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Exercises, 19th December

10.1 Let $(X_n, \mathcal{A}_n)_{n\in\mathbb{N}_0}$ be a martingale on $(\Omega, \mathcal{A}, P)$ with $X_n \in L^2(P)$ for all n.

a) (1 point) Show that the increments $\Delta X_n := X_n - X_{n-1}$ are pairwise uncorrelated.

b) (3 ponits) Prove that the sequence $(\frac{1}{n}X_n)_{n\in\mathbb{N}}$ converges in probability and in $L^2(P)$ if $\sup_n E[X_n^2] < \infty$.

10.2 a) (2 points) Let $(X_n, \mathcal{A}_n)_{n\in\mathbb{N}_0}$ be a martingale on $(\Omega, \mathcal{A}, P)$ with $X_0 = 0$ and $X_n \in L^2(P)$ for all n, and let $\Delta X_n := X_n - X_{n-1}$. Assume further that

$$\sum_{n=0}^{\infty} E[(\Delta X_n)^2] < \infty.$$

Show that the martingale $(X_n)_{n\in\mathbb{N}_0}$ converges in probability and in $L^2(P)$ to some random variable X_∞. Compute the expectation and the variance of X_∞.

b) (2 ponits) Consider the geometric series with random sign, i.e. the sequence

$$X_0 := 0, \qquad X_n := \sum_{k=1}^{n} \frac{1}{k} Y_k, \qquad n = 1, 2, \ldots$$

where Y_n, $n \in \mathbb{N}$ are independent with

$$Y_n = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2}. \end{cases}$$

Is this series convergent?

10.3 Let D_k $(k = 1, 2, \ldots)$ be an adapted process and π_k $(k = 1, 2, \ldots)$ a predictable process on $(\Omega, (\mathcal{A}_n)_{n\in\mathbb{N}_0}, \mathcal{A}, P)$ such that

$$\pi_k \geq \frac{1}{\alpha} \log E[e^{\alpha D_k} \mid \mathcal{A}_{k-1}] \qquad (k = 1, 2, \ldots) \tag{1}$$

for some fixed $\alpha > 0$. We interpret D_k as the payment for occured losses and π_k as the insurance premium in the period k of some portfolio consisting of insurance contracts. Then

$$Y_k := R + \sum_{k=1}^{n} \pi_k - \sum_{k=1}^{n} D_k \qquad (n = 0, 1, \ldots)$$

denotes the value process of the portfolio with initial value $Y_0 = R > 0$, and

$$\rho := \min \{ n \geq 0 \mid Y_n \leq 0 \}$$

is the time of "ruin".

a) (3 ponits) Show that

$$P[\rho < \infty] \leq e^{-\alpha R}. \tag{2}$$

b) (2 points) Motivate the assumption that $D_k = c_k Z_k$ with

$$P[Z_k = l \mid \mathcal{A}_{k-1}] = \frac{\lambda_k^l}{l!} e^{-\lambda_k} \qquad (l = 1, 2, \ldots),$$

where (c_k) and (λ_k) are predictable processes. Determine (π_k) such that (1) holds with "=" (and thus also (2) holds).

10.4 Let $Y_0 = R > 0$, and let $Y_n \in L^1$ $(n = 1, 2, \ldots)$ be i.i.d. random variables. For $0 < \beta < 1$ define

$$R_n := \sum_{k=0}^{n} \beta^{k-n} Y_k \quad (n = 0, 1, \ldots), \quad X := \sum_{k=1}^{\infty} \beta^k Y_k.$$

a) (1 point) Prove that $(R_n)_{n=0,1,\ldots}$ solves the recursive equation

$$R_{n+1} = \frac{1}{\beta} R_n + Y_{n+1} \qquad n = 0, 1, \ldots$$

with initial value $R_0 = R$, and that $\lim_n \beta^n R_n = R + X$, i.e. $R_n \sim \beta^{-n}(R + X)$ for large n.

b) (2 points) Let F be the distribution function of X and let $u(x) := F(-x)$. Show that $(u(R_n))_{n=0,1,\ldots}$ is a martingale, more precise

$$u(R_n) = P[X \leq -R \mid \mathcal{A}_n] \qquad n = 0, 1, \ldots.$$

c) (2 points) Show that for $\zeta := \min\{n \geq 0 \mid R_n \leq 0\}$ the probability of ruin satisfies the inequality

$$P[\zeta < \infty] \leq \frac{u(R)}{u(0)}.$$

The problems 10.1 -10.4. should be solved at home and delivered at Wednesday, the 9th January, before the beginning of the tutorial.

We wish you a merry Christmas and a happy New Year!