Omega^k(M;\mathbb{R})$ the equation $\Delta\alpha = \omega$ has a solution $\alpha \in \Omega^k(M;\mathbb{R})$ if and only if $\langle \omega, \beta \rangle = 0$ for all $\beta \in \mathcal{H}^k$.

<u>Proof</u>: The fact that $\mathcal{H}^k$ is finite-dimensional immediately follows from Proposition 3.5 as otherwise $\mathcal{H}^k$ would contain a sequence $(\alpha_j)_{j \in \mathbb{N}}$

so that $\alpha_1, \alpha_2, \ldots$ are pairwise orthonormal and such a sequence cannot have a Cauchy subsequence.

To prove the claimed orthogonal decompositions, it suffices to check only the first one: $\Omega^k(M;\mathbb{R}) = \Delta(\Omega^k(M;\mathbb{R})) \oplus H^k$. Indeed, using Corollary 3.4 and the definition of Δ, this last assertion implies that

$d(\Omega^{k-1}(M;\mathbb{R})) = d\delta(\Omega^k(M;\mathbb{R}))$ and $\delta(\Omega^{k+1}(M;\mathbb{R})) = \delta d(\Omega^k(M;\mathbb{R}))$. Since $\langle d\alpha, \delta\beta\rangle = \langle d^2\alpha, \beta\rangle = 0$ for all $\alpha \in \Omega^{k-1}(M;\mathbb{R})$, $\beta \in \Omega^{k+1}(M;\mathbb{R})$ by the proof of Proposition 3.3, we conclude

$$\Delta(\Omega^k(M;\mathbb{R})) = d\delta(\Omega^k(M;\mathbb{R})) \oplus \delta d(\Omega^k(M;\mathbb{R})) = d(\Omega^{k-1}(M;\mathbb{R})) \oplus \delta(\Omega^{k+1}(M;\mathbb{R})).$$

Since $H^k \subset \Omega^k(M;\mathbb{R})$ is finite-dimensional, there is an orthogonal sum decomposition

$$\Omega^k(M;\mathbb{R}) = (H^k)^\perp \oplus H^k$$

and we must show that $(H^k)^\perp =$

$\Delta(\Omega^k(M;\mathbb{R}))$. The inclusion $\Delta(\Omega^k(M;\mathbb{R})) \subseteq (H^k)^\perp$ readily follows from $\langle \Delta\omega, \alpha\rangle = \langle \omega, \Delta\alpha\rangle = 0$ for $\omega \in \Omega^k(M;\mathbb{R})$ and $\alpha \in H^k$. We now show the converse inclusion $(H^k)^\perp \subseteq \Delta(\Omega^k(M;\mathbb{R}))$. To this end, we first prove:

<u>Lemma 3.9</u> There exists a constant $C \in \mathbb{R}$, so that $\|\beta\| \leq C\|\Delta\beta\|$ for all $\beta \in (H^k)^\perp$.

<u>Proof of Lemma 3.9</u>: Assume otherwise. Then there exist a sequence $(\beta_j)_{j\in\mathbb{N}} \subset (H^k)^\perp$ so that $\|\beta_j\| = 1$ for all $j \in \mathbb{N}$ and $\|\Delta\beta_j\| \xrightarrow[j\to\infty]{} 0$. Applying Proposition 3.6, a subsequence of $(\beta_j)_{j\in\mathbb{N}}$, denoted again by $(\beta_j)_{j\in\mathbb{N}}$, is a Cauchy sequence and thus $\ell(\beta) := \lim_{j\to\infty} \langle \beta_j, \beta\rangle$ is a well-defined bounded linear functional

$$\ell : \Omega^k(M;\mathbb{R}) \to \mathbb{R}.$$

Since

$$\ell(\Delta\beta) = \lim_{j\to\infty} \langle \beta_j, \Delta\beta \rangle =$$

$$\lim_{j\to\infty} \langle \Delta\beta_j, \beta \rangle = 0 \quad \text{for all } \beta,$$

ℓ is a weak solution of $\Delta\alpha = 0$.
Applying Proposition 3.5, there exists $\alpha \in \Omega^k(M;\mathbb{R})$, so that $\ell(\beta) = \langle \alpha, \beta \rangle$ for all β. Since $\langle \alpha - \beta_j, \beta \rangle \xrightarrow[j\to\infty]{} 0$ for all β, we have $\beta_j \xrightarrow[j\to\infty]{} \alpha$ and from $(\beta_j)_{j\in\mathbb{N}} \subset (H^k)^\perp$ and $\|\beta_j\| = 1$ we conclude $\alpha \in (H^k)^\perp$ and $\|\alpha\| = 1$.
This is a contradiction to the fact that α is a solution of $\Delta\alpha = 0$.
$\square$

Let now $\omega \in (H^k)^\perp$. Define $\ell : \Delta(\Omega^k(M;\mathbb{R})) \to \mathbb{R}$ by

$$\ell(\Delta\beta) := \langle \omega, \beta \rangle \quad \text{for } \beta \in \Omega^k(M;\mathbb{R}).$$

As $\langle \omega, \underbrace{\beta_1 - \beta_2}_{\in H^k} \rangle = 0$ for $\Delta\beta_1 = \Delta\beta_2$, ℓ is well-defined. It follows from Lemma 3.9 that ℓ is bounded: denoting for $\beta \in \Omega^k(M;\mathbb{R})$ by $\beta^\perp$

the orthogonal projection to $(H^k)^\perp$, we have

$$|\ell(\Delta\beta)| = |\ell(\beta^\perp)| = |\langle \omega, \beta^\perp \rangle|$$
$$\leq \|\omega\| \cdot \|\beta^\perp\| \leq C\|\omega\| \cdot \|\Delta\beta^\perp\| =$$
$$C\|\omega\| \cdot \|\Delta\beta\|.$$

Applying the Hahn-Banach theorem, it follows that ℓ extends to a bounded linear functional $\tilde{\ell} : \Omega^k(M;\mathbb{R}) \to \mathbb{R}$. By construction, $\tilde{\ell}$ is a weak solution of $\Delta\alpha = \omega$ and thus applying Proposition 3.5, there exists $\alpha \in \Omega^k(M;\mathbb{R})$ so that $\Delta\alpha = \omega$. Thus $\omega \in \Delta(\Omega^k(M;\mathbb{R}))$. This completes the proof of Theorem 3.8. $\square$

It follows from the Hodge decomposition theorem that every de Rham cohomology class contains a unique harmonic representative. In order to make this explicit, define the <u>Green's operator</u>

$G: \Omega^k(M;\mathbb{R}) \to (H^k)^\perp$ by prescribing $\alpha = G(\omega) \in (H^k)^\perp$ to be the unique solution of $\Delta\alpha = \omega^\perp$.

