Stochastic Processes I Winter term 2007/2008
(Stochastik II)
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Exercises, 24th October

2.1 (4 points) Assume X and Z are independent random variables, where X is standard Gaussian distributed and $P(Z = 1) = P(Z = -1) = \frac{1}{2}$. Define $Y := Z \cdot X$.

a) Calculate the distribution function of Y.

b) Show that X and Y are uncorrelated.

c) Calculate the characteristic function of the random vector (X, Y).

d) Discuss the assertion, that "uncorrelated Gaussian random variables are independent".

2.2 (4 points) Assume $X = (X_1, \dots X_n)^T$ is an n-dimensional standard Gaussian vector. Which distribution has $Y = \| X \|^2 = \sum_{k=1}^{n} X_k^2$?
If $X \sim N_n(0, \Sigma)$ with Σ regular, calculate the distribution of $Y = X^T \Sigma^{-1} X$.

2.3 (2 points) Show, that every symmetric nonnegative definite 2×2-matrix Σ can be written as

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

with $\sigma_1, \sigma_2 \geq 0$ and $\rho \in [-1, 1]$.

2.4 (6 points) Let $n \in \mathbb{N}$ with $n \geq 2$ and let $X_1, X_2, \ldots, X_n$ be an i.i.d. sequence of random variables such that

$$P[X_j = l] = p_l, \quad l = 1, \ldots, k.$$

a) Show that the vector $N := (N_1, , \ldots, N_k)$ with

$$N_j := \sum_{m=1}^{n} \mathbb{1}_{\{j\}}(X_m), \quad j = 1, \ldots, k$$

has the multinomial distribution with

$$P[N = (n_1, \ldots, n_k)] = \frac{n!}{n_1! \cdots n_k!} p_1^{n_1} \cdots p_k^{n_k}$$

if $n_j \geq 0$, $\sum_{j=1}^{k} n_j = n$, and $P[N = (n_1, \ldots, n_k)] = 0$ otherwise.

b) Prove that the characteristic function of N is given by

$$\varphi(u) = \left(\sum_{j=1}^{k} p_j e^{iu_j} \right)^n, \qquad u = (u_1, \ldots, u_k)^T \in \mathbb{R}_k.$$

c) Calculate $E[N_j]$, $\mathrm{Var}(N_j)$ and $\mathrm{Cov}(N_j, N_l)$ for $j, l \in \{1, \ldots, k\}$.

d) Determine the covariance martix Σ of

$$\frac{N}{\sqrt{p}} := \left(\frac{N_1}{\sqrt{np_1}}, \ldots, \frac{N_k}{\sqrt{np_k}} \right).$$

e) Construct a vector $v := (v_1, \ldots, v_k)^T$ such that $\Sigma v = 0$ and verify that

$$\langle \frac{N}{\sqrt{p}} - E\left[\frac{N}{\sqrt{p}}\right], v \rangle = 0 \quad P\text{-a.s..}$$

The problems 2.1 -2.4. should be solved at home and delivered at Thursday, the 1st November, before the beginning of the lecture.