Stochastic Processes I Winter term 2007/2008
(Stochastik II)
Prof. Dr. Uwe Küchler
Dipl. Math. Irina Penner

Exercises, 7th November

4.1 (5 points) Let $Y : \Omega \to \Omega'$ be a map from some set Ω to a measurable space $(\Omega', \mathcal{A})$. Let further $Z : \Omega \to [-\infty, +\infty]$. Show that Z is measurable with respect to the σ-algebra $\sigma(Y)$ generated by Y on Ω if and only if there exists a $[-\infty, +\infty]$-valued measurable function g on $(\Omega', \mathcal{A})$ such that

$$Z(\omega) = g(Y(\omega)).$$

Hint: Prove the claim first for simple functions Z.

4.2 (4 points) Let X be a nonnegative random variable on $(\Omega, \mathcal{A}, P)$ and let $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{A}$. Show that

$$H_X := \big\{ \omega \in \Omega \;\big|\; E[X|\mathcal{H}](\omega) > 0 \big\}$$

is the smallest set in $\mathcal{H}$ containing $\{X > 0\}$, i.e. for all $H \in \mathcal{H}$ such that $\{X > 0\} \subseteq H$ it holds $H_X \subseteq H$ P-a.s..

4.3 (4 points) Let X and Y be independent and identically distributed randon variables such that $E[|X|] < \infty$. Show that

$$E[X|X+Y] = E[Y|X+Y] = \frac{X+Y}{2} \quad P\text{-a.s..}$$

4.4 (5 points) Let T_0 and T_1 be independent random variables, both exponentially distributed with parameter $\alpha > 0$. Let further $X := \min(T_0, T_1)$.

a) Show that

$$E[X|T_0] = \frac{1}{\alpha}(1 - e^{-\alpha T_0}).$$

b) Prove that the best linear estimator $\widehat{X}$ of X based on T_0 is given by

$$\widehat{X} = \frac{1}{4}(T_0 + \frac{1}{\alpha}).$$

The problems 4.1 -4.4. should be solved at home and delivered at Wednesday, the 14th November, before the beginning of the tutorial.