Stochastic Processes I Winter term 2007/2008
(Stochastik II)
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Exercises, 17th October

1.1 (4 points) Assume that Q is a probability measure on $(\mathbb{R}^n, \mathscr{B}_n)$. Then the function $\hat{Q}$, defined by

$$\hat{Q}(u) := \int_{\mathbb{R}^n} \mathrm{e}^{i<u,x>}\, Q(dx), \quad u \in \mathbb{R}^n,\ i = \sqrt{-1}$$

is called the characteristic function (shortly: c.f.) of the measure Q. The characteristic function of an n-dimensional random vector X is defined to be the c.f. of its distribution P^X.

a) Prove that $\hat{Q}$ is a bounded, continuous function with $\hat{Q}(0) = 1$.

b) Let X be an n-dimensional random vector, A an $n \times m$-matrix and a an m-dimensional vector. Calculate the c.f. of $AX + a$ in terms of the c.f. of X.

c) Show that every c.f. is nonnegative definite, i.e.

$$\sum_{k,l=1}^{m} \lambda_k \bar{\lambda}_l \hat{Q}(u_k - u_l) \geq 0$$

holds for all $m \geq 1$, all complex $\lambda_1, \ldots, \lambda_m$ and all $u_1, \ldots, u_m \in \mathbb{R}^n$.

d) Prove that the function

$$\varphi(u) = \mathrm{e}^{-|u|},\ u \in \mathbb{R}^1$$

is nonnegative definite.

1.2 (4 points) Let f be the density of a $\Gamma(\alpha, \lambda)$-distribution $(\alpha, \lambda > 0)$, i.e.

$$f(x) = \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} \mathrm{e}^{-\lambda x} \mathbb{1}_{(0,\infty)}(x), \quad x \in \mathbb{R}^1.$$

Calculate the Laplace transform

$$\hat{F}(u) := \int_0^\infty \mathrm{e}^{-ux} f(x) dx, \quad u \geq 0$$

of the $\Gamma(\alpha, \lambda)$-distribution and show, that

$$\ln \hat{F}(u) = \alpha \int_0^\infty (\mathrm{e}^{-uy} - 1) \frac{\mathrm{e}^{-\lambda y}}{y} dy$$

holds for all $u \geq 0$.

1.3 (6 points) Let $T = [0, \infty)$. We say that a subset D of $R^T = \{x_t : x_t \in \mathbb{R}^1, t \in T\}$ has the property C, if there exists a countable set $M_D := \{t_1, t_2, \ldots\} \subset T$ such that

$$x \in D \Longleftrightarrow \exists y \in D : \forall t_k \in M_D, x_{t_k} = y_{t_k}, k \geq 1.$$

a) Show that the subsets D of R^T with the property C form a σ-algebra $\mathfrak{A}$.

b) Check the assertion that for all $B \in \mathscr{B}_1, t \in T$ the set $D(t, B) := \{x \in R^T : x_t \in B\}$ belongs to $\mathfrak{A}$.

c) Prove, that $C([0, \infty)) = \{y \in R^T | t \to y_t \quad \text{is continuous}\}$ does not belong to $\mathfrak{A}$.

The problems 1.1 -1.3 should be solved at home and delivered at Thursday, the 25th October, before the beginning of the lecture.