Morse Homotopy and Topological Conformal Field Theory

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July 22, 2013

Two main goals:

- Present a new construction which uses flow graphs to recover interesting information about a manifold.
- Give theorems explaining in algebraic language what is recovered.

Here flow graphs are the graph analogues of flow trajectories.

Fix a graph G and a for every edge e of G, a flow $\Phi_e(t, \cdot)$ on M. Want to study the space $\mathcal{M}_G$ of all continuous maps $\gamma : G \to M$ such that

$$\gamma_e(t_0 + t) = \Phi_e(t, \gamma_e(t_0)).$$

Classical Morse Theory	The Graph Approach
Gradient flow of a Morse function f	$\{\Phi_e\}_{e \in E(G)}$-different flows on M, chosen 'in general position'
Spaces of Flow Trajectories	Spaces $\mathcal{M}_G$ of Flow Graphs
The Morse Complex $C^*(f)$	Field-theoretic structures

Spaces of flow graphs were studied by Ralph Cohen and his collaborators (Betz&Cohen (1994), Cohen&Godin (2004), Cohen&Norbury (2012)).

An alternative approach was pursued by K. Fukaya (1993, 1996).

Given a graph (i. e. one-dimensional CW-complex), partition the univalent vertices into n_+ *inputs* and n_- *outputs*.

Idea: Want to study the spaces $\mathcal{M}_G$ in order to associate to G an operation (i. e. a linear map)

$$(H_*(M))^{\otimes n_+} \to (H_*(M))^{\otimes n_-}.$$

In fact, to obtain more interesting structure, also take into account graph automorphisms.

Theorem (R. Cohen, P. Norbury)

For each graph G there is a corresponding linear map

$$q_G : H_*^{Aut(G)}(M^{\times n_+}) \to H^{Aut(G)}_{*+\chi(G)d-n_+d}(M^{\times n_-}),$$

and these maps are compatible with gluing graphs at their ends and with respect to morphisms $G_1 \to G_2$.

On $Aut_0(G) \subset Aut(G)$ - subgroup of automorphisms which fix the univalent vertices, this reduces to

$$q_G^0 : H_*(BAut_0(G)) \otimes H_*(M^{\times n_+}) \to H_*(M^{\times n_-}).$$

In several simple cases, the operation q_G can be identified explicitly.

- The Y-graph with $n_+ = 2$, $n_- = 1$. Then q_G^0 is the cup product.
- The Y-graph with $n_+ = 1$, $n_- = 2$. Here $Aut(G) = \mathbb{Z}/2$ and $q_G : H_*(B\mathbb{Z}/2) \otimes H_*(M) \to H_*^{\mathbb{Z}/2}(M \times M)$ is the equivariant diagonal.

Dually, $q_G^* : H^*_{\mathbb{Z}/2}(M \times M; \mathbb{Z}/2) \to H^*(B\mathbb{Z}/2; \mathbb{Z}/2) \otimes H^*(M; \mathbb{Z}/2)$ defines the Steenrod squares:

$$q_G^*(x \otimes x) = \sum_{0 \leq j \leq n} a^j \otimes Sq^{n-j}(x),$$

where $x \in H^n(M; \mathbb{Z}/2)$ and $a \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ is the generator.

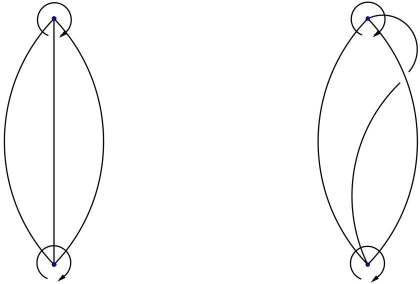
- $G = Q$-graph with $n_+ = 1$. Here $Aut(G) = \mathbb{Z}/2$, $q_G : H_*(B\mathbb{Z}/2; \mathbb{Z}/2) \otimes H_{d-*}(M; \mathbb{Z}/2) \to \mathbb{Z}/2$ or dually, $q_G^* \in H^*(B\mathbb{Z}/2; \mathbb{Z}/2) \otimes H^{d-*}(M; \mathbb{Z}/2)$. This turns out to compute the Stiefel-Whitney classes:

$$q_G^* = \sum_{0 \leq j \leq d} a^j \otimes w_{n-j}(M).$$

Two key properties: compatibility with momorphisms $G_1 \to G_2$ and with gluing of graphs. Ralph Cohen et al. showed that e. g. the Cartan formula can be viewed as a consequence of these properties.

