Solutions to **Final Exam**–Sample Test

Notice of Non–Guarantee: There may be small mistakes and typos in the solutions. If anyone finds them, please, let me know (via email), so I can make the adjustments before Wednesday, 15. Good Luck!!!

1. Describe all solutions of the following differential equations for functions $y = y(x)$. Use an appropriate method.

$$(y/x + 6x) + (\log x - 2)y' = 0$$

Solution: The equation is of the form $F(x, y) + G(x, y)y'$. One checks that $F_y(x, y) = 1/x = G_x(x, y)$. Hence the equation is **exact** and we may find a function $H = H(x, y)$ which satisfies $H_x(x, y) = F(x, y)$ and $H_y(x, y) = G(x, y)$ as follows

$$\begin{aligned} H(x, y) &= \int F(x, y)dx = y \log x + 3x^2 + C(y) \\ H_y(x, y) &= \log x + C'(y) = \log x - 2 (= G(x, y)) \\ C(y) &= -2y. \end{aligned}$$

Hence solutions $y = y(x)$ are given as level sets of the function $H(x, y) = y \log x + 3x^2 - 2y$.

$$t^3 y' + 4t^2 y = e^{-t}, \quad y(-1) = 0$$

Solution: Without the right–hand–side this is a linear equation. Hence we first solve the *homogeneous* equation

$$t^3 y' + 4t^2 y = 0,$$

find secondly the general solution using **variation of parameters** and, at last, plug in the initial value to determine the (until then) undetermined constant.

The homogeneous equation is a separable equation (typically) and leads to

$$\begin{aligned} \frac{y'}{y} &= -\frac{4}{t} \\ \log y &= -4 \log t + C \\ y &= Ct^{-4}. \end{aligned}$$

Now write the prospective solution of the *original* equation as $y = C(t)t^{-4}$ and plug it in. We obtain

$$t^3(C'(t)t^{-4} - 4C(t)t^{-5}) + 4t^2C(t)t^{-4} = C'(t)t^{-1} = e^{-t}$$
$$C'(t) = te^{-t}$$
$$C(t) = \int te^{-t}dt = -te^{-t} - \int -e^{-t}dt = -te^{-t} - e^{-t} + D,$$

wheras the last line is partial integration and D is the free constant. Now since $y(-1) = (-(-1))e - e + D = D$ we have to set $D = 0$ and the solution is

$$y(t) = -te^{-t} - e^{-t}.$$

$$y' = x^2/(y(1+x^3))$$

Solution: This is a **separable** equation. We obtain

$$\frac{y'}{y} = \frac{x^2}{1+x^3}$$
$$\log y = \int \frac{x^2}{1+x^3}dx.$$

The integral can be solved using substitution $t = x^3$, and with $dt = 3x^2dx$ we get

$$\int \frac{x^2}{1+x^3}dx = \int \frac{1}{3}\frac{1}{1+t}dt$$
$$= \frac{1}{3}\log(1+t) + C = \frac{1}{3}\log(1+x^3) + C$$

Hence we end up with

$$y = Ce^{\frac{1}{3}\log(1+x^3)} = C(1+x^3)^{\frac{1}{3}}.$$

2. Find the Wronskian of two solutions of the following differential equation.

$$t^2y'' - t(t+2)y' + (t+2)y = 0.$$

Solution: First we have to *normalize* the equation:

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2} = 0.$$

Then the Wronskian of any pair of solutions is given by

$$W(t) = Ce^{-\int(-\frac{t+2}{t})dt}.$$

The integrand is $\frac{t+2}{t} = 1 + \frac{2}{t}$ and hence the integral is equal to $t + 2\log t + c$. Thus we obtain

$$W(t) = Ce^{t+2\log t} = Ct^2e^t.$$

Notice that the constant of the undetermined integral in the exponent dos not change the result.
3. Find all solutions of the following differential equations. You are free to use either the method of undetermined coefficients (if it is appropriate) or variation of parameters.