<u>Proposition 3.10</u> G commutes with every linear operator $T: \Omega^k(M;\mathbb{R}) \to \Omega^l(M;\mathbb{R})$ which commutes with Δ. In particular, G commutes with d, δ and Δ.

<u>Proof</u>: By definition,

$$G = \left(\Delta|_{(H^k)^\perp}\right)^{-1} \circ \pi_{(H^k)^\perp} \cdot \quad (*)$$

Since $T\Delta = \Delta T$, $T(H^k) \subseteq H^k$ and since $(H^k)^\perp = \Delta(\Omega^k(M;\mathbb{R}))$, $T((H^k)^\perp) \subseteq (H^k)^\perp$. Thus

$$T \circ \pi_{(H^k)^\perp} = \pi_{(H^k)^\perp} \circ T \quad (**) \text{ and}$$

$$T \circ \left(\Delta|_{(H^k)^\perp}\right) = \left(\Delta|_{(H^k)^\perp}\right) \circ T.$$

From the latter equality we obtain

$$T \circ \left(\Delta|_{(H^k)^\perp}\right)^{-1} = \left(\Delta|_{(H^k)^\perp}\right)^{-1} \circ T \quad (***).$$

Combining $(*)$, $(**)$ and $(***)$, the claim follows. $\square$

Recall from Section 1 that the de Rham cohomology of M is given by

$$H^k_{dR}(M;\mathbb{R}) = \ker d_k / \operatorname{im} d_{k-1}$$

where $\ker d_k$ and $\operatorname{im} d_k$ are the kernel resp. the image of d on $\Omega^k(M;\mathbb{R})$.

<u>Proposition 3.11</u> Every de Rham cohomology class on a closed oriented manifold contains a unique harmonic form.

<u>Proof</u>: It follows from the definition of the Green's operator and the Hodge decomposition Theorem that for every $\alpha \in \Omega^k(M;\mathbb{R})$

$$\alpha^\perp = \Delta(G\alpha) = d\delta G\alpha + \delta d G\alpha.$$

Here $\alpha^\perp$ denotes the orthogonal projection of α to the complement $(\mathcal{H}^k)^\perp \subset \Omega^k(M;\mathbb{R})$ of $\mathcal{H}^k$.

Using Proposition 3.10, we conclude

$$\alpha^\perp = d\delta G\alpha + \delta G d\alpha$$

and thus for α closed,

$$\alpha^\perp = d\delta G\alpha.$$

We conclude that
$$H(\alpha) := \alpha - \alpha^{\perp} = \alpha - d\delta G\alpha$$
is a harmonic form in the de Rham cohomology class of α.
Now suppose that two harmonic k-forms β_1 and β_2 lie in the same de Rham cohomology class, i.e. there exists a $(k-1)$-form γ with $\beta_1 - \beta_2 = d\gamma$. Then
$$\langle (\beta_1 - \beta_2), d\gamma \rangle = \langle \delta\beta_1 - \delta\beta_2, \gamma \rangle = \langle 0, \gamma \rangle = 0$$
and thus $d\gamma = 0$, i.e. $\beta_1 = \beta_2$.
This proves the uniqueness part of the claim. $\square$

In particular, using the fact following from the Hodge decomposition theorem that the space $\mathcal{H}^k$ of harmonic k-forms is finite-dimensional, we conclude:

<u>Corollary 3.12</u> The de Rham cohomology groups of a closed oriented manifold are finite-dimensional.

As another application, we can use the Hodge decomposition theorem in order to describe Poincaré duality on de Rham cohomology. To this end, define a bilinear form

$$H^k_{dR}(M;\mathbb{R}) \times H^{n-k}_{dR}(M;\mathbb{R}) \longrightarrow \mathbb{R}$$

$$([\alpha],[\beta]) \longmapsto \int_M \alpha \wedge \beta. \qquad (*)$$

The fact that the form is well-defined follows from Stokes' Theorem: If $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, then $\alpha - \alpha' = dw$ and $\beta - \beta' = d\sigma$ for some $w \in \Omega^{k}(M;\mathbb{R})$ and $\sigma \in \Omega^{n-k}(M;\mathbb{R})$ and we have

$$\int_M \alpha \wedge \beta = \int_M \alpha' \wedge \beta' + \int_M \alpha' \wedge d\theta + \int_M dw \wedge \beta'$$

$$+ \int_M d\sigma \wedge dw = \int_M \alpha' \wedge \beta' + \int_M d(\alpha' \wedge \theta + w \wedge \beta'$$

$$+ \sigma \wedge dw) = \int_M \alpha' \wedge \beta',$$ where we have used in the second equality the fact that $d\alpha = d\alpha' = 0$.

Proposition 3.13 The bilinear form $(*)$ is non-singular, i.e. the map

$$H^k_{dR}(M;\mathbb{R}) \longrightarrow \left(H^{n-k}_{dR}(M;\mathbb{R})\right)^*$$

$$[\alpha] \mapsto \left([\beta] \mapsto \int_M \alpha \wedge \beta \right)$$

defines an isomorphism between $H^k_{dR}(M;\mathbb{R})$ and the dual vector space of $H^{n-k}_{dR}(M;\mathbb{R})$.

Proof: Let $[\alpha] \in H^k_{dR}(M;\mathbb{R})$ be a non-zero de Rham cohomology class. Using Proposition 3.11, there exists a unique harmonic k-form $H(\alpha) \in \mathcal{H}^k$ with $[H(\alpha)] = [\alpha]$. Since $[\alpha] \neq 0$, we have $H(\alpha) \neq 0$. Consider $\beta := *\alpha \in \Omega^{n-k}(M;\mathbb{R})$. Using Proposition 3.3, $\Delta\beta = \Delta * \alpha = * \Delta\alpha = 0$ and in particular, by Corollary 3.4 $d\beta = 0$. Now we compute

$$\int_M \alpha \wedge \beta = \int_M \alpha \wedge * \alpha = \|\alpha\|^2 \neq 0.$$

It follows that the form $(*)$ is

non-singular.

