QUANTUM NETWORKS MODELLED BY GRAPHS

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Abstract. Quantum networks are often modelled using Schrödinger operators on metric graphs. To give meaning to such models one has to know how to interpret the boundary conditions which match the wave functions at the graph vertices. In this article we give a survey, technically not too heavy, of several recent results which serve this purpose. Specifically, we consider approximations by means of "fat graphs" — in other words, suitable families of shrinking manifolds — and discuss convergence of the spectra and resonances in such a setting.

1. Introduction

Quantum mechanics on metric graphs is a subject with a long history which can be traced back to the paper of Ruedenberg and Scherr [21] on spectra of aromatic carbohydrate molecules elaborating an idea of L. Pauling. A new impetus came in the eighties from the need to describe semiconductor graph-type structures, cf. [10], and the interest to these problems driven both by mathematical curiosity and practical applications is steadily growing; we refer to [13, 14] or the proceedings [4] for a bibliography to the subject.

Since quantum graphs are supposed to model various real graph-like structures with the transverse size which is small but non-zero, one has to ask naturally how close are such system to an "ideal" graph in the limit of zero thickness. This problem is not easy and a reasonable complete answer is known in case of "fat graphs" with Neumann boundary conditions and similar systems. A pioneering work in this area was done by Freidlin and Wentzell [11] and the papers [15] and [20] can be mentioned as important milestones. We managed to contribute to this problem in a series of papers, [8], [19] and [9], in which we improved the approximation using the intrinsic geometry of the manifold only, demonstrating the norm resolvent convergence, and finally extending the approximation also to resonances by means of complex scaling.

While these results provide in our opinion a solid insight into the Neumann-type situation, we must acknowledge as the authors that the three papers are long and rather technical, and some may find them not easy to read. This motivated us to write the present survey in which we intend to describe this family of approximation results without switching in the heavy machinery; let the reader judge whether we have succeeded.

Before proceeding let us mention that there is an encouraging recent progress in the more difficult Dirichlet case, see [2, 12, 17, 7], however, we will not discuss it here.

Let us briefly describe the contents of the paper. In the next section we describe the two basic objects of this paper, quantum graphs and graph-like manifolds (cf. Figure 1). Section 3 is devoted to convergence of the discrete spectrum summarizing the main results of Ref. [8]. An extension to non-compact graphs and a resolvent convergence coming from [19] is given in Section 4. Finally, in Section 5 describe the results of Ref. [9] showing how the resonances on quantum graphs and graph-like manifolds approximate each other.

2. Quantum networks and graphs

2.1. **Quantum graphs.** Suppose that X_0 is a connected metric graph given by (V, E, ∂, ℓ) where (V, E, ∂) is a usual graph, i.e., V denotes the set of vertices, E denotes the set of edges, $\partial\colon E \longrightarrow V \times V$ associates to each edge e the pair $(\partial_- e, \partial_+ e)$ of its initial and terminal point (and therefore

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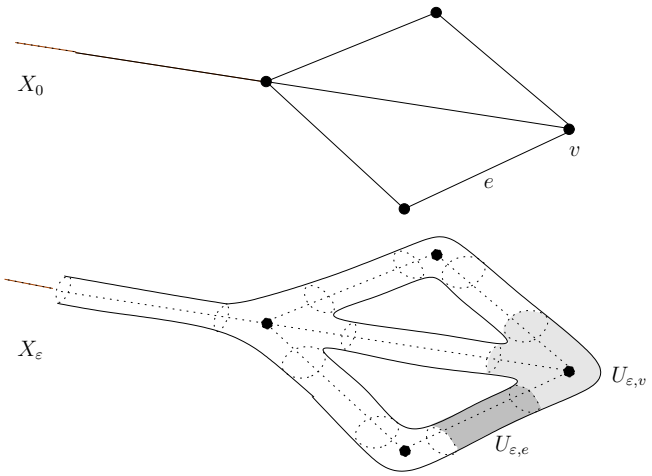

FIGURE 1. The metric graph X_0 with one external edge, five internal edges and four vertices and the associated graph-like manifold X_ε, here with cross section manifold $F = \mathbb{S}^1$.

an orientation). The space X_0 being a *metric* graph means that there is a *length function* $\ell\colon E \longrightarrow (0,\infty]$ associating to each edge e a length ℓ_e. We often identify the edge e with the interval $(0,\ell_e)$ and use $x = x_e$ as a coordinate. In addition, we denote $\mathrm{d}x = \mathrm{d}x_e$ the Lebesgue measure on e. In this way, X_0 becomes a topologically 1-dimensional space with singularities at the vertices. Moreover, X_0 carries a natural metric by defining the distance of two points to be the length of the shortest path in Γ joining these points.

We call an edge e *external* iff $\ell_e = \infty$ and *internal* otherwise and we denote the corresponding sets by E_{ext} and E_{int}. Properly speaking, the map ∂ is only defined on $E_{\mathrm{int}} \times E_{\mathrm{int}}$. For external edges, we do not consider the "end point" at infinity as a vertex of Γ, i.e., ∂e contains only the initial vertex $\partial_- e = \partial e$. For each vertex $v \in V$ we set

$$E_v^\pm := \{\, e \in E \,|\, \partial_\pm e = v \,\} \qquad \text{and} \qquad E_v := E_v^+ \uplus E_v^-,$$

i.e., $E_v^\pm$ consists of all edges starting $(-)$ resp. ending $(+)$ at v and E_v is their disjoint union. Note that the *disjoint* union is necessary in order to allow single-edge loops, i.e., edges having the same initial and terminal point. We adopt the following uniform bounds on the degree $\deg v := |E_v|$ and the length function ℓ:

$$1 \le \deg v \le d_0, \qquad v \in V, \tag{2.1a}$$

$$\ell_e \ge \ell_0, \qquad e \in E, \tag{2.1b}$$

where $1 \le d_0 < \infty$ and $0 < \ell_0 \le 1$. Needless to say, both the assumptions are fulfilled if $|E|$ and $|V|$ are finite.

We consider the Hilbert space defined naturally as the orthogonal sum,

$$\mathcal{H}_0 := \mathsf{L}_2(X_0) = \bigoplus_{e\in E} \mathsf{L}_2(e).$$

A metric graph X_0 becomes a *quantum* graph when it is equipped with a self-adjoint operator Δ_{X_0} in the Hilbert space $\mathsf{L}_2(X_0)$. The latter is assumed to act as

$$(\Delta_{X_0} f)_e := -f_e''$$

on each edge for $f \in \mathsf{H}^2(\Gamma^1)$, so a quantum graph is determined by fixing the *vertex boundary conditions* at each vertex, in order to turn the formal Laplacian into a self-adjoint operator. We will use the *free* (often called, not quite properly, *Kirchhoff*) boundary conditions: a function lies in the operator domain

$$\mathcal{H}_0^2 := \operatorname{dom}\Delta_{X_0} \qquad \text{iff} \qquad f \in \bigoplus_{e\in E} \mathsf{H}^2(e)$$

and the relations

$$f_{e_1}(v) = f_{e_2}(v), \qquad e_1, e_2 \in E_v \tag{2.2a}$$

$$\sum_{e\in E_v} \vec{f}'_e(v) = 0 \tag{2.2b}$$

are fulfilled for all $v \in V$ where

$$\vec{f}'_e(v) := \begin{cases} -f'_e(0), & \text{if } v = \partial_- e, \\ +f'_e(\ell_e), & \text{if } v = \partial_+ e \end{cases} \tag{2.3}$$

defines the *inward* derivative of f'_e at v. Under the assumptions (2.1), the operator Δ_{X_0} is self-adjoint in $\mathcal{H}_0$ (cf. [14]). The quadratic form associated with the operator Δ_{X_0} is $\mathfrak{d}_0(f) := \|(\Delta_{X_0})^{1/2} f\|^2$ and can be expressed as

$$\mathfrak{d}_0(f) = \|f'\|^2 = \sum_{e\in E} \|f'_e\|^2 \qquad \text{on} \qquad \mathcal{H}_0^1 := \mathsf{H}^1(X_0) := \mathsf{C}(X_0) \cap \bigoplus_{e\in E} \mathsf{H}^1(e).$$

