Exercises for Classical Mechanics and Symplectic Geometry, Herbst 2003

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1 Kepler's Laws

1.0.1 Conservative Systems

We fix some point $x_0 \in \mathbb{R}^n$ and set

$$U(x) := -\int_{x_0}^{x} F \cdot d\gamma$$

for some path connecting the two points.

Exercise 1 *Show that indeed*

$$F = -\nabla U.$$

Corollary 1 *A vector field F is conservative if and only if the work along each loop vanishes.*

Exercise 2 *Prove the above statement.*

Exercise 3 *Check whether the vector field $F_1 = x_2, F_2 = -x_1$ is conservative.*

Exercise 4 *Check whether the vector field*

$$F_1 = \frac{x_2}{x_1^2 + x_2^2} \qquad F_2 = \frac{-x_1}{x_1^2 + x_2^2}$$

is conservative on $\mathbb{R}^2 \setminus \{0\}$ or on $\mathbb{R}^2 \setminus \mathbb{R}_+$.

1.0.2 Conservation Laws

Exercise 5 *The motion of a charged particle of mass m and charge c in a magnetic field neglecting gravity is determined by*

$$m\ddot{x} = cV(x) \times \dot{x}.$$

By "$\times$" we denote as usual the vector product (also cross product) in $\mathbb{R}^3$. Show that the absolute value of the velocity $|\dot{x}|$ remains constant.

Exercise 6 *Prove that each central field is conservative on the complement of the origin.*

Exercise 7 *Show that all possible orbits of massive points in a central field lie in a plane.*

Exercise 8 *Study for instance conservative axial symmetric fields. Prove that the potential of such is of the form $U = U(r, z)$ where (r, φ, z) are the cylindrical coordinates, the z–axis is the fixed under the isometries which leave the vector field invariant. The angular momentum M_z with respect to the z-axis is defined as*

$$M_z := e_z \cdot x \times \dot{x}.$$

Show that this does not depend on the choice of the origin on the symmetry axis. How does it depend on the choice of a unit vector e_z which spans it? Prove that M_z is an invariant of motion in a system which is symmetric with respect to the z–axis.

Now, conservation of angular momentum gives $\dot{\varphi} = M/mr^2$ and we obtain the one-dimensional problem

$$\begin{aligned} m\ddot{r} &= -\frac{\partial U}{\partial r} + mr\frac{|M|^2}{m^2r^4} \\ &= -\frac{\partial U}{\partial r} + \frac{|M|^2}{mr^3}. \end{aligned}$$

Exercise 9 *Show that the total energy of the new system $\frac{m}{2}\dot{r}^2 + V(r)$ is equal to the total energy of the original system.*

2 Systems with n massive points

2.1 The Two-Body-Problem

Exercise 10 *Determine the size of the big half-axis of the ellipse around the center of mass of the earth-moon system on which the center of the earth lies. Does the center lie inside or outside of the earth?*

3 Calculus on Banach Spaces

3.1 Derivatives

Recall that a continous real function $f : B \to \mathbb{R}$, often referred to as functional, on a Banach space B is differentiable in $x \in B$ if there is a linear functional $f'(x) : B \to \mathbb{R}$ such that

$$\lim \|h\| \to 0 \frac{f(x+h) - f(x) - f'(x)(h)}{\|h\|} = 0.$$

Exercise 11 *Prove that $f'(x)$ is uniquely determined and continous as linear functional.*

Exercise 12 *Consider the set of real functions, $B = C^2(B^n; \mathbb{R})$, on the unit ball in $\mathbb{R}^n$ with two derivatives. We are given a function $L : \mathbb{R} \times \mathbb{R}^n \times B^n \to \mathbb{R}$ and define the functional $\mathcal{L} : B \to \mathbb{R}$*

$$\mathcal{L}(x) := \int_{B^n} L(f, \nabla f, x) d^n x.$$

Show that $\mathcal{L}$ is differentiable and compute its derivative.