$\square$

Finally, let us remark that the Hodge decomposition theorem implies an index formula. To this end, denote

$$\Omega^{even}(M;\mathbb{R}) := \bigoplus_{\substack{0\leq k\leq n\\ k \text{ even}}} \Omega^k(M;\mathbb{R})$$

and

$$\Omega^{odd}(M;\mathbb{R}) := \bigoplus_{\substack{0\leq k\leq n\\ k \text{ odd}}} \Omega^k(M;\mathbb{R})$$

and consider

$$(d+\delta) : \Omega^{even}(M;\mathbb{R}) \to \Omega^{odd}(M;\mathbb{R}).$$

<u>Proposition 3.14</u> The kernel and the cokernel of $(d+\delta)$ are finite-dimensional and with $\operatorname{ind}(d+\delta) := \dim\ker(d+\delta) - \dim\operatorname{coker}(d+\delta)$, we have

$$\operatorname{ind}(d+\delta) = \chi(M) := \sum_{k=0}^{n}(-1)^k b_k,$$

where $b_k = \dim H^k_{dR}(M;\mathbb{R})$.
Here $\operatorname{coker}(d+\delta) = \Omega^{odd}(M;\mathbb{R}) / \operatorname{im}(d+\delta)$.

Proof: Using Corollary 3.4,

$$\ker(d+\delta) = \bigoplus_{\substack{0 \le k \le n \\ k \text{ even}}} \mathcal{H}^k .$$

On the other hand, it follows from the Hodge decomposition theorem that the image of $(d+\delta)$ coincides with the image of

$$\Delta : \Omega^{\text{even}}(M;\mathbb{R}) \to \Omega^{\text{even}}(M;\mathbb{R})$$

and thus

$$\operatorname{coker}(d+\delta) \cong \bigoplus_{\substack{0 \le k \le n \\ k \text{ odd}}} \mathcal{H}^k .$$

The conclusion now follows since

$$\dim \mathcal{H}^k = \dim H^k_{dR}(M;\mathbb{R})$$

by Proposition 3.11. $\square$

3.2 Sobolev spaces

The purpose of this section is to establish the prerequisits needed for the proofs of Propositions 3.5 and 3.6. The proofs will work by reducing the statements from the case of a general manifold M to the local model of Euclidean space $\mathbb{R}^4$.

Let us denote by $\mathcal{P}$ the space of all smooth functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ that are 2π-periodic in every variable. Recall that for an n-tuple $\xi = (\xi_1, \dots, \xi_n)$ of integers the <u>Fourier coefficient</u> of $\varphi \in \mathcal{P}$ corresponding to ξ is given by

$$\varphi_\xi := \frac{1}{(2\pi)^n} \int_{(0,2\pi)^n} \varphi(x)\, e^{-ix\cdot\xi}\, dx .$$

Recall that the <u>Fourier series</u> $\sum_\xi \varphi_\xi \cdot e^{ix\cdot\xi}$ converges uniformly to φ. Moreover, for every $s \in \{0, 1, 2, \dots\}$ there exists a constant $C \in \mathbb{R}$ (depending on s and n), so that for all $\varphi \in \mathcal{P}$ we have the estimate

$$c \sum_{\xi} (1+|\xi|^2)^s |\varphi_\xi|^2 \le \sum_{\alpha_1+\dots+\alpha_n \le s} \left\| \frac{\partial^{\alpha_1+\dots+\alpha_n} \varphi}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right\|^2$$

$$\le \sum_{\xi} (1+|\xi|^2)^s \|\varphi_\xi\|^2$$

(see problem sheet 10).

It follows from the above that if S denotes the space of all sequences $(u_\xi \in \mathbb{C})_\xi$ of complex numbers indexed by n-tuples $\xi = (\xi_1, \dots, \xi_n)$ of integers, then $\varphi \longmapsto (\varphi_\xi)_\xi$ defines an embedding $\mathcal{P} \hookrightarrow S$.

We identify $\mathcal{P}$ with its image in S.

<u>Definition 3.15</u> For every integer $s \in \mathbb{Z}$, the <u>Sobolev space</u> H_s is the subspace of S given by

$$H_s = \left\{ u \in S \;\middle|\; \sum_{\xi} (1+|\xi|^2)^s |u_\xi|^2 < \infty \right\}.$$

We define an inner product on H_s by

$$\langle u, v \rangle_s := \sum_{\xi} (1+|\xi|^2)^s u_\xi \cdot v_\xi$$

for $u, v \in H_s$.

Note that we have $\mathcal{P} \subset H_s$ for every s. The key fact needed for the proof of the regularity statement made in Proposition 3.5 is that conversely, if $u \in S$ lies in H_s with s sufficiently large, then the Fourier series determined by u converges together with its derivatives up to a certain order.

<u>Proposition 3.16</u> (Sobolev Lemma)
Suppose that $u \in H_t$, where $t \geq \left[\frac{n}{2}\right] + 1 + m$ for some $m \in \mathbb{Z}, m \geq 0$. Then for all $\alpha = (\alpha_1, .., \alpha_n)$ with $\alpha_1 + .. + \alpha_n \leq m$ the series

$$D^\alpha u := \sum_\xi \xi^\alpha u_\xi e^{ix\cdot\xi}$$

converges uniformly. Here $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

<u>Remark</u>: Note that for $\varphi \in \mathcal{P}$, we have $(D^\alpha\varphi)_\xi = D^\alpha(\varphi)_\xi$ for all α; Moreover if $\varphi: \mathbb{R}^n \to \mathbb{R}$ where $D^\alpha\varphi = \frac{\partial^{\alpha_1 + \ldots + \alpha_n}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}\varphi$

is any continuous 2π - periodic function such that $\sum_{\xi} \xi^\alpha \varphi_\xi e^{ix\cdot\xi}$ converges uniformly for $\alpha_1 + \dots + \alpha_n \leq m$, then $\varphi \in C^m$ with $D^\alpha \varphi = \sum_{\xi} \xi^\alpha \varphi_\xi e^{ix\cdot\xi}$. Thus Proposition 3.16 implies that every element $u \in H_t$ for $t \geq \left[\frac{n}{2}\right] + 1 + m$ has the form $u_\xi = \varphi_\xi$ for some $\varphi \in C^m(\mathbb{R}^n, \mathbb{R})$.