2.2. **Graph-like manifolds.** Let us pass to a model of a quantum network which we will consider, corresponding to the idea that the graph has a small, but non-zero thickness. Let X_ε be a d-dimensional connected manifold with metric g_ε (cf. Figure 1). If X_ε has boundary, we denote it by ∂X_ε; let us stress that our discussion covers different kind of models, "full" fat graphs considered by Kuchment and Zeng [15, 16] where the boundary is present, as well as "hollow" or "sleeve-type" manifolds having no boundary. We assume that X_ε can be decomposed into open sets $U_{\varepsilon,e}$ and $U_{\varepsilon,v}$, i.e.,[1]

$$X_\varepsilon = \overline{\biguplus_{e\in E}} U_{\varepsilon,e} \;\overline{\uplus}\; \overline{\biguplus_{v\in V}} U_{\varepsilon,v}. \tag{2.4}$$

Denote the metric on X_ε by g_ε. We assume that $U_{\varepsilon,e}$ and $U_{\varepsilon,v}$ are isometric to

$$U_{\varepsilon,e} \cong (e \times F, g_{\varepsilon,e}) \qquad g_{\varepsilon,e} = \mathrm{d}x_e^2 + \varepsilon^2 h \tag{2.5a}$$

$$U_{\varepsilon,v} \cong (U_v, g_{\varepsilon,v}) \qquad g_{\varepsilon,v} = \varepsilon^2 g_v \tag{2.5b}$$

where (F, h) is a compact m-dimensional manifold with $m := (d-1)$, and (U_v, g_v) is an ε-independent d-dimensional manifold[2]. Note that $\partial U_{\varepsilon,v} \setminus \partial X_\varepsilon$ has $(\deg v)$-many components isometric to $(F, \varepsilon^2 h)$ denoted by $(\partial_e U_v, \varepsilon^2 h)$ for $e \in E_v$. We sometimes write $U_e := e \times F$.

Remark 1. For technical reasons (cf. Lemma 5) we assume that near $\partial_e U_v$, the (unscaled) manifold (U_v, g_v) has a collar neighborhood $((0, \ell_0/2) \times F, \mathrm{d}\hat{x}^2 + \varepsilon^2 h)$. If such a collar neighborhood happens to be too small, we can just change the decomposition (2.4) in such a way that we add a cylinder of length $\varepsilon\ell_0/2$ (the length taken in the edge coordinates x on U_e) from the edge neighborhood to the vertex neighborhood (becoming here a cylinder of length $\ell_0/2$ in the vertex coordinates on U_v since $\hat{x} = x/\varepsilon$).

[1] Here and in the following, the expression $A = \overline{\biguplus}_i A_i$ means that the A_i's are open (in A), mutually disjoint and the interior of $\bigcup_i \overline{A}_i$ equals A; recall that in an L_2-theory it is enough to have charts covering a set of full measure.

[2] We employ just these particular charts; there is no need for a complete system. Strictly speaking, Figure 1 shows a slightly different situation where the edge neighborhoods are shortened (cf. Remark 2 below).

The cross section manifold F has a boundary or does not have one, depending on the analogous property of X_ε. For simplicity, we suppose that $\mathrm{vol}_m F = 1$. Clearly, we have

$$\mathrm{d}U_{\varepsilon,e} = \varepsilon^m \,\mathrm{d}F\,\mathrm{d}x_e \tag{2.6}$$

for the Riemannian densities. We consider the Hilbert space

$$\mathcal{H}_\varepsilon = \mathsf{L}_2(X_\varepsilon)$$

and the Laplacian $\Delta_{X_\varepsilon} \geq 0$ (with Neumann boundary conditions if $\partial X_\varepsilon \neq \emptyset$) defined on

$$\mathcal{H}^2_\varepsilon := \mathrm{dom}\,\Delta_{X_\varepsilon}.$$

The associated quadratic form $\mathfrak{d}_\varepsilon(u) := \|(\Delta_{X_\varepsilon})^{1/2}u\|^2$ can be expressed as

$$\mathfrak{d}_\varepsilon(u) = \|\mathrm{d}u\|^2 \qquad \text{on} \qquad \mathcal{H}^1_\varepsilon := \mathsf{H}^1(X_\varepsilon)$$

where $\mathrm{d}u$ is the exterior derivative of u. Note that the quadratic form expression contains only the metric g_ε, but no derivatives of g_ε.

Remark 2. We have chosen the full edge length on the edge neighborhood although this assumption is not valid if X_ε is the $\varepsilon/2$-neighborhood of a metric graph X_0 *embedded* into a Euclidean space from which it inherited its metric. In such a situation, however, the metric $g_{\varepsilon,e}$ differs from the metric $\mathrm{d}x_e^2 + \varepsilon^2 h$ only by a small longitudinal error. Using the fact that the Laplacian on X_ε defined via its quadratic form depends only on g_ε (and not on its derivatives), it can be shown that a small (uniform) perturbation of the product structure has only a small effect on the Laplacian, its spectrum, etc.

In addition, we assume that the following uniformity conditions are valid,

$$c_{\mathrm{vol}} := \sup_{v\in V} \mathrm{vol}_d\, U_v < \infty, \qquad\qquad \lambda_2 := \inf_{v\in V} \lambda_2^{\mathrm{N}}(U_v) > 0, \tag{2.7}$$

where $\lambda_2^{\mathrm{N}}(U_v)$ denotes the second (i.e., first non-zero) Neumann eigenvalue of (U_v, g_v).

Roughly speaking, the requirements (2.7) mean that the region U_v remains small w.r.t. the vertex index — see the discussion in [19, Rem. 2.7]) for more details. Needless to say, these assumptions are trivially satisfied once the vertex set V is finite.

