Homework 7

Topology II

Winter 2016/17

Problem 1
(1) Explain the CW-structure of $\mathbb{C}P^n$ and $\mathbb{R}P^n$
(2) Study the notion of a Δ-complex and discuss examples and counterexamples (see Hatcher). Explain that each Δ–complex is a CW-complex.

Problem 2
The Euler characteristic of a topological space X is defined if the homology $H_*(X)$ is of finite rank and given by

$$\chi(X) := \sum_{k=0}^{\infty} (-1)^k \mathrm{rk}(H_k(X)).$$

(1) Describe $\chi(X)$ for a finite CW-complex or a finite Δ-complex X without knowledge on the boundary operator.
(2) Let $Y \to X$ be a finite covering with d preimages for any point in x and assume that X is a finite CW-complex. Then Y can also be given the structure of a finite CW-complex and the Euler characteristics are related by $\chi(Y) = d\chi(X)$.
(3) Let X, Y be CW-complexes. Show that $X \times Y$ is also a CW-complex. If X, Y are finite CW-complexes then $X \times Y$ can also be finite and $\chi(X \times Y) = \chi(X)\chi(Y)$.

Problem 3
(1) Compute the homology of $\vee_{\alpha \in A} X_\alpha$ of spaces X_α with a point x_α with a contractible neighbourhood.
(2) Compute the homology of the following spaces using apropriate CW-structures:
(a) S^2 with north- and south-pole identified.
(b) The space obtained by deleting the interior of two disjoint closed discs in the interior of the closed unit discs and identifying all three boundary components using clockwise parametrization with $[0, 1]$ of constant length.
(c) The space obtained from S^2 by identifying $x \sim -x$ on the equator and the space obtained from S^3 by doing the same on the equatorial 2-sphere.

Problem 4
Let (X, A) be a CW-pair. Show that there is a relative cellular chain complex whose groups are given by $H_k(X^k, X^{k-1} \cup A^k)$ and whose boundary operator is defined similarly to the one for the cellular complex of X and A. Show how it fits into a long exact sequence as relative homology. Show that the cellular relative homology is isomorphic to the relative homology, and that all these isomorphism commute with all maps in the two long exact sequences, i.e. are natural.