Proof of Proposition 3.16

We first consider the case $m = 0$. In order to show uniform convergence of $\sum_{\xi} u_\xi e^{ix\cdot\xi}$, it suffices to check that $\sum_{\xi} |u_\xi| < \infty$. Using Cauchy - Schwarz, we have for $N \in \mathbb{N}$

$$\sum_{|\xi| < N} |u_\xi| = \sum_{|\xi| < N} (1 + |\xi|^2)^{-t/2} (1 + |\xi|^2)^{t/2} |u_\xi|$$

$$\leq \left(\sum_{|\xi| < N} (1 + |\xi|^2)^{-t} \right)^{1/2} \left(\sum_{|\xi| < N} (1 + |\xi|^2)^{t} |u_\xi|^2 \right)^{1/2}$$

$$\leq \left(\sum_{|\xi| < N} (1 + |\xi|^2)^{-t} \right)^{1/2} \|u\|_t \text{, where}$$

$$\|u\|_t = \sqrt{\langle u,u\rangle_t} = \Big(\sum_\xi (1+|\xi|^2)^t |u_\xi|^2\Big)^{1/2} < \infty.$$

Now it remains to check that

$$\sum_\xi (1+|\xi|^2)^{-t} < \infty \quad \text{for} \quad t \geq \left[\frac{n}{2}\right]+1.$$

Recall that the sum is over all tuples $\xi = (\xi_1, .., \xi_n) \in \mathbb{Z}^n$.

Denoting for $j \in \mathbb{Z}$, $j \geq 0$

$$S_j := \{(\xi_1,..,\xi_n) \in \mathbb{Z}^n \mid \max_{1\leq i\leq n} |\xi_i| = j\},$$

we have $|S_j| \leq 2n(2j+1)^{n-1}$.

Moreover, since $|\xi|^2 \geq j^2$ for $\xi \in S_j$,

$$\sum_{\xi\in S_j} (1+|\xi|^2)^{-t} \leq \frac{2n(2j+1)^{n-1}}{(1+j^2)^t} \leq C j^{n-1-2t}$$

for some constant C dependending only on n. Thus

$$\sum_\xi (1+|\xi|^2)^{-t} \leq 1 + C\sum_{j=1}^{\infty} j^{n-1-2t} < \infty$$

for $2t+1-n > 1$, i.e. $t \geq \left[\frac{n}{2}\right]+1$.

To conclude the case of a general $m \geq 0$, we use the fact that $D^\alpha : (u_\xi)_\xi \mapsto (\xi^\alpha u_\xi)_\xi =: D^\alpha (u_\xi)_\xi$

for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ and with $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ defines a bounded linear operator

$$D^\alpha : H_{s+[\alpha]} \to H_s,$$

where $[\alpha] = \alpha_1 + \dots + \alpha_n$, as follows from the estimate

$$\sum_\xi (1+|\xi|^2)^s |\xi^\alpha u_\xi|^2 =$$

$$\sum_\xi (1+|\xi|^2)^s |\xi|^{2\alpha} |u_\xi|^2 \leq \sum_\xi (1+|\xi|^2)^s (1+|\xi|^2)^{[\alpha]} |u_\xi|^2$$

$$= \sum_\xi (1+|\xi|^2)^{s+[\alpha]} |u_\xi|^2.$$

It follows that if $t \geq \left[\frac{n}{2}\right] + 1 + m$, then $D^\alpha u \in H_{\underbrace{t-[\alpha]}_{\geq [\frac{n}{2}]+1}}$ for $[\alpha] \leq m$ and thus by the previous step, $\sum_\xi \xi^\alpha u_\xi e^{ix\cdot\xi}$ converges uniformly. $\square$

<u>Corollary 3.17</u> For $t \geq \left[\frac{n}{2}\right] + 1$, there exists a constant C, so that

$$\|D^\alpha \varphi\|_\infty \leq C \|\varphi\|_{t+[\alpha]}$$

for all $\varphi \in \mathcal{P}$, $\alpha \in \mathbb{Z}_{\geq 0}^n$.

Proof: Using the proof of Proposition 3.16 and the fact that $\sum_\xi \varphi_\xi e^{ix\cdot\xi}$ converges uniformly to φ for every $\varphi \in \mathcal{P}$, we conclude

$$\|\varphi\|_\infty \leq C \, \|\varphi\|_t$$

for $t \geq \left[\frac{n}{2}\right] + 1$ with some $C > 0$.
The claim follows using the fact that $D^\alpha : H_{s+[\alpha]} \to H_s$ is bounded.

$\square$

It is immediate from the definition of the Sobolev spaces that for $s \leq t$, there is a natural embedding $H_t \hookrightarrow H_s$. An important fact, known as the Rellich Lemma, is that for $s < t$, this embedding is compact.

Proposition 3.18 (Rellich Lemma)
Let $(u_k)_{k\in\mathbb{N}} \subset H_t$ be a sequence with $\|u_k\|_t \leq 1$ for all k.
Then for every $s < t$, there exists a subsequence of $(u_k)_{k\in\mathbb{N}}$ which converges in H_s.

Proof: It follows from

$$\sum_\xi (1+|\xi|^2)^t \, |(u_k)_\xi|^2 \leq 1$$

that $(1+|\xi|^2)^{t/2}|(u_k)_\xi| \leq 1$ for each fixed ξ and all $k\in\mathbb{N}$.
Thus for every ξ, the sequence $\left((1+|\xi|^2)^{t/2}(u_k)_\xi\right)_{k\in\mathbb{N}} \subset \mathbb{C}$ has a convergent subsequence. Using a standard diagonalization argument, we may assume, after passing to a subsequence, that $((1+|\xi|^2)^{t/2}(u_k)_\xi)_{k\in\mathbb{N}}$ converges in $\mathbb{C}$ for every ξ.
We claim that $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in H_s for $s<t$.
We have for $N\in\mathbb{N}$

$$\|u_k-u_j\|_s^2 = \sum_{|\xi|<N}(1+|\xi|^2)^{s-t}(1+|\xi|^2)^t |(u_k)_\xi-(u_j)_\xi|^2 + \sum_{|\xi|\geq N}(1+|\xi|^2)^{s-t}(1+|\xi|^2)^t |(u_k)_\xi-(u_j)_\xi|^2 =: I + II.$$

We have

$$II \leq N^{2(s-t)}\sum_{|\xi|\geq N}(1+|\xi|^2)^t\left(|(u_k)_\xi|^2 + 2|(u_k)_\xi||(u_j)_\xi| + |(u_j)_\xi|^2\right) \leq 4N^{2(s-t)},$$

where the last inequality follows

using the assumption $\|u_k\|_t \leq 1$.
Now given $\varepsilon > 0$, fix $N_0 \in \mathbb{N}$ with $4 N_0^{2(s-t)} < \frac{\varepsilon}{2}$. Then for $N = N_0$ we have $II < \varepsilon/2$ and

$$I \leq \sum_{|\xi| < N_0} (1+|\xi|^2)^t \, |(u_k)_\xi - (u_j)_\xi|^2 .$$

Since the latter is a sum over finitely many ξ and since $((u_k)_\xi)_{k \in \mathbb{N}}$ converges for every fixed ξ, there exists $K \in \mathbb{N}$ with $I \leq \varepsilon/2$ for $j, k \geq K$. It follows that $\|u_j - u_k\|_s^2 < \varepsilon$ for $j, k \geq K$. Thus $(u_k)_{k \in \mathbb{N}} \subset H_s$ is a Cauchy sequence.