There is also a theorem which explains how to compute the operations q_G from the knowledge of certain equivariant diagonal maps in homology.

A *ribbon structure* on a graph is a cyclic ordering of the half-edges at every vertex.



- $\Gamma =$ graph G together with a ribbon structure.
- Σ - the oriented surface associated to Γ.
- Univalent vertices of $\Gamma \Leftrightarrow$ marked points on $\partial\Sigma$.
- $Met_0(\Gamma)$-space of metric structures on Γ.
- $\mathcal{M}_\Sigma$-space of complex structures on Σ.
- From here on k is a field of characteristic zero.

Theorem (J. L. Harer, R. C. Penner, K. Strebel, K. Igusa,...)

There is a homeomorphism

$$\mathcal{M}_\Sigma \simeq \cup_\Gamma Met_0(\Gamma)/\sim,$$

where the union is over all ribbon graphs Γ whose associated surface is Σ. The equivalence relation is generated by:

- *Collapsing edges of length zero.*
- *$Aut(\Gamma)$-action.*

Fix a Morse function f on M and $(\mathbf{p}_+, \mathbf{p}_-) \in Crit^{n_+ + n_-}(f)$.

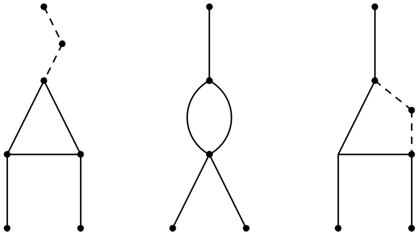
Associate to each edge e of Γ a (one-parameter-family) of vector fields x_e on M, so that $x_e = \nabla f$ in a neighbourhood of the univalent vertices. We refer to $\mathbf{x} := (x_e)_{e \in E(\Gamma)}$ as the *vector field data*.

$$\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-) := \{(\ell, \gamma) : \ell \in Met(\Gamma), \gamma : \Gamma \to M \text{ is continuous,}$$

$$\dot{\gamma}|_e(t) = x_e(t, \gamma|_e(t)) \text{ and convergence}$$

$$\text{to } (\mathbf{p}_+, \mathbf{p}_-) \text{ along external edges}\}.$$

$\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ is *not* compact because of the following:



1) Breaking along external edges,
2) Collapsing of internal edges,
3) Breaking along internal edges.

$\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-) :=$ partial compactification obtained by adding strata of type 1) and 2) (but not 3)). Formally,

$$\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-) =$$

$$\bigcup_{\mathbf{q}_+,\mathbf{q}_-,\widetilde{\Gamma}\prec\Gamma} \overline{\mathcal{M}}(\mathbf{p}_+,\mathbf{q}_+)\times\mathcal{M}_{\widetilde{\Gamma},\mathbf{x}}(\mathbf{q}_+,\mathbf{q}_-)\times\overline{\mathcal{M}}(\mathbf{q}_-,\mathbf{p}_-),$$

where $\widetilde{\Gamma}\prec\Gamma$ means that $\widetilde{\Gamma}$ is obtained from Γ by collapsing edges.

$\pi_\Gamma : \overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)\to\mathcal{M}_\Sigma$ defined by forgetting γ.

Proposition

1. *There exists vector field data* $\mathbf{x}$ *so that* $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ *is a manifold with corners of dimension*

$$|\mathbf{p}_-| - |\mathbf{p}_+| + \chi(\Sigma)d - n_-d + |E(\Gamma)|$$

2. *The map* $\pi_\Gamma : \overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-) \to \mathcal{M}_\Sigma$ *is proper.*

$C_*^{BM}(\mathcal{M}_\Sigma) :=$ chain complex generated by pairs (P, π), where P is an oriented manifold with corners and $\pi : P \to \mathcal{M}_\Sigma$ a proper continuous map; $\partial(P, \pi) = (\partial P, \pi|_{\partial P})$.

The homology of $(C_*^{BM}(\mathcal{M}_\Sigma), \partial)$ is the *Borel-Moore homology* $H_*^{BM}(\mathcal{M}_\Sigma)$. i. e. the homology of 'locally finite chains'.

Over a field of characteristic zero, $H_*^{BM}(\mathcal{M}_\Sigma) \simeq H^{\dim \mathcal{M}_\Sigma - *}(\mathcal{M}_\Sigma)$.