$$y'' + 2y' + y = 3e^{-t}$$

Solution: The characteristic equation $0 = r^2 + 2r + 1 = (r+1)^2$ has a double zero at $r = -1$. hence $\{e^{-t}, te^{-t}\}$ is a fundamental set of solutions.We use the method of **undetermined coefficients**. Since the right hand side is a multiple of the first solution we suppose that a solution is of the form $y(t) = At^2e^{-t}$. The power of t^2 is the smallest power such that this is not a multiple of one of the fundamental solutions. Plugging that into the equation we get

$$\begin{aligned} 2Ae^{-t} - 2Ate^{-t} - 2Ate^{-t} + At^2e^{-t} + 2(2Ate^{-t} - At^2e^{-t}) + At^2e^{-t} \\ = 2Ae^{-t} = 3e^{-t}. \end{aligned}$$

hence we find that $A = \frac{3}{2}$ and the general solution of the equation is $y(t) = (\frac{3}{2}t^2 + c_1 + c_2t)e^{-t}$.

$$y'' + 4y = t^2 + 3e^t$$

Solution: The characteristic equation gives $0 = r^2 + 4$. The solutions are $r = \pm 2i$ and thus $\{\cos 2t; \sin 2t\}$ is a fundamental set of solutions. Again we use **undetermined coefficients**. We find special solutions for each of the right–hand–sides t^2 and $3e^t$, respectively. For the first we suppose that $y = A + Bt + Ct^2$ and find that

$$2C + 4(A + Bt + Ct^2) = (2C + 4A) + 4Bt + 4Ct^2 = t^2.$$

Hence $C = \frac{1}{4}, B = 0$ and $A = -\frac{1}{8}$. The corresponding special solution is $-\frac{1}{8} + \frac{1}{4}t^2$. For $3e^t$ we assume $y = Ae^t$ and find

$$Ae^t + 4Ae^t = 3e^t,$$

and hence $A = \frac{3}{5}$ and the special solution is $\frac{3}{5}e^t$. Finally we take the sum of the two special solutions. The general solution is

$$y(t) = -\frac{1}{8} + \frac{1}{4}t^2 + \frac{3}{5}e^t + c_1 \cos 2t + c_2 \sin 2t.$$

$$y'' + 4y = 1$$

Solution: The set of fundamental solutions is the same as in the previous problem. We use here **variation of parameters**, even though the method of undetermined coefficients would work as well. We write

$y = c_1(t)\cos 2t + c_2(t)\sin 2t$ and assume the first of the following two equations while we obtain the second via plugging into the equation:

$$\begin{aligned} c_1'(t)\cos 2t + c_2'(t)\sin 2t &= 0 \\ -2c_1'(t)\sin 2t + 2c_2'(t)\cos 2t &= 1. \end{aligned}$$

Solving the linear system we get

$$\begin{aligned} c_1'(t) &= \frac{-\sin 2t}{2} \\ c_2'(t) &= \frac{\cos 2t}{2}. \end{aligned}$$

This yields via integration by t

$$\begin{aligned} c_1(t) &= \frac{\cos 2t}{4} \\ c_2(t) &= \frac{\sin 2t}{4}. \end{aligned}$$

That gives $y = \frac{1}{4}(\cos^2 2t + \sin^2 2t) + c_1 \cos 2t + c_2 \sin 2t = \frac{1}{4} + c_1 \cos 2t c_2 \sin 2t$. (*The clever student would have guessed it!!*).

$$x^2 y'' - 3xy' + 4y = 0, \quad x > 0$$

Solution: This is an **Euler equation**. Its characteristic equation is $0 = r(r-1) - 3r + 4 = r^2 - 4r + 4$ with $r = 2$ as the only (double) solution. Hence $\{x^2, x^2 \log x\}$ is a fundamental set of solutions and $y(x) = x^2(c_1 + c_2 \log x)$ is the genneral form of a solution.