Finally, note that each H_s is a Hilbert space as the map $(u_\xi)_\xi \longmapsto ((1+|\xi|^2)^{-t/2} u_\xi)$ defines an isomorphism between $(H_s, \langle\cdot,\cdot\rangle_s)$ and the Hilbert space of all sequences $(w_\xi \in \mathbb{C})_{\xi \in \mathbb{Z}^n}$ with $\sum_\xi |w_\xi|^2 < \infty$.

In order to approach the applications to the Laplace - Beltrami operator, we next introduce the notion of a differential operator.

Definition 3.19 A (linear) differential operator L of order ℓ acting on $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ is given by an $m \times m$-matrix $(L_{ij})_{i,j=1}^{m}$ with entries of the form

$$L_{ij} = \sum_{[\alpha]=0}^{\ell} a_{ij}^{\alpha} D^{\alpha},$$

where the sum is over all tuples $\alpha = (\alpha_1, .., \alpha_n) \in \mathbb{Z}_{\geq 0}^{n}$, $[\alpha] = \alpha_1 + .. + \alpha_n$, $D^\alpha = \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} .. \partial x_n^{\alpha_n}}$ and $a_{ij}^{\alpha} \in C^\infty(\mathbb{R}^n, \mathbb{R})$ are functions with $a_{ij}^{\alpha} \not\equiv 0$ for some $i,j \in \{1, \dots, m\}$ with $[\alpha] = \ell$.

We say that L is periodic if a_{ij}^{α} are 2π-periodic in every variable.

Denote by $\mathcal{P}^m$ the space of all smooth functions $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ that are 2π-periodic in every variable.

If L is a periodic linear differential, then L acts on $\mathcal{P}^m$ by

$$L\varphi := \left(\sum_j L_{1j}\varphi_j, \dots, \sum_j L_{mj}\varphi_j \right).$$

We define the <u>formal adjoint</u> of L by $L^*_{ij} := \sum_{|\alpha|=0}^{\ell} D^\alpha a^\alpha_{ji}$, it follows using integration by parts that

$$\langle L\varphi, \psi \rangle = \langle \varphi, L^*\psi \rangle$$

for all $\varphi, \psi \in \mathcal{P}^m$.

The following Proposition, whose proof is left as an exercise (see problem sheet 11), shows that a differential operator extends to a bounded operator between (suitable) Sobolev spaces.

<u>Proposition 3.20</u> Let L be a periodic differential operator of order ℓ and $s \in \mathbb{Z}$. There exists constants c and C', where c depends on n, m, ℓ, s and the absolut values of the top order coefficients of L, and where c' depends on n, m, ℓ, s and the coefficients of L aswell as their derivatives up to order ℓ, so that

$$\|L\varphi\|_s \le C\|\varphi\|_{s+l} + C'\|\varphi\|_{s+l-1}$$

for all $\varphi \in \mathcal{D}^m$. In particular, $\|L\varphi\| \le C''\|\varphi\|_{s+l}$ for all $\varphi \in \mathcal{D}^m$ with some constant C''. Thus L extends to a bounded linear operator $H_{s+l} \to H_s$.

We are interested in a particular class of differential operators. Given a differential operator L acting on $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$, we write

$$L = P_l(D) + \dots + P_0(D),$$

where $P_j(D)$ is a differential operator of the form $P_j(D) = \sum_{[\alpha]=j} a^\alpha D^\alpha$, $a^\alpha \in C^\infty(\mathbb{R}^n, \mathbb{R})$. For $\xi \in \mathbb{R}^n$, we denote by $P_j(\xi)$ the matrix obtained by substituting $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ for D^α, i.e. $P_j(\xi) = \sum_{[\alpha]=j} a^\alpha \xi^\alpha \in \mathbb{R}^{m\times m}$.

<u>Definition 3.21</u> We say that L is <u>elliptic</u> at the point $x \in \mathbb{R}^n$ if

the matrix $P_L(\xi) \in \mathbb{R}^{m \times m}$ is non-singular at x. L is called elliptic if L is elliptic at every point $x \in \mathbb{R}^n$.

Define the symbol of L as the map σ_L which assigns to every vector $\xi \in \mathbb{R}^n$

$$\sigma_L(\xi) \in C^\infty(\mathbb{R}^n, \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m))$$

given by

$$\sigma_L(\xi)(v)_x = L(\varphi^e u)(x)$$

where $u : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function with $u(x) = v$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ a smooth function such that $\varphi(x) = 0$ and $d\varphi(x) = \xi$. The symbol is well-defined (i.e. $\sigma_L(\xi)(v)$ is independent of the choice of u and φ) and L is elliptic at x if and only if $\sigma_L(\xi)_x : \mathbb{R}^m \to \mathbb{R}^m$ is a vector space isomorphism for every $\xi \neq 0$ (see problem sheet 11).

Elliptic differential operators have a special useful property with regard to their action on the Sobolev spaces:

<u>Proposition 3.22</u> Let L be a periodic differential operator of order ℓ acting on $\mathcal{P}^m$. There exists a constant $C > 0$ such that

$$\|u\|_{s+\ell} \leq C\left(\|Lu\|_s + \|u\|_s\right)$$

for all $u \in H_{s+\ell}$. ("fundamental inequality")

<u>Proof</u>: Using the fact that $\mathcal{P}$ is dense in H_s, it suffices to prove the desired estimate for all $u = \varphi \in \mathcal{P}^m$. We first consider the case when $L = L_0$ has constant coefficients and $L_0 = P_\ell(D)$.

Using the assumption of ellipticity, $\|P_\ell(\xi)v\|^2 > 0$ for $\xi, v \in \mathbb{R}^n \setminus \{0\}$.