Exercise 13 Length of a curve. *Let $B = C^2([t_0, t_1]; \mathbb{R}^n)$. Let $g_{ij}(x), \quad x \in \mathbb{R}^n$ be a family of symmetric, positive definite $n \times n$–matrices, i.e. a Riemannian metric or structure. It defines a family of euclidean products via $g_x(v, w) = \sum_{ij} v_i w_j$. They are usually considered to be inner products on the tangent space (the velocity vectors) at a point $x \in \mathbb{R}^n$, the location. Then the norm of such a velocity v is given by $\|v\|^2 = g_x(v, v)\|$. The length of a curve γ parametrized as a $\gamma \in B$ is defined to be*

$$l(\gamma) := \int_{t_0}^{t_1} \|\dot\gamma(t)\| dt.$$

Show that this definition is independent of the parametrization of γ.
Compute its derivative on the euclidean plane, i.e. $n = 2$ and $g_{ij} = \delta_{ij}$.

3.2 Critical Points

Proposition 2 *A Kurve $x = x(t)$ is extremal for $\mathcal{L}(x) := \int_{t_0}^{t_1} L(x, \dot x, t) dt$ on the curves in $\mathbb{R}^n$ starting at $x(t_0) = x_0$ and ending at $x(t_1) = x_1$ iff*

$$f(t) = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot x}\Big) - \frac{\partial L}{\partial x} = 0.$$

Exercise 14 *Show that the extremality condition does not depend on the choice of coordinates on* $\mathbb{R}^n$. *Which quantities in the definition of the Lagrangian are transformed? Show that the extremal points for the length functional in the euclidean plane are lines. Compute the equation for a line in polar coordinates.*
(*) How does a Lagrangian, defined for curves, change under time transformations, i.e. diffeomorphisms acting on the parametrization of the paths?

4 Euler–Lagrange Equations

Exercise 15 *Determine the elements of a Lagrangian system for a free massive point in* $\mathbb{R}^3$, *i.e. in a system without forces. What are the critical points of the functional? Compare that to the length functional.*
(*) Is that true for a general Riemannian structure on $\mathbb{R}^3$*? Why?*

Exercise 16 *Determine the elements of a Lagrangian system for a massive point in a central field in the plane in polar coordinates. What is the role of the two coordinates in the Lagrangian?*

5 The Legendre–Transform

Exercise 17 *Compute the Legendre transform of* $f(x) = x^\alpha/\alpha$ *for* $\alpha > 0$.

Exercise 18 *What is the Legendre transform of* $f(x) + c$ *if the Legendre transform of* f *is* g.

Exercise 19 *Derive Young's inequality for* $f(x) = e^x$.

Exercise 20 *Show that these conditions define* g *uniquely and completely, provided that the differential* df_x *attains all values in* $(\mathbb{R}^n)^*$ *as* x *runs through* $\mathbb{R}^n$. *Show that all other statements and corollaries remain true.*

Exercise 21 *Compute the Legendre transform* g *of a positive definite quadratic form* $f(x) = \sum g_{jk}x_jx_k$. *Show that for the dual variable* p *it satisfies*

$$g(p) = f(x).$$

6 hamilton's Equations

Exercise 22 *Show the inverse statement: Given a Hamiltonian function* $H = H(p, q, t)$ *which is strongly convex in* p *then the Hamilton equations transform into the Lagrange equations for* L *being the Legendre transform* $L(q, \dot{q}, t) = p\dot{q} - H(p, q, t)$*, where* $\dot{q} = \partial H/\partial p$.

7 Conservation of Energy

Exercise 23 *Sketch the graph of a differentiable potential* $U = U(q)$ *with at least three critical points. Indicate the structure of the dynamics of the corresponding Hamiltonian system in the phase space*

7.1 Ergodic Systems

Exercise 24 *Consider the series* $1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, \ldots$ *of first digits in the series* 2^n. *Does* 7 *occur at all? Which digit occurs more often,* 7 *or* 8*?*

8 Lagrange–Mechanics on Manifolds

Exercise 25 *Show that* $SO(3)$ *is* not *homeomorphic (continously bijective) to* $\mathbb{S}^2 \times \mathbb{S}^1$. *Hint: Show that* $SO(3)$ *is he same as the real projective space* $\mathbb{RP}^3$. *Compare loops in each of the two spaces in question.*

9 Manifolds

Exercise 26 *Show that the product of two manifolds* $M \times N$ *carries naturally the structure of a manifold.*

Exercise 27 *Represent* $SO(3)$ *as the zero set of a set of functions on a euclidean space and show that these satisfy the condition of the theorem.*

Exercise 28 Lie groups. *Show that the maps*

$$(A, B) \in SO(3) \times SO(3) \mapsto (AB) \in SO(3)$$
$$A \in SO(3) \mapsto A^{-1} \in SO(3)$$

are differentiable.