4. Solve the given differential equation by means of power series about $x_0 = 0$

$$y'' - xy' - y = 0.$$

Solution:

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$
$$y'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1}$$
$$xy'(x) = \sum_{k=0}^{\infty} k a_k x^k$$
$$y''(x) = \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2}$$
$$y''(x) = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k$$
$$y'' - xy' - y = \sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} - (k+1)a_k] x^k = 0$$

Hence $(k+2)(k+1)a_{k+2} - (k+1)a_k = 0$ or $a_{k+2} = \frac{a_k}{k+2}$. If $y_1(0) = 1, y_1'(0) = 0$ and $y_2(0) = 0$ and $y_2'(0) = 1$ then

$$y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{2 \cdot 4}x^4 + \frac{1}{2 \cdot 4 \cdot 6}x^6 + \cdots$$
$$y_2(x) = x + \frac{1}{3}x^3 + \frac{1}{3 \cdot 5}x^5 + \frac{1}{3 \cdot 5 \cdot 7}x^7 + \cdots .$$

A general solution will have the form $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

5. For the following system of differential equations

$$x' = x - x^2 - xy$$
$$y' = 3y - xy - 2y^2$$

(a) determine all critical points,
(b) find the corresponding linear system at each critical point,
(c) find the eigenvalues and eigenvectors of each of these linear systems and draw a coclusion about the behavior of solutions with initial values close to the critical point,
(d) sketch a phase portrait using nullclines.

Solution: (a) Denote by $F(x, y) = x - x^2 - xy = x(1 - x - y)$ and $G(x, y) = 3y - xy - 2y^2 = y(3 - x - 2y)$ the right–hand–sides of the two equations for x' and y', respectively. Now $F(x, y) = 0$ if and only if either $x = 0$ or $1 - x - y = 0$, and $G(x, y) = 0$ if and only if $y = 0$ or $3 - x - 3y = 0$. This leads to four systems of linear equations like for example

$$x = 0$$
$$3 - x - 3y = 0.$$

Each has (at most) one solution. Hence we obtain the four critical points $(0,0), (1,0), (0, \frac{3}{2})$ and $(-1, 2)$.

(b) The linearized system at a critical point (x_0, y_0) is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Now $F_x(x,y) = 1 - 2x - y$, $F_y(x,y) = -x$, $G_x(x,y) = -y$ and $G_y(x,y) = 3 - x - 4y$. Hence the matrices of the linearized system at $(0,0), (1,0), (0, \frac{3}{2})$ and $(-1, 2)$, respectively, are

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & 0 \\ -\frac{3}{2} & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -2 & -4 \end{pmatrix}.$$

(c) $(x_0, y_0) = (0,0)$: The eigenvalues are (obviously) 1 and 3, the corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Hence this is a (proper) node, a source and unstable for the linear system. The same is therefore true for the original system.

$(x_0, y_0) = (1,0)$. This matrix is upper triangular. Therefore the eigenvalues are -1 and 2. The corresponding eigenvectors are determined by the systems

$$\begin{pmatrix} -1 - r & -1 \\ 0 & 2 - r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for $r = -1$ and $r = 2$. One finds $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$, respectively. (Notice that the vector must be nonzero and is unique up to scaling only!!). The eigenvalues determine that this point is a saddle point and unstable for the linearized and hence for the original system.

$(x_0, y_0) = (0, \frac{3}{2})$: This time the matrix is lower triangular and therefore the eigenvalues are $-\frac{1}{2}$ and -3, respectively and we have to solve

$$\begin{pmatrix} -\frac{1}{2} - r & 0 \\ -\frac{3}{2} & -3 - r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for both eigenvalues r to find the corresponding eigenvector. We find $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. Since both eigenvalues are negative this is a proper node, a sink and stable for the linear and thus for the original system.