Using compactness of the unit sphere $S^{n-1} \subset \mathbb{R}^n$, there exists $\tilde{C} > 0$ such that

$$\|P_\ell(\xi)v\|^2 \geq \tilde{C} \quad \text{for all}$$

$\xi, v \in \mathbb{R}^n$, with $\|v\| = \|\xi\| = 1$.
we conclude

$$\| P_\ell(\xi) v\|^2 \geq \tilde{C} \ \|\xi\|^{2\ell} \ \|v\|^2$$

for all $u, \xi \in \mathbb{R}^n$. Using the assumption that $L = L_0$ has constant coefficients, we conclude for $\varphi \in \mathcal{P}$

$$\|L_0 \varphi\|_s^2 = \sum_\xi \|P_\ell(\xi) \varphi_\xi\|^2 \ (1 + |\xi|^2)^s$$

$$\geq \tilde{C} \ \sum_\xi |\xi|^{2\ell} \ |\varphi_\xi|^2 (1 + |\xi|^2)^s$$ and

hence $\left(\|L_0 \varphi\|_s + \|\varphi\|_s\right)^2 \geq \|L_0 \varphi\|_s^2 +$

$$\|\varphi\|_s^2 \geq \sum_\xi |\varphi_\xi|^2 \ (1 + |\xi|^2)^s \ (1 + \tilde{C} |\xi|^{2\ell})$$

$$\geq C' \ \sum_\xi |\varphi_\xi|^2 \ (1 + |\xi|^2)^{s+\ell} = C' \|\varphi\|_{s+\ell}^2.$$

Consider now the case of a general periodic elliptic operator L of order ℓ. We first prove that for every $x \in \mathbb{R}^n$ there exists a neighbourhood $\mathcal{U} \subset \mathbb{R}^n$ of x, so that the claim of the Proposition holds for all $\varphi \in \mathcal{P}$ with $\operatorname{supp} \varphi \subset T(\mathcal{U})$, where $T(\mathcal{U}) = \{2\pi\xi + p \mid p \in \mathcal{U}, \xi \in \mathbb{Z}^n\}$ is the set of all periodic translates of elements of $\mathcal{U}$.

Denoting by $L_0 = (P_L)_x(D)$ the constant coefficient elliptic operator which is homogeneous of order ℓ with coefficients given by the top-order coefficients of L at the point x, we have by the previous step

$$\|\varphi\|_{s+\ell} \leq C\left(\|L_0\varphi\|_s + \|\varphi\|_s\right) \leq$$
$$C\left(\|L\varphi\|_s + \|(L_0-L)\varphi\|_s + \|\varphi\|_s\right). \quad (*)$$

Recall from Proposition 3.20 that

$$\|L\varphi\|_s \leq k\|\varphi\|_{s+\ell} + k'\|\varphi\|_{s+\ell-1} \quad (**)$$

for some constants k, k' independent of φ. Let $\varepsilon < \frac{1}{2Ck}$. Choose a neighbourhood W of x, so that the coefficients of the highest-order part of $L_0 - L$ have absolute value less than ε on W. Let $\tilde{L}$ be a periodic operator agreeing with $L_0 - L$ on some neighbhd U of x with $U \subseteq W$ and with highest order coefficients bounded by ε in absolute value everywhere. It follows from $(*)$ and $(**)$ by the choice of ε that for every $\varphi \in \mathcal{P}$ with support in $T(\mathcal{U})$

$$\|\varphi\|_{s+l} \leq C\left(\|L\varphi\|_s + \|\tilde{L}\varphi\|_s + \|\varphi\|_s\right)$$

$$\leq C\left(\|L\varphi\|_s + \frac{1}{2C}\|\varphi\|_{s+l} + k\|\varphi\|_{s+l-1} + \|\varphi\|_s\right).$$

In order to estimate $\|\varphi\|_{s+l-1}$, use the following result (see problem sheet 11)

Lemma 3.23 (Peter-Paul inequality)
For $t' < t < t''$ and $\varepsilon > 0$, there exists $C(\varepsilon) > 0$, so that

$$\|u\|_t^2 \leq \varepsilon\|u\|_{t''}^2 + C(\varepsilon)\|u\|_{t'}^2$$

for all $u \in H_{t''}$.

We obtain an estimate

$$\|\varphi\|_{s+l} \leq C\|L\varphi\|_s + \frac{3}{4}\|\varphi\|_{s+l} + C_1\|\varphi\|_s$$

for some $C_1 > 0$. This completes the proof of the claim in the case $\operatorname{supp}\varphi \subset \mathcal{U}$, where $\mathcal{U} \subset \mathbb{R}^n$ is as constructed above.

To prove the general case, note that the image of $\mathcal{U}$ under the natural projection $\mathbb{R}^n \to \mathbb{R}^n/_{2\pi\cdot\mathbb{Z}^n} \cong T^n$ is an open subset of the n-dimensional torus and that due to compactness T^4 can be covered by finitely many open subsets $\mathcal{U}_1, .., \mathcal{U}_k$ of this form. Fix smooth functions $\sigma_1, .., \sigma_k \in C^\infty(T^4; \mathbb{R})$ with $\operatorname{supp} \sigma_j \subset \mathcal{U}_j$ and $\sum_{j=1}^{k} \sigma_j^2 = 1$.

One readily checks that there is an estimate of the form

$$| \langle \sigma u, v \rangle_s - \langle u, \sigma v \rangle_s | \leq C_2 (\|u\|_s \|v\|_{s-1} + \|u\|_{s-1} \|v\|_s)$$

for all $u, v \in H_s$, $\sigma \in C^\infty(\mathbb{R}^4, \mathbb{R})$. It follows that

$$\|\varphi\|_{s+l}^2 = \langle \varphi, \varphi \rangle_{s+l} = \langle \sum_j \sigma_j^2 \varphi, \varphi \rangle_{s+l}$$

$$\leq \sum_j \langle \sigma_j \varphi, \sigma_j \varphi \rangle_{s+l} + C_3 \|\varphi\|_{s+l} \|\varphi\|_{s+l-1}$$

with some $C_3 > 0$.

Now since $\sigma_j \cdot \varphi$ has support in one of the subsets $\mathcal{U}$ constructed above, we obtain

$$\|\varphi\|_{s+l}^2 \leq C_4 \sum_j \| L(\sigma_j \varphi) \|_s^2 + C_5 \|\varphi\|_s^2 + C_3 \|\varphi\|_{s+l} \|\varphi\|_{s+l-1}$$

One can check that there is an estimate of the form

$$|\langle L(\nabla^2 u), Lu\rangle_s - \|L(\nabla u)\|_s^2| \leq$$

$$C_6 \cdot \|u\|_{s+\ell} \|u_{s+\ell-1}\|$$ for $u \in H^{s+\ell}$, $\nabla \in C^\infty(\mathbb{R}^4, \mathbb{R})$ and concludes

$$\|\varphi\|_{s+\ell}^2 \leq C_7 \sum_j \langle L(\nabla_j^2 \varphi), L\varphi\rangle_s$$

$$+ C_5 \|\varphi\|_s^2 + C_8 \|\varphi\|_{s+\ell} \|\varphi\|_{s+\ell-1}$$

$$= C_7 \|L\varphi\|_s^2 + C_5 \|\varphi\|_s^2 + C_8 \|\varphi\|_{s+\ell} \|\varphi\|_{s+\ell-1}$$

$$\leq C_7 \|L\varphi\|_s^2 + C_5 \|\varphi\|_s^2 + \frac{1}{2} \|\varphi_{s+\ell}\|^2 +$$

$$C_9 \|\varphi\|_{s+\ell-1}^2 \leq C_7 \|L\varphi\|_s^2 + C_9 \|\varphi\|_s^2 +$$

$$\frac{3}{4} \|\varphi_{s+\ell}\|^2,$$ where in the last inequality we used Lemma 3.23.

