Stochastic Processes II Summer term 2008
(Stochastische Analysis)
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Exercises, 14th May

5.1 (3 points) If $X = (X_t, t \geq 0)$ is a right-continuous positive supermartingale, prove that

$$P[\sup_t X_t > \lambda] \leq \frac{1}{\lambda} E[X_0], \quad \lambda > 0.$$

5.2 (3 points) Let $(\mathcal{A}_t, t \geq 0)$ be a filtration on some probability space $(\Omega, \mathcal{A}, P)$, let $\mathcal{A}_\infty := \bigvee_{t \geq 0} \mathcal{A}_t$ and let

$$\mathcal{N} := \left\{ N \subseteq \Omega \;\middle|\; \exists\, N' \in \mathcal{A}_\infty \text{ with } P[N'] = 0, N \subseteq N' \right\}.$$

We denote by

$$\mathcal{A}_t^P := \sigma(\mathcal{A}_t \cup \mathcal{N}), \quad t \geq 0$$

the augmented filtration. Show that

$$\mathcal{A}_t^P = \left\{ A \subseteq \Omega \;\middle|\; \exists\, A' \in \mathcal{A}_t \;:\; A \triangle A' \in \mathcal{N} \right\},$$

where $A \triangle A' := (A \backslash A') \cup (A' \backslash A)$.

5.3 (2+2+2 points) Let $(\mathcal{A}_n, n \in \mathbb{N})$ be a filtration on $(\Omega, \mathcal{A}, P)$ and let Q be a probability measure on $(\Omega, \mathcal{A})$ such that Q is *locally* absolutely continuous with respect to P, i.e.

$$Q_n := Q|_{\mathcal{A}_n} \ll P|_{\mathcal{A}_n} =: P_n \quad \text{for all} \quad n \in \mathbb{N}.$$

We denote by $L_n := \frac{dQ_n}{dP_n}$ the Radon-Nykodim-derivative of Q_n w.r.t. P_n, $(n \in \mathbb{N})$.

a) Show that $(L_n, \mathcal{A}_n, n \in \mathbb{N})$ is a nonnegative martingale.

b) Prove that Q is *globally* absolutely continuous with respect to P, i.e. $Q|_{\mathcal{A}_\infty} \ll P|_{\mathcal{A}_\infty}$, if and only if $(L_n, n \in \mathbb{N})$ is uniformly integrable w.r.t. P.

c) The *relative entropy* of Q w.r.t. P is defined as

$$H(Q|P) := \begin{cases} \int \log\left(\frac{dQ}{dP}\right) dQ & \text{if } Q \ll P, \\ +\infty & \text{otherwise.} \end{cases}$$

Show that Q is globally absolutely continuous with respect to P if

$$\sup_n H(Q_n|P_n) < \infty.$$

5.4 (2+1+2+2 points) Let $(\mathcal{A}_t)_{t\geq 0}$ be a right-continuous filtration. Show that

a) For any stopping time T

$$\mathcal{A}_T := \left\{ A \in \mathcal{A} \;\middle|\; A \cap \{T \leq t\} \in \mathcal{A}_t \text{ for all } t \geq 0 \right\}$$

is a σ-algebra, and T is $\mathcal{A}_T$-measurable.

b) If $S \leq T$ is another stopping time, then $\mathcal{A}_S \subseteq \mathcal{A}_T$.

c) If S, T are stopping times, then also $T \wedge S$ is a stopping time, and we have $\mathcal{A}_{T\wedge S} = \mathcal{A}_T \cap \mathcal{A}_S$.

d) If T_n $(n = 1, 2, \ldots)$ is a decreasing sequence of stopping times and $T = \lim_{n\to\infty} T_n$, then T is a stopping time and we have $\mathcal{A}_T = \bigcap_{n\geq 1} \mathcal{A}_{T_n}$.

The problems 5.1 -5.4 should be solved at home and delivered at Wednesday, the 21st May, before the beginning of the tutorial.