We can now complete the construction. Denote

$$Z^f_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-) = (\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-), \pi_\Gamma)$$

and

$$Z^f(\mathbf{p}_+,\mathbf{p}_-) := \sum_\Gamma Z^f_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-),$$

where the sum is over all Γ, so that each internal vertex of Γ has valency three. Define

$$F^f_\Sigma : (C^*(f))^{\otimes n_+} \to C^{BM}_*(\mathcal{M}_\Sigma; det^d) \otimes (C^*(f))^{\otimes n_-},$$

$$\mathbf{p}_+ \mapsto \sum_{\mathbf{p}_-} Z^f(\mathbf{p}_+,\mathbf{p}_-) \otimes \mathbf{p}_-.$$

Here det^d is a certain local system on $\mathcal{M}_\Sigma$.

Theorem

1. *F_Σ^f is a cochain map.*
2. *Different choices of the Morse function or of the vector field data lead to chain homotopic maps.*

1) implies that there are induced maps

$$HF_\Sigma^f : (H^*(M))^{\otimes n_+} \to H^*(\mathcal{M}_\Sigma; det^d) \otimes (H^*(M))^{\otimes n_-}.$$

Sketch of Proof

Consider $\partial(Z^f(\mathbf{p}_+, \mathbf{p}_-))$. This is a sum, where the summands correspond to boundary components of types 1) and 2).

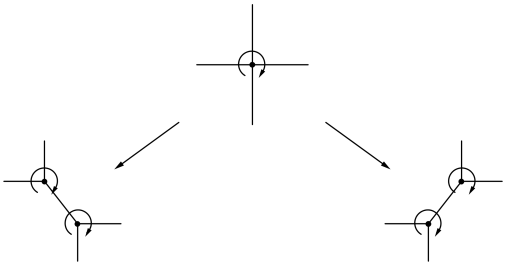
Components of type 1) yield

$$Z^f(d\mathbf{p}_+, \mathbf{p}_-) \pm Z^f(\mathbf{p}_+, d\mathbf{p}_-).$$

We have to show that the summands corresponding to boundary components of type 2) cancel.

These summands are of the form $Z^f_{\widetilde{\Gamma},\mathbf{x}}$, where $\widetilde{\Gamma}$ is obtained from a graph whose internal edges are trivalent by collapsing a single internal edge.

For each such $\widetilde{\Gamma}$, there are exactly two pairs (Γ_1, e_1) and (Γ_2, e_2) with $\Gamma_i/e_i \simeq \widetilde{\Gamma}$, $i = 1, 2$.



One checks that the boundary components from $Z^f_{\Gamma_1,\mathbf{x}}$ and $Z^f_{\Gamma_2,\mathbf{x}}$ enter the sum with the opposite sign.

One can show that the operations F_Σ^f are compatible with gluing together Riemann surfaces. This means that they fit together into what is called an (open) *topological conformal field theory*.

TCFTs have previously been studied from a more algebraic perspective by G. Segal, E. Getzler, M. Kontsevich and more recently, K. Costello.

A_∞-algebra=vector space equipped with linear maps

$$m_k : A^{\otimes k} \to A, \ \ k = 1, 2, 3, \dots$$

of degree $2 - k$ which satisfy for all $n \geq 1$

$$\sum (-1)^u m_{i+1+j}(a_1, \dots, a_i, m_k(a_{i+1}, \dots, a_{i+k}), a_{i+k+1}, \dots, a_n) = 0$$

where the sum is over $i, j \geq 0, k \geq 1, i + k + j = n$ and where $u = i + jk + k(|a_1| + \dots + |a_i|)$.

Combinatorial interpretation: Ways of putting two parentheses in a word on n letters:

- $n = 1$: $((a_1))$

$$m_1^2 = 0$$

- $n = 2$: $((a_1 a_2))$,$((a_1)a_2)$,$(a_1(a_2))$

$$m_1(m_2(a_1, a_2)) \pm m_2(m_1(a_1), a_2) \pm m_2(a_1, m_1(a_2)) = 0$$

- $n = 3$:

$$m_2(m_2(a_1, a_2), a_3) - m_2(a_1, m_2(a_2, a_3))$$
$$\pm m_1(m_3(a_1, a_2, a_3)) \pm m_3(m_1(a_1), a_2, a_3)$$
$$\pm m_3(a_1, m_1(a_2), a_3) \pm m_3(a_1, a_2, m_1(a_3)) = 0$$

There is also a corresponding notion of A_∞-morphisms.

