Stochastic Processes II Summer term 2008
(Stochastische Analysis)
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Exercises, 18th June

10.1 (4 points) Let X be an adapted continuous process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with continuous quadratic variation $\langle X\rangle_t$ and let A be an adapted continuous increasing process with $A_0 = 0$. Prove that the following are equivalent:

a) X is a local martingale with $\langle X\rangle_t = A_t$ P-a.s. for all $t \geq 0$.

b) The process

$$G_t^\alpha := \exp\left(\alpha X_t - \frac{1}{2}\alpha^2 A_t\right), \qquad t \geq 0,$$

is a local martingale for every $\alpha \in \mathbb{R}$.

10.2 (4+4 points) Let B be a Brownian motion, and let $h : [0,\infty) \to \mathbb{R}$ be a continuous function of finite variation. Show that:

a) The processes

$$M_t := \int_0^t h(s)dB_s, \qquad t \geq 0,$$

and

$$\mathcal{E}(M)_t := \exp(M_t - \frac{1}{2}\langle M\rangle_t), \qquad t \geq 0,$$

are continuous martingales.

b) The sets

$$\{\, \mathcal{E}(M)_\infty = 0\} \quad \text{and} \quad \{\, \langle M\rangle_\infty = \infty\}$$

coincide P-a.s..

10.3 (2+4+1 points) Let $\tau^n = \{0 = t_0^1 < \ldots < t_{k_n}^n = T\}\,(n = 1, 2, \ldots)$ a sequences of partitions of the interval $[0, T]$ with $|\tau^n| \to 0$. Show that

a) If $A : [0, T] \to \mathbb{R}$ is a continuous function of finite variation with $A_0 = 0$, then

$$\sum_{t_i^n \in \tau^n, t_i^n < T} A_{\theta_i^n}(A_{t_{i+1}^n} - A_{t_i^n}) \to \frac{1}{2} A_T^2$$

for any choice of the points $\theta_i^n \in [t_i^n, t_{i+1}^n]\,(i = 0, \ldots, k_n - 1, n \in \mathbb{N})$.

b) For a Brownian motion B and $\lambda \in [0, 1]$ both

$$S_\lambda^n := \sum_{t_i^n \in \tau^n, t_i^n < T} (\lambda B_{t_i} + (1 - \lambda) B_{t_{i+1}})(B_{t_{i+1}} - B_{t_i})$$

and

$$\bar{S}_\lambda^n := \sum_{t_i^n \in \tau^n, t_i^n < T} B_{\lambda t_i + (1-\lambda) t_{i+1}}(B_{t_{i+1}} - B_{t_i})$$

converge in L^2 to

$$\frac{1}{2} B_T^2 + \left(\frac{1}{2} - \lambda\right) T.$$

c) For which λ is $\frac{1}{2} B_t^2 + (\frac{1}{2} - \lambda) t\,(t \geq 0)$ a martingale?

The problems 10.1 -10.3 should be solved at home and delivered at Wednesday, the 25th June, before the beginning of the tutorial.