This completes the proof of Proposition 3.22. $\square$

We can now state and prove a statement about regularity of elliptic diff. operators in the periodic case.

Proposition 3.24 (Regularity of periodic elliptic differential operators).
Let L be a periodic differential operator of order l. Suppose that for some $u \in H_s$ $s \in \mathbb{Z}$, we have $Lu = v \in H_t$.
Then $u \in H_{t+l}$.

Proof: It suffices to check that if $u \in H_s$ and $v = Lu \in H_{s-l+1}$, then $u \in H_{s+1}$.
We consider for $h \in \mathbb{R}^n$, $h \neq 0$ the difference quotient
$$u^h(x) := \frac{u(x+h) - u(x)}{\|h\|}.$$
We want to show that under the assumptions of the Proposition, $\|u^h\|_s$ is bounded by a constant independent of s.
To this end, denote by L^h the differential operator obtained from L by replacing a^α by $(a^\alpha)^h_{(x)} = \dfrac{a^\alpha(x+h) - \alpha(x)}{|h|}$

Then for $u \in H_s$, $s \in \mathbb{Z}$

$$L(u^h) = (Lu)^h - L^h(T_h u)$$

where $(T_h u)(x) := u(x+h)$ for all x.

Using Proposition 3.22, we conclude

$$\|u^h\|_s \le C_1 \| L(u^h)\|_{s-\ell} + C_2 \|u^h\|_{s-\ell} \le C_1 \| (Lu)^h\|_{s-\ell} + C_1 \|L^h(T_h u)\|_{s-\ell} + C_2 \|u^h\|_{s-\ell}.$$

Due to periodicity, we have

$$\|L^h(T_h u)\|_{s-\ell} \le C_3 \|T_h(u)\|_s$$

for some constant C_3 independent of h. It is not difficult to see that $\|u^h\|_s \le \|u\|_{s+1}$ and $\|T_h(u)\|_s = \|u\|_s$. Thus we conclude

$$\|u^h\|_s \le C_1 \|Lu\|_{s-\ell+1} + C_4 \|u\|_s.$$

We claim that the fact that $\|u^h\|_s$ is bounded independently of h implies that $u \in H_{s+1}$. To this end, suppose that $\|u^h\|_s \le k$ for all h, where k is independent of h. Then by definition of $\|\cdot\|_s$,

$$\sum_\xi (1+|\xi|^2)^s |u_\xi|^2 \left| \frac{e^{ih\xi}-1}{|h|} \right|^2 \le k^2 \text{ for}$$

all $h \in \mathbb{R}^n$ (note that $(u^h)_\xi = \frac{e^{ih\xi}-1}{|h|} \cdot u_\xi$).

Putting $h := t \cdot e_i$, $t \in \mathbb{R}$, $1 \le i \le n$, we have

$$\left| \frac{e^{ih\xi}-1}{|h|} \right|^2 \xrightarrow[t \to 0]{} |\xi_i|^2$$

and thus we conclude

$$\sum_{|\xi| \le N} (1+|\xi|^2)^s |u_\xi|^2 |\xi_i|^2 \le k^2$$

for every $N \in \mathbb{N}$. Thus

$$\sum_{|\xi| \le N} (1+|\xi|^2)^{s+1} |u_\xi|^2 \le n k^2 + \|u\|_s^2$$

and since N was arbitrary, $u \in H_{s+1}$ follows. This completes the proof of Proposition 3.24.

<u>Proof of Proposition 3.5</u>

We must show that if $\ell : \Omega^k(M;\mathbb{R}) \to \mathbb{R}$ is a bounded linear functional so that $\ell(\Delta\varphi) = \langle f, \varphi \rangle$ for all $\varphi \in \Omega^k(M;\mathbb{R})$, then there exists $u \in \Omega^k(M;\mathbb{R})$ with $\ell(w) = \langle u, w \rangle$ for all $w \in \Omega^k(M;\mathbb{R})$.

Using partition of unity, it suffices to check that every point $x \in M$ has a neighbourhood $p \in U \subset M$

so that for $\omega \in \Omega^n(M;\mathbb{R})$ with $\operatorname{supp} \omega \subset \mathcal{U}$, $\ell(\omega) = \langle u_\ell, \omega \rangle$ for some $u_\ell \in \Omega^n(\mathcal{U};\mathbb{R})$. Working in local coordinates, we may assume $\mathcal{U} = \mathbb{R}^n$ and that the inner product on $\Omega^k(M;\mathbb{R})$ corresponds to $\langle \varphi, \psi \rangle_A = \int_{\mathbb{R}^n} \varphi^T A \psi \, dx =: \langle \varphi, A\psi \rangle$

for $\varphi, \psi \in \Omega^k_0(\mathbb{R}^n, \mathbb{R}) = C^\infty_0(\mathbb{R}^n, \mathbb{R}^m)$

$m = \binom{n}{k}$, where $A \in C^\infty(\mathbb{R}^n, \mathbb{R}^{m \times m})$.