Recall also one more domain related to the Laplacian for $\varepsilon \geq 0$. We denote by

$$\mathcal{H}^k_\varepsilon := \mathrm{dom}(\Delta_{X_\varepsilon} + 1)^{k/2}$$

the scale of Hilbert spaces associated with the self-adjoint, non-negative operator Δ_{X_ε}, together with its natural norm

$$\|u\|_k := \|(\Delta_{X_\varepsilon} + 1)^{k/2}u\|.$$

For negative exponents, we set

$$\mathcal{H}^{-k}_\varepsilon := \left(\mathcal{H}^k_\varepsilon\right)^*;$$

note that $\mathcal{H}^{-k}_\varepsilon$ can be again viewed as the completion of $\mathcal{H}_\varepsilon \subset \mathcal{H}^{-k}_\varepsilon$ with respect to the appropriate norm, in this case $\|u\|_{-k} = \|(\Delta_{X_\varepsilon} + 1)^{-k/2}u\|$.

3. Convergence of discrete spectrum

Let us start with a simple thing, a convergence result using a variational characterization of discrete eigenvalues. We assume that X_0 is a compact metric graph, in other words,

$$|E| < \infty \quad \text{and} \quad \ell_e < \infty \quad \text{for all } e \in E.$$

In this case, the graph-like manifold X_ε is also compact, and the spectrum of Δ_{X_ε}, $\varepsilon \geq 0$, is purely discrete. We denote by $\lambda_k(\varepsilon)$ the k-th eigenvalue of Δ_{X_ε}, $k \geq 1$, repeated with respect to the multiplicity. The min-max variational characterization is then

$$\lambda_k(\varepsilon) = \inf_{L_k} \sup_{u\in L_k\setminus\{0\}} \frac{\mathfrak{d}_\varepsilon(u)}{\|u\|^2} \tag{3.1}$$

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Theorem 24. *Assume that the metric graph X_0 is finite and non-compact, cf.* (5.1). *Then the complex dilated Laplacians H_0^θ and H_ε^θ are $\mathcal{O}(\varepsilon^{1/2})$-close, i.e.*

$$\|JH_0^\theta - H_\varepsilon^\theta J\|_{2,\theta\to-2,\theta} = \|R_\varepsilon^\theta J - JR_0^\theta\| = \mathcal{O}(\varepsilon^{1/2}),$$

where $\mathcal{O}(\varepsilon^{1/2})$ depends on θ.

Proof. The proof is essentially the same as for the non-dilated case. First we define operators on the scales of order one, specifically

$$J^{1,\theta}\colon \mathcal{H}_0^{1,\theta} \longrightarrow \mathcal{H}_\varepsilon^{1,\theta} \qquad \text{and} \qquad J^{1,\theta\prime}\colon \mathcal{H}_\varepsilon^{1,\theta} \longrightarrow \mathcal{H}_0^{1,\theta}$$

in exactly the same way as in (3.4) and (3.5). Recall that we do not consider the boundary points between the internal and external parts as vertices, i.e. the graph-like manifold does not have a vertex neighborhood there. Hence

$$T_\varepsilon^\theta J^{1,\theta} = J^1 T_0^\theta \qquad \text{and} \qquad T_0^\theta J^{1,\theta\prime} = J^{1\prime} T_\varepsilon^\theta,$$

and then we have

$$R_\varepsilon^\theta J - JR_0^\theta = R_\varepsilon^\theta\big[(J - J^{-1,\theta})H_0^\theta + (J^{-1,\theta}H_0^\theta - H_\varepsilon^\theta J^{1,\theta}) + H_\varepsilon^\theta(J^{1,\theta} - J)\big]R_0^\theta$$

where $J^{-1,\theta} := (J^{1,\overline{\theta}\prime})^*$. The last difference at the right-hand side can be estimated by

$$\begin{aligned}\|R_\varepsilon^\theta H_\varepsilon^\theta(J^{1,\theta} - J)R_0^\theta\| &\le \|\mathrm{id}_\varepsilon - R_\varepsilon^\theta\| \cdot \|(J^{1,\theta} - J)R_0^\theta\| \\ &\le (1 + \|R_\varepsilon^\theta\|)\|T_\varepsilon^{-\theta}\| \cdot \|(J^1 - J)(\Delta_0 + 1)^{-1/2}\|\,\|(\Delta_0 + 1)^{1/2}T_0^\theta R_0^\theta\| \\ &\le (1 + C^\theta)\mathrm{e}^{-\mathrm{Re}\,\theta/2}\mathcal{O}(\varepsilon^{1/2})C^\theta\end{aligned}$$

using Lemma 14 and Lemma 23, and the first one can be treated similarly. For the remaining term we observe that in order to prove

$$\|R_\varepsilon^\theta(J^{-1,\theta}H_0^\theta - H_\varepsilon^\theta J^{1,\theta})R_0^\theta\| = \|J^{-1,\theta}H_0^\theta - H_\varepsilon^\theta J^{1,\theta}\|_{2,\theta\to-2,\theta} \le \delta$$

it suffices to show

$$\big|\langle H_0^{\overline{\theta}} f, J^{1,\theta\prime}u\rangle - \langle J^{1,\theta}f, H_\varepsilon^\theta u\rangle\big| \le \widetilde{\delta}\|T_0^{\overline{\theta}} f\|_1 \|T_\varepsilon^\theta u\|_1 \tag{5.4}$$

for $f \in \mathcal{H}_0^{2,\overline{\theta}}$ and $u \in \mathcal{H}_\varepsilon^{2,\theta}$. From the compatibility, we obtain $\|T_\varepsilon^\theta u\|_1 \le C^\theta\|u\|_{2,\theta}$ and similarly for f; in particular, we can choose $\delta = (C^\theta)^2\widetilde{\delta}$, however, the estimate (5.4) is almost the same as in the non-dilated case given in Lemma 15. □

As in the non-dilated case, on can develop a functional calculus for the pairs of operators H_0^θ and H_ε^θ – cf. [9, App. B]. Since now the operators are not self-ajoint, we only have a *holomorphic* functional calculus. In particular, we can show

$$\|\mathbb{1}_D(H_\varepsilon^\theta)J - J\mathbb{1}_D(H_0^\theta)\| \le \mathcal{O}(\varepsilon^{1/2}) \quad \text{and} \quad \|\mathbb{1}_D(H_\varepsilon^\theta) - J\mathbb{1}_D(H_0^\theta)J^*\| \le \mathcal{O}(\varepsilon^{1/2})$$

for the spectral projections, provided D is an open disc containing a single discrete eigenvalue $\lambda(0)$ of H_0^θ. From here our main result on resonances follows:

Theorem 25. *Assume that the metric graph X_0 is finite and non-compact, cf.* (5.1). *If $\lambda(0)$ is a resonance of the Laplacian Δ_{X_0} with a multiplicity $m > 0$, then for a sufficiently small $\varepsilon > 0$ there exist m resonances $\lambda_1(\varepsilon), \dots, \lambda_m(\varepsilon)$ of Δ_{X_ε}, satisfying $\mathrm{Im}\,\lambda_j(\varepsilon) < 0$ and not necessarily mutually different, which all converge to $\lambda(0)$ as $\varepsilon \to 0$. The same is true in the case when $\lambda(0)$ is an embedded eigenvalue of Δ_{X_0}, except that then only $\mathrm{Im}\,\lambda_j(\varepsilon) \le 0$ holds in general.*

Finally, if the multiplicity of $\lambda(0)$ is one with a normalized eigenfunction $u^\theta(0)$ (corresponding to a resonance *or* embedded eigenvalue *for Δ_{X_0}), then there exists a normalized eigenfunction $u^\theta(\varepsilon)$ (related to the respective entity for Δ_{X_ε}) on the graph-like manifold) such that*

$$\|Ju^\theta(0) - u^\theta(\varepsilon)\| = \mathcal{O}(\varepsilon^{1/2}).$$

where the infimum is taken over all k-dimensional subspaces L_k of $\mathsf{H}^1(X_\varepsilon)$.

