Stochastic Processes I Winter term 2007/2008
(Stochastik II)
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Exercises, 28th November

7.1 a) (1 point) Let $(X_n)_{n\in\mathbb{N}}$ be an adapted integrable stochastic process on some probability space $(\Omega, \mathcal{A}, P)$ with the filtration $(\mathcal{A}_n)_{n\in\mathbb{N}}$. Show that $(X_n, \mathcal{A}_n)_{n\in\mathbb{N}}$ is a martingale if and only if $E[X_{n+1}|\mathcal{A}_n] = X_n$ for all $n \in \mathbb{N}$.

b) (3 points) Let $(X_n, \mathcal{A}_n)_{n\in\mathbb{N}}$ be a martingale such that $X_n \geq 0$ P-a.s.. Prove that for P-almost all $\omega \in \Omega$ it holds

$$X_n(\omega) = 0 \quad \text{for some } n \quad \Longrightarrow \quad X_{n+k}(\omega) = 0 \quad \text{for all } k = 0, 1, \ldots.$$

7.2 Let $T = \mathbb{N}_0$ or $T = [0, \infty)$ and let $(\mathcal{A}_t)_{t\in T}$ be a filtration on some probability space $(\Omega, \mathcal{A}, P)$. Assume further that τ and σ are stopping times with respect to $(\mathcal{A}_t)_{t\in T}$. We define

$$\mathcal{A}_\tau := \left\{ A \in \mathcal{A} \;\middle|\; A \cap \{\tau \leq t\} \in \mathcal{A}_t \text{ for all } t \in T \right\}.$$

a) (1 point) Show that $\tau \wedge \sigma := \min(\tau, \sigma)$ and $\tau \vee \sigma := \max(\tau, \sigma)$ are stopping times with respect to $(\mathcal{A}_t)_{t\in T}$.

b) (2 points) Let $\varphi : T \to T$ be an increasing function such that $\varphi(t) \geq t$ for all $t \in T$. Show that $\varphi(\tau)$ is a stopping time with respect to $(\mathcal{A}_t)_{t\in T}$.

c) (2 points) Show that if $\sigma \leq \tau$ we have $\mathcal{A}_\sigma \subseteq \mathcal{A}_\tau$.

7.3 (4 points) Let $(X_n)_{n=0,1,\ldots}$ and $(Y_n)_{n=0,1,\ldots}$ be two martingales on some probability space $(\Omega, \mathcal{A}, P)$ with the filtration $(\mathcal{A}_n)_{n\in\mathbb{N}}$, and let τ be a stopping time such that $X_\tau = Y_\tau$ P-a.s. on the set $\{\tau < \infty\}$. Prove that the process

$$Z_n := X_n \, I_{\{\tau > n\}} + Y_n \, I_{\{\tau \leq n\}}, \quad n = 0, 1, \ldots$$

is again a martingale.

7.4 Let $(X_i, i \in I)$ be a family of random variables on $(\Omega, \mathcal{A}, P)$.

a) (3 points) Let $g : [0, \infty) \to \mathbb{R}$ be an increasing function such that

$$\lim_{x \to \infty} \frac{x}{g(x)} = 0.$$

Prove that $(X_i, i \in I)$ is uniformly integrable if

$$\sup_{i \in I} E[g(|X_i|)] < \infty.$$

b) (1 point) Show that $(X_i, i \in I)$ is uniformly integrable if

$$\sup_{i \in I} E[|X_i|^p] < \infty$$

for some $p > 1$.

The problems 7.1 -7.4. should be solved at home and delivered at Wednesday, the 5th December, before the beginning of the tutorial.