$(x_0, y_0) = (-1, 2)$: We have to compute the *characteristic polynomial* to determine the eigenvalues (they are not so obvious as in the previous cases):

$$\det\begin{pmatrix} 1-r & 1 \\ -2 & -4-r \end{pmatrix} = (1-r)(-4-r)-(-2)1 = r^2+3r-2.$$

The roots of this polynomial are $r_{1/2} = -\frac{3}{2} \pm \frac{\sqrt{17}}{2}$. For the corresponding eigenvectors we have to solve the linear system corresponding to the matrix we computed the determinant for, with r being the eigenvalue. Keeping our fingers crossed we *know* that if we solve one of the two equations the other is automatically satisfied (if we computed the eigenvalues correctly!!). Hence the eigenvectors are $\begin{pmatrix} 1 \\ r-1 \end{pmatrix}$ for each of the eigenvalues, or more precisely $\begin{pmatrix} 1 \\ -\frac{5}{2} \pm \frac{\sqrt{17}}{2} \end{pmatrix}$. One eigenvalue is positive while the other is negative (just look at the formula for the roots of the quadratic polynomial!). Hence the critical point is a saddle point and unstable for the linearized system and so it has the same properties for the nonlinear system we want to study.

(d) I will most likely not be able to scan a picture of the phase diagram in time. However, the nullclines consist of the lines determined by the factors for F and G and the analysis using them is therefore similar to the competing species model we discussed in class and which can be found in the handout. Only difference: We allow negative values of x and y!!

6. Show that the following functions are Liapunov functions for the critical point $(0,0)$ and analyze the stability properties for each of following systems of differential equations near $(0,0)$:

(a) $V(x,y) = x^4 + y^4$ for

$$\begin{aligned} x' &= -x^3 + y^3 \\ y' &= -x^3 - y^3 \end{aligned}$$

Solution: Check first that V is decreasing along solutions.

$$\begin{aligned} \dot{V}(x,y) &= V_x(x,y)F(x,y) + V_y(x,y)G(x,y) \\ &= 3x^3(-x^3+y^3) + 3y^3(-x^3-y^3) \\ &= -3x^6 + 3x^3y^3 - 3x^3y^3 - 3y^6 = -3(x^6+y^6). \end{aligned}$$

$\dot{V}(x,y)$ is negative definite. On the other hand V is positive definite. Hence Liapunov second method tell sus that $(0,0)$ is a sink and stable.

(b) $V(x, y) = x^4 - y^4$ for

$$x' = -x^3 + y^3$$
$$y' = x^3 + y^3.$$

Solution:

$$\begin{aligned}\dot{V}(x, y) &= V_x(x, y)F(x, y) + V_y(x, y)G(x, y)\\ &= 3x^3(-x^3 + y^3) - 3y^3(x^3 + y^3)\\ &= -3x^6 + 3x^3y^3 - 3x^3y^3 - 3y^6 = -3(x^6 + y^6).\end{aligned}$$

$\dot{V}(x, y)$ is negative definite. On the other hand V is negative for all $(0, y)$ which can be chosen arbitrarily close to the origin $(0, 0)$. Hence Liapunov's second method tells us that $(0, 0)$ is an unstable point.

7. The following autonomous system of differential equations is expressed in polar coordinates. Determine all periodic solutions, all limit cycles, and determine their stability properties:

$$r' = r(r - 1)(r - 2)$$
$$\theta' = -1.$$

Solution: $\theta(t) = -t + t_0$. Hence the solution turns clockwise around the origin. The right–hand–side for r' is *positive* for $r \in (0, 1)$ and $r > 2$ and negative for $r \in (1, 2)$ like a little analysis of the graph of the cubic polynomial $r(r-1)(r-2)$ reveals. We have periodic solutions where $r' = 0$, i.e. $r = 1$ and $r = 2$ ($r = 0$ is the stationary solution which could be considered as a periodic solution). Since r is increasing when $r' > 0$ and decreasing if $r' < 0$ we see that $r = 1$ is a stable periodic solution while $r = 2$ is an unstable periodic solution. The critical (stationary) point $(0, 0)$ corresponding to $r = 0$ is unstable.