Our main result in this section, coming from [20, 15, 8], is the following:

Theorem 3. *Assume that the metric graph X_0 is compact, i.e., the graph is finite and each edge has a finite length. Then the k-th (Neumann) eigenvalue of the Laplacian on the graph-like manifold $\lambda_k(\varepsilon)$ converges to the k-th eigenvalue of the Laplacian on the metric graph X_0 with the free boundary conditions. Specifically, we have*

$$\lambda_k(\varepsilon) - \lambda_k(0) = \mathcal{O}(\varepsilon^{1/2})$$

where the error term $\mathcal{O}(\varepsilon^{1/2})$ depends on the index k.

The proof is based on an abstract comparison result. In order to compare the respective eigenvalues of the quantum graph and the graph-like manifold, we need identification maps expressed in terms of the quadratic forms, namely

$$J^1\colon \mathcal{H}_0^1 \longrightarrow \mathcal{H}_\varepsilon^1 \qquad \text{and} \qquad J^{1\prime}\colon \mathcal{H}_\varepsilon^1 \longrightarrow \mathcal{H}_0^1. \tag{3.2}$$

We have the following abstract eigenvalue comparison result (see, e.g., [8, Lem. 2.1]):

Lemma 4. *Assume that there are $\delta_1 = \delta_1(\varepsilon)$ and $\delta_2 = \delta_2(\varepsilon)$ such that*

$$\mathfrak{d}_0(f) + \delta_1\|f\|_1^2 \ge \mathfrak{d}_\varepsilon(J^1 f), \qquad \|f\|^2 - \delta_1\|f\|_1^2 \le \|J^1 f\|^2, \tag{3.3a}$$

$$\mathfrak{d}_\varepsilon(u) + \delta_2\|u\|_1^2 \ge \mathfrak{d}_0(J^{1\prime}u), \qquad \|u\|^2 - \delta_2\|u\|_1^2 \le \|J^{1\prime}u\|^2 \tag{3.3b}$$

for all $f \in \mathcal{H}_0^1$ and $u \in \mathcal{H}_\varepsilon^1$. Then

$$\frac{-2(1+\lambda_k(0))(1+\delta_1)}{1 - \big(\delta_1 + \delta_2(1+\delta_1)\big)(1+\lambda_k(0))}\,\delta_2 \le \lambda_k(\varepsilon) - \lambda_k(0) \le \frac{2(1+\lambda_k(0))}{1 - \delta_1(1+\lambda_k(0))}\,\delta_1$$

where the upper bound is $\mathcal{O}(\delta_1)$ depending on $\lambda_k(0)$ and the lower bound is $\mathcal{O}(\delta_2)$ depending additionally on δ_1.

In our concrete example, the $\mathcal{H}_0^1 \to \mathcal{H}_\varepsilon^1$ identification operator can be chosen as

$$J^1 f(z) := \begin{cases} \varepsilon^{-m/2} f_e(x) & \text{if } z = (x,y) \in U_e, \\ \varepsilon^{-m/2} f(v) & \text{if } z \in U_v \end{cases} \tag{3.4}$$

Note that the definition makes sense since functions in $\mathcal{H}_0^1$ are continuous. For the map in the opposite direction, we first introduce the following averaging operators

$$(N_e u)(x) := \langle \mathbb{1}, u_e(x,\cdot)\rangle_F = \int_F u_e(x,y)\,\mathrm{d}F(y),$$

$$C_v u := \langle \mathbb{1}_v, u_v\rangle_{U_v} = \frac{1}{\mathrm{vol}_d\, U_v}\int_{U_v} u\,\mathrm{d}U_v$$

for $u \in \widetilde{\mathcal{H}} = \mathsf{L}_2(X_\varepsilon)$. What they yield is nothing else than the Fourier coefficient corresponding to the first (transverse) eigenfunction $\mathbb{1}(y) = 1$ on F and $\mathbb{1}_v(z) = (\mathrm{vol}_d\, U_v)^{-1}$ on U_v, respectively; note that these eigenfunctions are constant and that $\mathrm{vol}_m F = 1$. We set

$$(J^{1\prime}u)_e(x) := \varepsilon^{m/2}\Big(N_e u(x) + \sum_{v\in\partial e}\rho(\mathrm{dist}(x,v))\big(C_v u - N_e u(v)\big)\Big) \tag{3.5}$$

for $x \in e$, where $\rho\colon \mathbb{R} \longrightarrow [0,1]$ is a smooth function with

$$\rho(0) = 1 \quad \text{and} \quad \rho(r) = 0 \quad \text{for all } r \ge \ell_0/2. \tag{3.6}$$

The interpolating contribution related to $C_v u - N_e u(v)$ is needed in order to make $J^{1\prime}u$ continuous at each vertex. The following lemma ensures that the error coming from this correction remains small:

Lemma 5. *We have*

$$\varepsilon^m |C_v u - N_e u(v)|^2 \le \frac{8}{\ell_0}\Big(1+\frac{1}{\lambda_2}\Big)\varepsilon \|\mathrm{d}u\|^2_{\varepsilon,v}$$

for all $u \in \mathcal{H}^1_\varepsilon$ and $v \in \partial e$, where $\|\cdot\|_{\varepsilon,v}$ denotes the L_2-norm on $U_{\varepsilon,v}$.

Proof. Due to Remark 1, each component $\partial_e U_v$ of ∂U_v has a collar neighborhood $(0,\ell_0/2)\times F$ of length $\ell_0/2$ (in the unscaled coordinates of U_v). The Cauchy-Schwarz inequality and the Sobolev trace estimate (see e.g. [14, Lem. 8])

$$|f(0)|^2 \le \frac{8}{\ell_0}\int_0^{\ell_0/2} \big(|f(s)|^2 + |f'(s)|^2\big)\,\mathrm{d}s \tag{3.7}$$

yield (with $f(s) = f_y(s) = C_v u - u(s,y)$ in the collar coordinates, and then integration over $y \in F$)

$$|C_v u - N_e u(v)|^2 \le \int_F |C_v u - u(0,\cdot)|^2\,\mathrm{d}F \le \frac{8}{\ell_0}\big(\|C_v u - u\|_v^2 + \|\mathrm{d}u\|_v^2\big),$$

where $C_v u$ is considered to be a constant function on U_v and $\|\cdot\|_v$ is the L_2-norm on U_v. Now $C_v u - u$ is orthogonal to the first (constant) eigenfunction of the Neumann Laplacian on U_v, and the min-max principle ensures that the squared norm of $C_v u - u$ can be estimated by $(\lambda_2^{\mathrm{N}}(U_v))^{-1}\|\mathrm{d}u\|^2_{U_v}$. Using the scaling $g_{\varepsilon,v} = \varepsilon^2 g_v$ of the metric and (2.7), we obtain the desired estimate. □

We also have to make sure that eigenfunctions u of Δ_{X_ε} belonging to eigenvalues *bounded* with respect to ε, cannot concentrate on the vertex neighborhoods:

Lemma 6. *We have*

$$\|u\|^2_{\varepsilon,v} \le c_{\mathrm{vx}}\varepsilon\big(\|\mathrm{d}u\|^2_{\varepsilon,v} + \|u\|^2_{\varepsilon,e} + \|\mathrm{d}u\|^2_{\varepsilon,e}\big)$$

for $u \in \mathcal{H}^1_\varepsilon$, where e is any edge adjacent to the vertex v. The constant c_{vx} in this inequality depends only on ℓ_0, λ_2 and c_{vol}.