A *cyclic structure* on A is a non-degenerate inner product which is compatible with the operations m_k. A is called *minimal* if $m_1 = 0$.

Examples

1) For a Morse function f on a manifold M, one defines $m_k : (C^*(f))^{\otimes k} \to C^*(f)$ by counting the zero-dimensional components of spaces of flows over ribbon *trees*. Cyclic structure: $\langle p, q \rangle = \pm\delta_{pq}$.

Here $m_1 = 0$ means that f is perfect.

2) $\Omega^*(M)$, $m_1 = d$, $m_2(\alpha, \beta) = \alpha \wedge \beta$, $m_k \geq 0$ for $k > 2$ and $\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta$.

We can 'destill' from $\Omega^*(M)$ a finite-dimensional A_∞-algebra A_{dR} with $m_1 = 0$ using classical Hodge theory and the so-called homological perturbation lemma.

Kontsevich (1994):

$$A \longrightarrow F_\Sigma^A$$

Finite-dimensional minimal cyclic A_∞-algebra	Operations analogous to the ones constructed above

The *ribbon graph complex* $\mathcal{G}^*_\Sigma$ is the vector space generated by the ribbon graphs Γ whose associated surface is Σ, graded by the number of edges. The differential $d : \mathcal{G}^*_\Sigma \to \mathcal{G}^{*+1}_\Sigma$ is defined by expanding vertices.

Proposition (M. Kontsevich, K. Igusa,...)

The cohomology of $(\mathcal{G}^*_\Sigma, d)$ is isomorphic to $H^*(\mathcal{M}_\Sigma)$.

Kontsevich used the structure constants of A to associate to each generator Γ of $\mathcal{G}^*_\Sigma$ a number r^A_Γ.

Roughly, one assigns to each half-edge an element of a basis of A; for each vertex of valency $k+1$ one takes the expression $\langle m_k(\cdot, \ldots, \cdot), \cdot \rangle$ and for each edge the expression $\langle \cdot, \cdot \rangle$. One multiplies all these expressions and sums up over all basis elements.

Theorem (M. Kontsevich '94)

Suppose that A is minimal, i. e. $m_1 = 0$. Then the map

$$\Gamma \mapsto r_\Gamma^A$$

defines a cocycle in $Hom(\mathcal{G}_\Sigma^*, k)$.

For $n_+ + n_- > 0$ we get cochain maps

$$F_\Sigma^A : A^{\otimes n_+} \to Hom(\mathcal{G}_\Sigma^*, k) \otimes A^{\otimes n_-},$$

$$\mathbf{a}_+ \mapsto \sum_{\mathbf{a}_-} \left(\Gamma \mapsto r_\Gamma^A(\mathbf{a}_+, \mathbf{a}_-) \right) \mathbf{a}_-.$$

Recall that we have a Morse-A_∞-algebra A_f. If f is perfect, then we can apply to it Kontsevich's construction. The next Theorem states that the result is equivalent to the more direct construction via flow graphs given above.

Recall that $Z^f(\mathbf{p}_+, \mathbf{p}_-) := \sum_\Gamma Z^f_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ denotes the sum of geometric chains corresponding to flows of all the trivalent ribbon graphs Γ.

Theorem

Assume that f is perfect. There is a subcomplex of the complex $C_*(\mathcal{M}_\Sigma)$ of singular chains in $\mathcal{M}_\Sigma$, which is isomorphic to $\mathcal{G}^*_\Sigma$, and so that the intersection product of $Z^f(\mathbf{p}_+, \mathbf{p}_-)$ with the chain C_Γ corresponding to a generator Γ is given by

$$C_\Gamma \cdot Z^f(\mathbf{p}_+, \mathbf{p}_-) = r_\Gamma^{A_f}(\mathbf{p}_+, \mathbf{p}_-).$$

Finally, we can compare with A_{dR}.

Kontsevich's construction is compatible with A_∞-morphisms: If there is a quasi-isomorphism $A \to B$, then the associated homological operations HF_Σ^A and HF_Σ^B coincide (A. Hamilton, A. Lazarev '06)

Theorem

There is a quasi-isomorphism $A_{dR} \to A_f$ as cyclic A_∞-algebras.

This extends a result of V. K. A. M. Guggenheim from the 70s.