Stochastic Processes I Winter term 2007/2008
(Stochastik II)
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Exercises, 16th January 2008

12.1 (2 points) Let (X_n) be a Markov chain on $\{0, 1, \dots, 5\}$ with transition matrix

$$\mathbb{P} = \begin{pmatrix} 0,1 & 0,2 & 0 & 0 & 0 & 0,7 \\ 0,5 & 0,1 & 0 & 0 & 0 & 0,4 \\ 0 & 0 & 0,5 & 0,5 & 0 & 0 \\ 0 & 0 & 0,7 & 0,3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0,4 & 0,6 \\ 0,1 & 0,9 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Determine the irreducible and the closed subsets of recurrent states. Which states are transient?

12.2 (3 points) Assume $\mathbb{P}$ is a transition matrix, A, B are matrices with

$$AB = I (I = \text{ unit matrix})$$

and Λ is a diagonal matrix, such that

$$\mathbb{P} = B\Lambda A$$

(One says, $\mathbb{P}$ is diagonalizable).

a) Show that for all $n \geq 1$ it holds

$$\mathbb{P}^n = B\Lambda^n A$$

b) Prove that every stochastic matrix

$$\mathbb{P} = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}, \quad (\alpha, \beta \in [0,1])$$

is diagonalizable and calculate $\mathbb{P}^n$.

c) Under which conditions does $\mathbb{P}^n$ converge for $n \to \infty$?
Compute the limit in this case.

12.3 (4 points) Let $(Z_n, n \geq 0)$ be a branching process with

$$P(Z_1 = k) = f_k, \quad k \geq 0, \quad \text{and} \quad \sum_{k \geq 1} k f_k =: m < \infty.$$

Let π be the probability that $\lim_{n \to \infty} Z_n = 0$. Show that π is the smallest solution of

$$\varphi(s) = s, \quad 0 \leq s \leq 1,$$

where $\varphi(s) = \sum_{k=0}^{\infty} s^k f_k$ and prove that $\pi < 1$ if $m > 1$.
Hint: Show first that

$$\pi_n := P(Z_n = 0) = \varphi^{(n)}(0),$$

where

$$\varphi^{(n)}(s) = E\left[s^{Z_n}\right] = \varphi^{(n-1)}(\varphi(s)), \quad \varphi^{(1)}(s) = \varphi(s), \quad s \geq 0.$$

Then prove that $\pi_n \uparrow \pi$, and $\varphi(\pi) = \pi$. Since φ is strictly convex, the equation $\varphi(s) = s$ has at most two real solutions, one of them is $s = 1$, the other is denoted by ξ. Show that $\varphi(0) = f_0 = \pi_1 < \xi \wedge 1$, and thus $\varphi^{(n)}(0) < \xi \wedge 1$ in virtue of $\varphi'(s) > 0, s \geq 0$. Finish the proof.

12.4 (2 points) Let $(X_n, n \geq 0)$ be a Markov chain with state space S. Define $T_i := \inf\{k \geq 1 | X_k = i\}$ with $\inf \emptyset := \infty$.
Show that the following relations hold for all $i, j \in S$ with $i \neq j$:

(i) $T_j \leq T_i + T_j \circ \Theta_{T_i} \quad \text{on} \quad \{T_i < \infty\}$,

(ii) $T_j = T_i + T_j \circ \Theta_{T_i} \quad \text{on} \quad \{T_i < T_j \leq \infty\}$.

The problems should be solved at home and delivered at Wednesday, January 23rd, before the beginning of the tutorial.