Proof. We employ the estimate

$$\|u\|_{\varepsilon,v} \le \|u - C_v u\|_{\varepsilon,v} + \sqrt{\mathrm{vol}_d\, U_{\varepsilon,v}}\big(|C_v u - N_e u(v)| + |N_e u(v)|\big).$$

The first summand can be treated as in the previous proof, the second by Lemma 5, and the last one by a Sobolev trace estimate on the *edge* neighborhood similar to (3.7), namely $\varepsilon^m|N_e u(v)|^2 \le 8(\|u\|^2_{\varepsilon,e} + \|\mathrm{d}u\|^2_{\varepsilon,e})/\ell_0$. □

Proof of Theorem 3. It remains to show that the conditions (3.3) are fulfilled. We do not give the details here referring to [8, Sec. 5]. The proof of (3.3a) is simple, and it works even with $\delta = 0$, hence one obtains a stronger estimate, $\lambda_k(\varepsilon) \le \lambda_k(0)$.

For the opposite inequality, we need to verify (3.3b). To this end we need Lemma 5 in the norm and a quadratic form estimate. The norm estimate uses in addition Lemma 6, and the estimate

$$\|u\|^2_{\varepsilon,e} - \|Nu\|^2_{\varepsilon,e} \le \mathcal{O}(\varepsilon)\|u\|^2_1$$

which follows from (4.2). For the quadratic form estimate we need the simple Cauchy-Schwarz bound $\|\mathrm{d}u\|^2_{\varepsilon,e} \le \|(Nu)'\|^2_{\varepsilon,e}$. □

Remark 7. Similar results can be obtained for more general situations when the vertex and edge neighborhoods scale at different rates, cf. [8, 16], and in certain situations also for the Dirichlet Laplacian, cf. [18], where, however, the resulting graph operator is decoupled.

This is crucial for the above mentioned reformulation. Recall that by the most common definition a *resonance* is a pole in a meromorphic continuation of the resolvent over the cut corresponding to the essential spectrum into the "unphysical sheet" of the Riemann energy surface [6]. Rotating the essential spectrum one can reveal these singularities; this allows us to identify a *resonance* of $H^0_\varepsilon = \Delta_{X_\varepsilon}$ with a complex L_2-eigenvalue of the dilated operator H^θ_ε for $\operatorname{Im}\theta$ large enough. Notice also that such a definition is consistent: it does not depend on where we cut the spaces into an interior and exterior part; of course, as far as the interior part remains compact.

5.2. **Convergence result.** In order to demonstrate convergence properties of resonances, we need to introduce a scale of Hilbert spaces associated to the non-selfadjoint operator H^θ_ε. In particular, we set $\mathcal{H}^{2,\theta} := \operatorname{dom} H^\theta_\varepsilon$ with the norm $\|u\|_{2,\theta} := \|(H^\theta_\varepsilon + 1)u\|$ and $\mathcal{H}^{-2,\theta} := (H^{\overline{\theta}}_\varepsilon)^*$ with the dual norm; for details we refer to [9, App. A].

Definition 21. We say that an operator $J\colon \mathcal{H}_0 \longrightarrow \mathcal{H}_\varepsilon$ is *δ-quasi-unitary* with respect to H^θ_ε, iff $J^*J = \mathrm{id}_0$, $\|J\| = 1$, and

$$\|JJ^* - \mathrm{id}_\varepsilon\|_{2,\theta\to 0} = \|(JJ^* - \mathrm{id}_\varepsilon)R^\theta_\varepsilon)\| \le \delta,$$

where $R^\theta_\varepsilon := (H^\theta_\varepsilon + 1)^{-1}$ and $\|A\|_{2,\theta\to 0}$ is the operator norm of $A\colon \mathcal{H}^{2,\theta}_\varepsilon \longrightarrow \mathcal{H}^0_\varepsilon$.

It is much easier to use the δ-quasi-unitarity with respect to the non-dilated operator (via its quadratic form) as in Definition 8. In particular, we would like to compare the non-dilated scale $\mathcal{H}^1_\varepsilon$ of order 1 and the dilated scale $\mathcal{H}^{2,\theta}_\varepsilon$ of order 2:

Definition 22. Δ_ε and H^θ_ε are *compatible* if there is a family of bounded, invertible operators T^θ_ε on $\mathcal{H}_\varepsilon$ such that $T^{-\theta}_\varepsilon = (T^\theta_\varepsilon)^{-1}$, $T^{\overline{\theta}}_\varepsilon = (T^\theta_\varepsilon)^*$ and $T^{-\theta}_\varepsilon(\mathcal{H}^1_\varepsilon) \subset \mathcal{H}^{2,\theta}_\varepsilon$. Given $\varepsilon \ge 0$, we say that Δ_ε and H^θ_ε are *uniformly* compatible with respect to ε, if there is a constant C^θ, *independent* of ε, such that

$$\|T^\theta_\varepsilon\|_{2,\theta\to 1} = \|(\Delta_\varepsilon + 1)^{1/2}T^\theta_\varepsilon R^\theta_\varepsilon\| \le C^\theta \qquad \text{and} \qquad \|R^\theta_\varepsilon\| \le C^\theta.$$

In the situation we consider there is a natural candidate for T^θ_ε, namely

$$T^\theta_\varepsilon u := u_{\mathrm{int}} \oplus \mathrm{e}^{\theta/2}u_{\mathrm{ext}}.$$

In contrast to U^θ_ε, the operators T^θ_ε are defined also for *complex* values of θ. Again, as for the analyticity, the proof of uniform compatibility in our example needs some technical preliminaries which we skip here. In essence, one needs to define the resolvent R^θ_ε as a bounded operator from $\mathcal{H}^{-1,\theta}$ to $\mathcal{H}^{1,\theta}$ with ε-independent bound, where

$$\mathcal{H}^{1,\theta} := T^\theta(\mathcal{H}^1) = \Big\{ f \in \mathsf{H}^1(X_{\varepsilon,\mathrm{int}}) \oplus \mathsf{H}^1(X_{\varepsilon,\mathrm{ext}}) \;\Big|\; u_{\mathrm{ext}} = \mathrm{e}^{\theta/2}u_{\mathrm{int}} \text{ on } \Gamma_\varepsilon \Big\}$$

is an appropriate space of order 1 for H^θ_ε, not necessarily related to $\operatorname{dom}(H^\theta_\varepsilon + 1)^{1/2}$.

