Stochastic Processes II Summer term 2008
(Stochastische Analysis)
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Exercises, 30th April

3.1 (5 points) Let $(x_t)_{t\geq 0}$ be a continuous real-valued function with continuous quadratic variation $\langle x\rangle$. Show that for $b \in C^1(\mathbb{R})$ the function

$$G_t := \exp\left(\int_0^t b(x_s)dx_s - \frac{1}{2}\int_0^t b^2(x_s)d\langle x\rangle_s\right) \quad t \geq 0$$

satisfies the equation

$$dG = Gb(x)dx,$$

i.e. it holds

$$G_t = 1 + \int_0^t G_s b(x_s)dx_s \quad t \geq 0,$$

where the Itô-integral on the right-hand side is well defined.

3.2 (4 points) Let $(x_t)_{t\geq 0}$ and $(y_t)_{t\geq 0}$ be two continuous real-valued functions with continuous quadratic variations $\langle x\rangle$ and $\langle y\rangle$, and assume that there exists the covariation $\langle x, y\rangle$ and is continuous. Show that then for $F, G \in C^1$ the covariation $\langle F(x), G(y)\rangle$ also exists, and that

$$\langle F(x), G(y)\rangle_t = \int_0^t F'(x_s)G'(y_s)d\langle x, y\rangle_s \quad t \geq 0.$$

3.3 (4 points) Let $(x_t)_{t\geq 0}$ be a continuous real-valued function with continuous quadratic variation $\langle x\rangle$. Prove that the counterparts of the sine und cosine functions in the Itô-calculus framework, i.e. the solutions S und C of the Itô-equations

$$dS(x) = C(x)dx, \qquad dC(x) = -S(x)dx,$$

are given by

$$S(x)_t = \exp\left(\frac{1}{2}\langle x\rangle_t\right)\sin(x_t), \qquad C(x)_t = \exp\left(\frac{1}{2}\langle x\rangle_t\right)\cos(x_t).$$

Verify that the functions S, C and the Itô-exponential

$$G(\alpha x)_t := \exp\left(\alpha x_t - \frac{1}{2}\alpha^2 \langle x\rangle_t\right)$$

satisfy the relation

$$G(ix)_t = C(x)_t + iS(x)_t.$$

The problems 3.1 -3.3 should be solved at home and delivered at Wednesday, the 7th May, before the beginning of the tutorial.