Moreover, Δ corresponds to an second order elliptic differential operator L acting on $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ and ℓ corresponds to $\ell(A^{-1}\cdot) : C^\infty_0(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$. Fix $V \subset \mathbb{R}^n$ open, so that $\bar{V}$ is compact and denote $\tilde{\ell} := \ell|_{C^\infty_0(V, \mathbb{R}^m)}$. Then $\tilde{\ell} : C^\infty_0(V, \mathbb{R}^m) \to \mathbb{R}$ is bounded and using

$$\tilde{\ell}(L^* \varphi) = \ell(A^{-1} L^* \varphi) =$$

$$\ell(\Delta A^{-1}\varphi) = \langle f, A^{-1}\varphi \rangle_A = \langle f, \varphi \rangle$$

for all $\varphi \in C^\infty_0(V)$, $\tilde{\ell}$ is a weak solution to $Lu = f$

(Here $L^*\varphi = A\Delta A^{-1}\varphi$ is the formal adjoint of L w.r.t. $\langle\, ; \rangle_A$).
It follows from boundedness of $\tilde{\ell}$ and the Riesz representation theorem that there exists $\tilde{u} \in H_0$ s.t. $\tilde{\ell}(w) = \langle \tilde{u}, w \rangle$ for all $w \in H_0$.
Fix $W_0 \subset \mathbb{R}^n$ open with $p \in W_0$ and $\overline{W}_0 \subset U$ sufficiently small, so that there exists a periodic diff. operator $\tilde{L}$, s.t. $L|_{W_0} = \tilde{L}|_{W_0}$.
Choose $p \in W \subset \mathbb{R}^n$ open with $\overline{W} \subset W_0$ and open subsets $W_1, W_2, \ldots$ s.t. $\overline{W}_j \subset W_{j-1}$, $\overline{W} \subset W_j$ for all j. Moreover, for $j = 1, 2, \ldots$ let $\sigma_j \in C^\infty(\mathbb{R}^n, [0,1])$ be a fct with $\sigma_j|_{W_j} \equiv 1$, supp $\sigma_j \subset W_{j-1}$.
Consider $v_1 := \sigma_1 \tilde{u} \in H_0$. We have

$$\tilde{L} v_1 = w_1 \tilde{L} \tilde{u} + \Pi_1 \tilde{u}, \quad (*)$$

$$\Pi_1 = \tilde{L} w_1 - w_1 \tilde{L}.$$

We have $\sigma_1 \tilde{L} \tilde{u} = \sigma_1 f \in C_0^\infty(W_0) \subset H_s$ for every s. Now since Π_1 has order 1, we have $\Pi_1 \tilde{u} \in H_{-1}$. Thus the RHS of $(*)$ belongs to H_{-1}. It follows from Proposition 3.24 that $v_1 \in H_1$

No we take $v_2 := \sigma_2 \tilde{u}$. We have

$$\tilde{L} v_2 = \sigma_2 \tilde{L} \tilde{u} + \sigma\!\!\!/_2 \tilde{u} = w_2 \tilde{L}\tilde{u} + \sigma\!\!\!/_2 v_1$$

$$\sigma\!\!\!/_2 = \tilde{L}\sigma_2 - \sigma_2 \tilde{L} \qquad (**)$$

In the last equality we used the fact that $\sigma\!\!\!/_2$ has support in W_1 and $\tilde{u} = v_1$ on W_1).

Arguing as before, the RHS of $(**)$ lies in H_0, thus $v_2 \in H_2$. Continuing inductively, we conclude that $\tilde{u}|_W$ lies in H_s for every s. Applying the Sobolev embedding Theorem, it follows that $\tilde{u}|_W$ is smooth.

$\square$

Sketch of Proof of Proposition 3.6

Given a sequence $(\alpha_k)_{k\in\mathbb{N}} \subset \Omega^+(M;\mathbb{R})$ s.t. $(\alpha_k)_{k\in\mathbb{N}}$ and $(\Delta\alpha_k)_{k\in\mathbb{N}}$ are bounded, it follows from Proposition 3.22 that

$$\|\alpha_k\|_1 \leq C\,(\|\Delta\alpha_k\| + \|\alpha_k\|)$$

and thus $(\alpha_k)_{k\in\mathbb{N}}$ is bounded in H_1. Using Proposition 3.18, a subsequence of $(\alpha_k)_{k\in\mathbb{N}}$ converges in H_0.

Finally, we show that the Laplace-Beltrami operator Δ on a closed oriented Riemannian manifold is elliptic.
To this end, we must check that

$$\Delta(\varphi^2\alpha)_x \neq 0$$

for all $x \in M$, $\alpha \in \Omega^k(M;\mathbb{R})$ and $\varphi \in C^\infty(M;\mathbb{R})$, so that $\alpha(x) = 0$,

$\varphi(x) = 0$ and $d\varphi(x) =: \xi \neq 0$.

Since

$$\Delta\alpha = (-1)^{n(k+1)+1} d{*}d{*} + (-1)^{nk+1} {*}d{*}d,$$

we have using $\varphi(x) = 0$

$$(d{*}d{*})(\varphi^2\alpha)(x) = \left(d{*}d\left((\varphi^2){*}\alpha\right)\right)(x)$$
$$= \left(2d{*}(\varphi\, d\varphi\wedge{*}\alpha)\right)(x) = 2d\left(\varphi{*}(d\varphi\wedge{*}\alpha)\right)(x)$$
$$= 2\left(d\varphi\wedge{*}(d\varphi\wedge{*}\alpha)\right)(x) = 2\xi\wedge\left({*}(\xi\wedge{*}(\alpha(x)))\right).$$

Analogously,

$$({*}d{*}d)(\varphi^2\alpha)(x) = 2{*}\left(\xi\wedge\left({*}(\xi\wedge\alpha(x))\right)\right).$$

Thus $\Delta(\varphi^2\alpha)(x) =$

$$2\left[(-1)^{kn+1}{*}\left(\xi\wedge{*}(\xi\wedge(\cdot))\right) + (-1)^{n(k+1)+1}\xi\wedge\left({*}(\xi\wedge(\cdot))\right)\right](\alpha(x))$$

$=: (\dagger)$

One readily checks that the formal adjoint of

$$\xi\wedge(\cdot) : \Lambda^k(T_x^*M) \to \Lambda^{k+1}(T_x^*M)$$

is $(-1)^{kn} * (\xi \wedge *(\cdot))$. Denoting

$$A := \xi \wedge (\cdot) : \Lambda^k(T_x^* M) \longrightarrow \Lambda^{k+1}(T_x^* M)$$

and

$$B := \xi \wedge (\cdot) : \Lambda^{k-1}(T_x^* M) \to \Lambda^k(T_x^* M),$$

we have

$$(+) = -2 \cdot \left[A^* A + B B^* \right] (\alpha(x)).$$

Since $\langle (A^*A + BB^*)(\alpha(x)), \alpha(x) \rangle_{T_x^* M}$

$$= \langle A\alpha(x), A\alpha(x) \rangle_{T_x^* M} + \langle B^* \alpha(x), B^* \alpha(x) \rangle_{T_x^* M},$$

$\Delta(\varphi^2 \alpha)(x) = 0$ implies $A\alpha(x) = B^* \alpha(x) = 0$.
It is readily seen that kernel of A coincides with the image of B.
On the other hand, the adjoint B^* of B is injective on the image of B. Thus $A\alpha(x) = B^* \alpha(x) = 0$ implies $\alpha(x) = 0$. This establishes the ellipticity of Δ. $\square$