Lemma 23. *For a given $\varepsilon > 0$, the operators Δ_{X_ε} and its complex dilated counterparts H^θ_ε are (uniformly) compatible. In particular, the map J as defined in* (4.1) *is $\mathcal{O}(\varepsilon^{1/2})$-quasi-unitary with respect to H^θ_ε, where $\mathcal{O}(\varepsilon^{1/2})$ depends on θ.*

Proof. The compatibility is demonstrated in [9, App. C]. The second assertion follows from

$$\begin{aligned}\|(JJ^* - \mathrm{id}_\varepsilon)R^\theta_\varepsilon)\| &\le \|T^{-\theta}_\varepsilon\| \cdot \|(JJ^* - \mathrm{id}_\varepsilon)(\Delta_\varepsilon + 1)^{-1/2})\| \cdot \|(\Delta_\varepsilon + 1)^{1/2}T^\theta_\varepsilon R^\theta_\varepsilon\| \\ &\le \mathrm{e}^{-\operatorname{Re}\theta/2}\mathcal{O}(\varepsilon^{1/2})C^\theta\end{aligned}$$

where we employed the fact that $T^\theta_\varepsilon J = JT^\theta_0$ and Lemma 9. □

Now we are in position to state our first convergence result of this section:

one-parameter unitary groups on $\mathcal{H}_0 = \mathsf{L}_2(X_0)$ and $\mathcal{H}_\varepsilon = \mathsf{L}_2(X_\varepsilon)$, respectively, acting non-trivially on the *external* part only. We call the operator

$$H_\varepsilon^\theta := U_\varepsilon^\theta \Delta_{X_\varepsilon} U_\varepsilon^{-\theta}$$

for $\varepsilon \geq 0$ the *dilated* Laplacian on X_ε with the domain $\operatorname{dom} H_\varepsilon^\theta := U_\varepsilon^\theta(\mathcal{H}_\varepsilon^2)$ for *real* θ. A simple calculation shows that $(H_\varepsilon^\theta u)_e = (\Delta_{X_\varepsilon} u)_e$ holds for internal edges and that

$$(H_0^\theta f)_e = -\mathrm{e}^{-2\theta} f_e'' \qquad \text{and} \qquad (H_\varepsilon^\theta u)_e = -\mathrm{e}^{-2\theta}\partial_{xx} u_e + \frac{1}{\varepsilon^2}\Delta_F u_e \tag{5.2}$$

is true for external edges with the domain $\mathcal{H}_\varepsilon^{2,\theta} := \operatorname{dom} H_\varepsilon^\theta$ given by

$$\mathcal{H}_\varepsilon^{2,\theta} := \Big\{ f \in \mathsf{H}^2(X_{\varepsilon,\mathrm{int}}) \oplus \mathsf{H}^2(X_{\varepsilon,\mathrm{ext}}) \,\Big|\, u_{\mathrm{ext}} = \mathrm{e}^{\theta/2} u_{\mathrm{int}}, \quad \vec{u}'_{\mathrm{ext}} = \mathrm{e}^{3\theta/2}\vec{u}'_{\mathrm{int}} \text{ on } \Gamma_\varepsilon \Big\}. \tag{5.3}$$

Here, $\mathsf{H}^2(X_{\varepsilon,\bullet})$ are the functions from $\operatorname{dom}\Delta_{X_\varepsilon}$ restricted to $X_{\varepsilon,\bullet}$, i.e. without any condition at Γ_ε. In addition, $\vec{u}'_\bullet$ is defined as the longitudinal derivative on the common boundary Γ_ε oriented *away* from the internal part.

The expression of H_ε^θ now can be generalized to *complex* θ in the strip

$$S_\vartheta = \big\{ \theta \in \mathbb{C} \,\big|\, |\operatorname{Im}\theta| < \vartheta/2 \big\}$$

where $0 \leq \vartheta < \pi$; we call the corresponding H_ε^θ the *complex dilated* Laplacian. The operators $\{H_\varepsilon^\theta\}_\theta$ form a family with spectrum contained in the common sector

$$\Sigma_\vartheta := \big\{ z \in \mathbb{C} \,\big|\, |\arg z| \leq \vartheta \big\}.$$

Moreover, we can determine the essential spectrum coming from the external part. Note that the branches of the essential spectrum associated with the higher transverse eigenvalues $\lambda_k(F)$, $k \geq 2$, on the graph-like manifold all vanish as $\varepsilon \to 0$.

Lemma 18. *Let the metric graph be finite and non-compact, i.e.* (5.1) *is fulfilled. Then*

$$\sigma_{\mathrm{ess}}(H_0^\theta) = \mathrm{e}^{-2\theta}[0,\infty) \qquad \textit{and} \qquad \sigma_{\mathrm{ess}}(H_\varepsilon^\theta) = \frac{1}{\varepsilon^2}\bigcup_{k\in\mathbb{N}} \big(\lambda_k(F) + \mathrm{e}^{-2\theta}[0,\infty)\big).$$

In particular, since $\lambda_1(F) = 0$, we have

$$\sigma_{\mathrm{ess}}(H_\varepsilon^\theta) \cap B = \mathrm{e}^{-2\theta}[0,\infty) \cap B$$

for any bounded set $B \subset \mathbb{C}$ provided $\varepsilon > 0$ is small enough.

In addition, we have the following important result:

Theorem 19. *For z not contained in the ϑ-sector Σ_ϑ, the resolvents*

$$R_\varepsilon^\theta(z) := (H_\varepsilon^\theta - z)^{-1}$$

depend analytically on $\theta \in S_\vartheta$.

This is a highly non-trivial fact since H_ε^θ is neither of type A nor of type B, in other words, both the sesquilinear form and the operator domains depend on θ even for real θ. To put it differently, the (non-smooth) exterior scaling as defined here is a very singular perturbation of the operator $\Delta_{X_\varepsilon} = H_\varepsilon^0$. The main idea is to compare R_ε^θ with the resolvent of a *decoupled* operator, where one imposes Dirichlet boundary conditions at Γ_ε. An inspiration for such an idea was used in [5]; for a full proof in our situation we refer to [9]. As a consequence, we have the following result on the discrete spectrum.[3]

Lemma 20. *The discrete spectrum of H_ε^θ is locally constant in θ. As a consequence, discrete (complex) eigenvalues are "revealed" if $\operatorname{Im}\theta$ is positive and large enough. The same is true for eigenvalues embedded into the continuous spectrum $[0,\infty)$ of the Laplacian; in this case it is sufficient to have $\operatorname{Im}\theta > 0$.*

[3]One can derive also other spectral properties, e.g., $\sigma_{\mathrm{sc}}(H_\varepsilon^\theta) = \emptyset$ – see [9, Prop. 5.8] for more details.