9.1 Tangent and cotangent bundles

Exercise 29 *Show that the definition of* T_xM *and the operations do not depend on the coordinate chosen. Show that this defines a vector space structure on* T_xM*. Show that* f_* *is a linear map between vector spaces.*

Exercise 30 *Let* $f : M \to N$ *be a smooth map between manifolds* M *and* N*. Show smoothness of* $f_* : TM \to TN$*.*

Exercise 31 *(*) Show that* $\cup_x(T_xM \times T_xM)$ *carries the canonical structure of a smooth manifold (fibre product). Show that*

$$(v, w) \in \cup_x(T_xM \times T_xM) \mapsto (v + w) \in TM$$

is a smooth map between manifolds.

Exercise 32 *Compute the transition functions of the associated charts in the cotangent bundle of a smooth manifold.*

Exercise 33 *Let* $f : M \to \mathbb{R}$ *be a smooth function on the manifold* M*. Show that its differential* df_x *defines a smooth map* $df : M \to T^*M$ *between manifolds. Let* $\pi : TM \to M$ *denote the natural projection which assigns to a cotangent vector its base point. Show that this is a smooth map between manifolds. compute* $\pi \circ df$*.*

10 Lagrangian Dynamical Systems on Manifolds

Exercise 34 *Compute the Riemann structure induced on* $\mathbb{S}^2 := \{x \in \mathbb{R}^3 \mid |x| = 1\}$ *in terms of stereographic and angular coordinates.*

Exercise 35 *Compute* δl_γ *in a coordinate chart. This is the differential equation for geodesics.*

Exercise 36 *Find the geodesics on* $\mathbb{S}^2 \subset \mathbb{R}^3$ *given above uisng stereographic and angular coordinates.*

Exercise 37 *Show that the motion* $\gamma : \mathbb{R} \to M$ *of a free massive point* $U \equiv 0$ *in a Riemannian manifold is given by the geodesic with the same initial point and direction in* M*. Show that the* velocity $|\dot{\gamma}|_{g(\gamma(t))}$ *is preserved.*

11 D'Alembert's Principle

Exercise 38 *Determine this force for the triangle of the second example and give a quantitative interpretation along the lines of the spherical pendulum.*

Exercise 39 *Determine the geodesics of $\mathbb{S}^2 \subset \mathbb{R}^3$ as defined above from d'Alembert's principle.*

12 Noether's Theorem

$$I(q, \dot{q}) := \frac{\partial L}{\partial \dot{q}} \frac{dh_s(q)}{d\tau}\Big|_{\tau=0}.$$

Exercise 40 *Express the first integral without using local coordinates.*

13 Mechanics of the Rigid Body

Exercise 41 *Assume that the earth has the perfect shape of a round ball. Show that then both quantities M_0 and μ are conserved quantities.*

Exercise 42 *Show that the momentum of a free rigid body is preserved, i.e. its center of mass is non–accelerated.*

Exercise 43 *Show that the eigenvalues (I_1, I_2, I_3) of the inertia tensor of a rigid body satisfy the triangle inequality*

$$I_1 \leq I_2 + I_3, \quad I_2 \leq I_3 + I_1, \quad I_3 \leq I_1 + I_2.$$

Exercise 44 *Let $\Omega \in \mathbb{R}^3$ be the angular velocity vector of a rotating rigid body. Show that the total angular momentum M is parallel to the normal to the ellipsoid of inertia at the intersection of the line through Ω with it.*

Exercise 45 *Determine the ellipsoid of inertia for your favorite platonic body with respect to its center.*