4. Resolvent convergence

Next we would like to go further and prove also results for non-compact graphs, and also convergence of eigenfunctions or resolvents. To do so we need some more notation. We write $\mathcal{H}_\varepsilon$ and $\Delta_\varepsilon = \Delta_{X_\varepsilon}$ for the ε-dependent spaces, $\varepsilon > 0$. We stress that the parameter ε enters only through the quantity $\delta = \delta_\varepsilon > 0$ and one can interpret it as a label for the second Hilbert space involved – see also the appendix in [19] for the concept of a "distance" between two Hilbert spaces and associated non-negative operators. For brevity, we set

$$R_\varepsilon := (\Delta_\varepsilon + 1)^{-1}, \qquad \Delta_\varepsilon := \Delta_{X_\varepsilon}, \qquad \varepsilon \geq 0.$$

Definition 8. We say that an operator $J\colon \mathcal{H}_0 \longrightarrow \mathcal{H}_\varepsilon$ is *δ-quasi-unitary* with respect to Δ_ε, iff $J^*J = \mathrm{id}_0$, $\|J\| = 1$, and

$$\|JJ^* - \mathrm{id}_\varepsilon\|_{1\to 0} = \|(JJ^* - \mathrm{id}_\varepsilon)R_\varepsilon^{1/2})\| \leq \delta,$$

where id_ε is the identity on $\mathcal{H}_\varepsilon$, and $\|A\|_{1\to 0}$ is the operator norm of $A\colon \mathcal{H}_\varepsilon^1 \longrightarrow \mathcal{H}_\varepsilon^0$.

In our particular situation, we will employ the quasi-unitary operator

$$Jf := \{f_e \otimes \mathbb{1}_\varepsilon\}_e \oplus \{0_v\}_v, \tag{4.1}$$

where $\mathbb{1}_\varepsilon = \varepsilon^{-m/2}\mathbb{1}$ is the lowest normalized eigenfunction on $(F, \varepsilon^2 h)$ and in turn 0_v is the zero function on U_v. The quasi-unitarity is stated in the following lemma:

Lemma 9. *The map J defined by (4.1) is δ_ε-quasi-unitary, where $\delta_\varepsilon = \mathcal{O}(\varepsilon^{1/2})$ depends only on ℓ_0, c_{vol} and λ_2.*

Proof. A simple calculation shows that $J^*J = \mathrm{id}_0$, $\|J\| = 1$, and that

$$\|JJ^*u - u\|^2 = \sum_{e\in E} \|N_e u - u\|_{\varepsilon,e}^2 + \sum_{v\in V} \|u\|_{\varepsilon,v}^2.$$

The function $N_e u(x) - u(x,\cdot)$ is orthogonal to the constant function on F, and by the min-max principle we infer that

$$\|N_e u - u\|_{\varepsilon,e}^2 \leq \frac{\varepsilon^2}{\lambda_2(F)} \|\mathrm{d}_F u\|_{\varepsilon,e}^2 \leq \frac{\varepsilon^2}{\lambda_2(F)} \|\mathrm{d}u\|_{\varepsilon,e}^2 \tag{4.2}$$

where $\lambda_2(F)$ is the first non-zero (Neumann) eigenvalue on F, and d_F is the derivative with respect to the transverse variable(s). The estimate of the sum over the vertex contributions follows from Lemma 6. □

We also need a tool to compare the Laplacians on $\mathcal{H}_0$ and $\mathcal{H}_\varepsilon$. To this end we put:

Definition 10. We say that Δ_ε and Δ_0 are *δ-close* w.r.t. the map $J\colon \mathcal{H}_0 \longrightarrow \mathcal{H}_\varepsilon$ iff

$$\|J\Delta_0 - \Delta_\varepsilon J\|_{2\to -2} = \|R_\varepsilon J - JR_0\| \leq \delta,$$

where $\|A\|_{2\to -2}$ denotes the operator norm of $A\colon \mathcal{H}_0^2 \longrightarrow \mathcal{H}_\varepsilon^{-2}$.

Remark 11. Note that a 0-quasi-unitary map is indeed unitary. Furthermore, if Δ_0 and Δ_ε are 0-close with respect to a 0-quasi-unitary map J, then Δ_0 and Δ_ε are unitarily equivalent. In this sense, the concept of quasi-unitarity and closeness provides a quantitative way to measure how far a pair of operators is from being unitarily equivalent.

In order to show that the operators Δ_ε and Δ_0 are δ-close, it is often easier to deal with the respective quadratic form domains as we have already done when demonstrating the convergence of the discrete spectrum. We thus want to compare the identification operators on the scale of order 1 with the quasi-unitary map J:

Definition 12. We say that the identification maps (3.2) are *δ-compatible* with the map $J\colon \mathcal{H}_0 \longrightarrow \mathcal{H}_\varepsilon$ iff

$$\|J - J^1\|_{1\to 0} = \|(J - J^1)R_0^{1/2}\| \leq \delta \quad \text{and} \quad \|J^* - J^{1\prime}\|_{1\to 0} = \|(J^* - J^{1\prime})R_\varepsilon^{1/2}\| \leq \delta.$$

By means of an adjoint we obtain from $J^{1\prime}$ a natural map

$$J^{-1} := (J^{1\prime})^* \colon \mathcal{H}_0^{-1} \longrightarrow \mathcal{H}_\varepsilon^{-1}.$$

Now it is easy to derive the following criterion for δ-closeness:

Lemma 13. *Assume that J^1 and $J^{1\prime}$ are δ-compatible w.r.t. the map J, and that*

$$\|J^{-1}\Delta_0 - \Delta_\varepsilon J^1\|_{1\to-1} = \|R_\varepsilon^{1/2}(J^{-1}\Delta_0 - \Delta_\varepsilon J^1)R_0^{1/2}\| \le \delta. \tag{4.3}$$

Then Δ_ε and Δ_0 are 3δ-close with respect to J.

We first check that the identification maps J^1 and $J^{1\prime}$ are indeed $\mathcal{O}(\varepsilon^{1/2})$-compatible with respect to the map J:

Lemma 14. *The maps J^1 and $J^{1\prime}$ as defined in* (3.4) *and* (3.5) *are δ_ε-compatible with J, where $\delta_\varepsilon = \mathcal{O}(\varepsilon^{1/2})$ depends only on ℓ_0, d_0, c_{vol} and λ_2.*

Proof. We have

$$\|(J - J^1)f\|^2 = \varepsilon^{-m}\sum_{v\in V}(\mathrm{vol}_d U_{\varepsilon,v})|f(v)|^2 = \varepsilon\sum_{v\in V}(\mathrm{vol}_d U_v)|f(v)|^2.$$

By a standard Sobolev estimate – see, e.g., estimate (3.7) or [19, Lem. 2.4]) – we can estimate the latter sum by $8\varepsilon c_{\mathrm{vol}}\|f\|_1^2/\ell_0$. For the other identification operator we have

$$\|(J^* - J^{1\prime})u\|^2 = \sum_{e\in E}\sum_{v\in\partial e}\int_0^{\ell_0/2}\rho(r)^2\,\varepsilon^m\big|C_v u - N_e u(v)\big|^2\,\mathrm{d}r.$$

Using now Lemma 5 and reordering the sum, we obtain the additional factor d_0, the maximum degree of a vertex, and the second estimate follows as well. □

Next, we will now indicate briefly how to prove the closeness of the Laplacians:

Lemma 15. *The Laplacians Δ_ε and Δ_0 are δ_ε-close with respect to the map J defined in* (4.1) *where $\delta_\varepsilon = \mathcal{O}(\varepsilon^{1/2})$ depends only on ℓ_0, d_0 and λ_2.*

Proof. We check the condition (4.3) of Lemma 13 which reduces to estimating

$$\big|\mathfrak{d}_0(f, J^{1\prime}u) - \mathfrak{d}_\varepsilon(J^1 f, u)\big| \le \sum_{e\in E}\sum_{v\in\partial e}\|f'\|_e\int_0^{\ell_0/2}|\rho'(r)|^2\,\mathrm{d}r\,\varepsilon^{m/2}\big|C_v u - N_e u(v)\big|.$$

in terms of $\delta_\varepsilon\|f\|_1\|u\|_1$; the claim follows from Lemma 5 and Cauchy-Schwarz. □

Putting together the previous results, we come to the following conclusion:

Theorem 16. *Adopt the uniformity conditions* (2.1) *and* (2.7). *Then the Laplacians Δ_{X_ε} and Δ_{X_0} are $\mathcal{O}(\varepsilon^{1/2})$-close with respect to the quasi-unitary map J defined in* (4.1), *i.e.*

$$\|(\Delta_{X_\varepsilon} + 1)^{-1}J - J(\Delta_{X_0} + 1)^{-1}\| \le \mathcal{O}(\varepsilon^{1/2}),$$

where the error term depends only on ℓ_0, d_0, c_{vol} and λ_2. In addition, we have

$$\|(\Delta_{X_\varepsilon} + 1)^{-1} - J(\Delta_{X_0} + 1)^{-1}J^*\| \le \mathcal{O}(\varepsilon^{1/2}). \tag{4.4}$$

Proof. The first estimate follows from Lemmata 13–15; the second one in turn is a consequence of the first estimate and Lemma 9. □

One can now develop the standard functional calculus for the Laplacians Δ_ε and Δ_0, and deduce estimates similar to the ones in Theorem 16, but with the resolvent replaced by more general functions $\varphi(\Delta_\varepsilon)$ of the Laplacians. Specifically, φ need to be measurable, continuous in a neighborhood of the spectrum of Δ_0, and the limit at infinity must exist. For example, one can control the heat operators via $\varphi_t(\lambda) = \mathrm{e}^{-t\lambda}$ or the spectral projectors via $\varphi = \mathbb{1}_I$. A proof of the following result can be found in the appendices of the paper [19], see also [9]:

Theorem 17. *Under the assumptions of the previous theorem, we have*

$$\|\mathbb{1}_I(\Delta_{X_\varepsilon})J - J\mathbb{1}_I(\Delta_{X_0})\| \le \mathcal{O}(\varepsilon^{1/2}) \quad \textit{and} \quad \|\mathbb{1}_I(\Delta_{X_\varepsilon}) - J\mathbb{1}_I(\Delta_{X_0})J^*\| \le \mathcal{O}(\varepsilon^{1/2})$$

for the spectral projections provided I is a compact interval such that $\partial I \cap \sigma(\Delta_{X_0}) = \emptyset$. In particular, if I contains a single eigenvalue $\lambda(0)$ of Δ_{X_0} with multiplicity one corresponding to an eigenfunction $u(0)$, then there is an eigenvalue $\lambda(\varepsilon)$ and an eigenfunction $u(\varepsilon)$ of Δ_{X_ε} such that

$$\|Ju(0) - u(\varepsilon)\| = \mathcal{O}(\varepsilon^{1/2}).$$

In addition, the spectra converge uniformly on $[0,\Lambda]$, i.e.

$$\sigma(\Delta_{X_\varepsilon}) \cap [0,\Lambda] \to \sigma(\Delta_{X_0}) \cap [0,\Lambda]$$

in the sense of Hausdorff distance on compact subsets of $[0,\Lambda]$. The same result is true if we consider only the essential or the discrete spectral components.

Naturally, the above stated spectral convergence reduces to the claim of Theorem 3 in the situation when the spectra are purely discrete.

5. Convergence of resonances

In the final section we will deal with the convergence of resonances in the present setting. It is useful to include into the considerations also eigenvalues embedded in the continuous spectrum, because it may happen that resonances of a "fat graph" converge to such an eigenvalue, as it can be seen, e.g., in a simple motivating example of the metric graph consisting of a single loop with a half-line "lead" attached [9].

A standard and successful method of dealing with resonances is based on the concept of *complex scaling*, often an *exterior* one. The method has its roots in the seminal papers [1, 3] and a lot of work was devoted to it; we refer to [9] for a sample bibliography. The main virtue is that it allows to reformulate treatment of resonances, i.e. poles of the analytically continued resolvent, and embedded eigenvalues, to analysis of discrete eigenvalues of a suitable *non-selfadjoint* operator. As we will see below the complex-scaling approach suits perfectly, in particular, to our convergence analysis.

In this section, we assume that the metric graph is finite, but non-compact, i.e.

$$|E_{\mathrm{int}}| < \infty \qquad \text{and} \qquad 0 < |E_{\mathrm{ext}}| < \infty\,, \tag{5.1}$$

which means, in particular, that the assumptions (2.1) and (2.7) are satisfied.

5.1. **Exterior scaling.** We decompose the metric graph X_0 and the graph-like manifold X_ε into an *interior* and *exterior* part $X_{\varepsilon,\mathrm{int}}$ and $X_{\varepsilon,\mathrm{ext}}$, respectively. For technical reasons, it is easier to do the cut not at the initial vertices ∂e of an external edge $e \in E_{\mathrm{ext}}$, but at a fixed distance, say one, from ∂e along e, and similarly for the graph-like manifold. We therefore consider the *internal* metric graph $X_{0,\mathrm{int}}$ consisting of all vertices, all edges of finite length and the edge parts $(0,1)$ for each external edge. The exterior metric graph $X_{0,\mathrm{ext}}$ is just the disjoint union of $|E_{\mathrm{ext}}|$-many copies of a half-line $[0,\infty)$, and we use the corresponding parametrization on an external edge. In other words, we do not regard the *boundary points* $\Gamma_0 = \partial X_{0,\mathrm{int}} \cap \partial X_{0,\mathrm{ext}}$ as vertices. Similarly, let Γ_ε be the common boundary of $X_{\varepsilon,\mathrm{int}}$ and $X_{\varepsilon,\mathrm{ext}}$; note that Γ_ε is isometric to $|E_{\mathrm{ext}}|$-many copies of $(F, \varepsilon^2 h)$.

Now we introduce the exterior dilation operator. For $\theta \in \mathbb{R}$ we define by

$$(U_0^\theta f)(x) := \begin{cases} f(x), & x \in X_{0,\mathrm{int}} \\ \mathrm{e}^{\theta/2} f(\mathrm{e}^\theta x), & x \in X_{0,\mathrm{ext}} \end{cases} \qquad \text{and}$$

$$(U_\varepsilon^\theta u)(z) := \begin{cases} u(z), & z \in X_{\varepsilon,\mathrm{int}} \\ \mathrm{e}^{\theta/2} u(\mathrm{e}^\theta x, y), & z = (x,y) \in X_{\varepsilon,\mathrm{ext}} \end{cases}$$