PERIODIC MANIFOLDS, SPECTRAL GAPS, AND EIGENVALUES IN GAPS

OLAF POST

Abstract. We investigate spectral properties of the Laplace operator on a class of non-compact Riemannian manifolds. We prove that for a given number N we can construct a periodic manifold such that the essential spectrum of the corresponding Laplacian has at least N open gaps. Furthermore, by perturbing the periodic metric of the manifold locally we can prove the existence of eigenvalues in a gap of the essential spectrum.

1. Introduction

There has been done many work in the analysis of periodic Schrödinger or divergence type operators. It is well-known that the spectrum of a Schrödinger-operator with periodic potential has band-gap structure under certain conditions (see e.g. [HH95]), i.e., the spectrum is the locally finite union of compact intervals and there exist an interval (a, b) not lying in the spectrum but with essential spectrum above and below the interval. Here, we want to give an example for a periodic Laplacian on a manifold *without* potential which has spectral gaps. Therefore we obtain the same qualitative results *only* by the periodic geometry. As in the Schrödinger case a decoupling procedure is responsible for the gaps. Related results can be found in [DH87] and [G97].

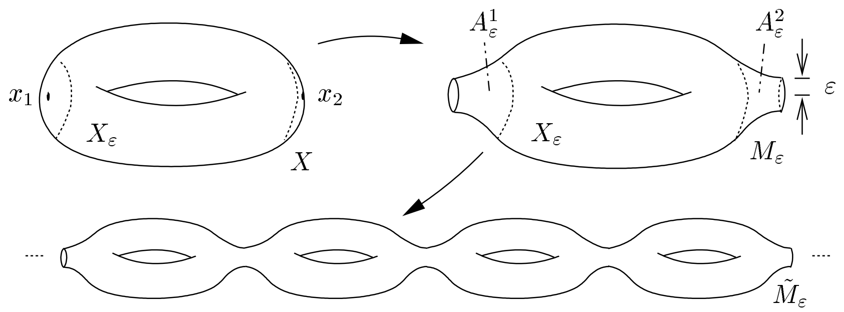

Figure 1. Construction of the periodic manifold $\tilde{M}_\varepsilon$

We start our construction of a periodic manifold with spectral gaps from a compact Riemannian manifold X (for simplicity without boundary). We choose

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two different points $x_1, x_2 \in X$ and attach to x_1 resp. x_2 a cylindrical neighbourhood A^1_ε resp. A^2_ε (cf. Figure 1) where the "free" boundary of A^i_ε is isometric to a sphere of radius $\varepsilon > 0$. We call the resulting manifold M_ε. By glueing together $\mathbb{Z}$ copies of the period cell M_ε we obtain a $\mathbb{Z}$-periodic manifold $\widetilde{M}_\varepsilon$. Our first main result is the following:

Theorem 1.1. *For each $N \in \mathbb{N}$ there exist at least N gaps in the spectrum of the Laplacian on the periodic manifold $\widetilde{M}_\varepsilon$ provided ε is small enough.*

The proof basically uses Floquet Theory for which we refer to the next section. More examples of periodic manifolds with spectral gaps can be found in [P00].

Next, we locally perturb the metric of a periodic manifold $\widetilde{M}$ with a spectral gap (a, b) to produce eigenvalues in the gap. Again, such effects are well studied in the case of Schrödinger or divergence type operators (see e.g. [DH86], [AADH94] or [HB00]). To simplify the notation, we only allow a conformal perturbation supported on a compact subset. More general settings (i.e., infinite range perturbations and non-conformal perturbations) can be found in [P00].

Here, the perturbation is a blow-up of some compact area, i.e., the manifold $\widetilde{M}$ is perturbed by conformal factors $\rho_\tau \colon \widetilde{M} \longrightarrow]0, \infty[$ starting from the constant function 1 for $\tau = 0$ and growing up to infinity only on a compact area as $\tau \to \infty$ (outside this area nothing is changed). The *Decomposition Principle* (see Theorem 4.2) assures that a spectral gap (a, b) of $\Delta_{\widetilde{M}}$ remains a spectral gap in the essential spectrum of $\Delta_{\widetilde{M}(\tau)}$ for all $\tau \geq 0$. Our second main result is the following:

Theorem 1.2. *Let $\lambda \in (a, b)$ be in a spectral gap. Then an infinite number of pairs (τ, u) with $\tau > 0$ and $u \neq 0$ such that $\Delta_{M(\tau)} u = \lambda u$ exist.*

The idea of the proof is quite simple (see [AADH94] or [HB00]). We show that the eigenfunctions of the full problem on $\widetilde{M}$ can be approximated by eigenfunctions of an approximating problem on M^n (consisting of n copies of the period cell M), see Theorem 4.3. On the compact manifold M^n we can apply the Min-max Principle to assure the existence of eigenfunctions of the approximating problem (Theorem 4.8).

2. Periodic Manifolds and Floquet Theory

For a Riemannian manifold M (compact or not) we denote by $L_2(M)$ the usual L_2-space of square integrable functions on M with respect to the volume measure on M. The corresponding norm will be denoted by $\|\cdot\|_M$. For $u \in C^\infty_c(M)$, the space of compactly supported smooth functions, we set

$$q_M(u) := \int_M |\mathrm{d}u|^2.$$

Here $\mathrm{d}u$ denotes the exterior derivate of u, which is a section of the cotangent bundle over M. The *Laplacian* Δ_M (for a manifold without boundary) is defined via the (closure of the) quadratic form, i.e., $q_M(u) = \langle \Delta_M u, u \rangle$ for $u \in C_c^\infty(M)$ (for details on quadratic forms see e.g. [RS80]). We therefore obtain a self-adjoint operator with spectrum lying in $[0, \infty[$.

If M is a compact manifold with (piecewise) smooth boundary $\partial M \neq \emptyset$ we can define the Laplacian with *Dirichlet* resp. *Neumann boundary conditions* in the same way. Here, we start from the (closure of the) quadratic form q_M defined on $C_c^\infty(M)$, the space of smooth functions with support *away* from the boundary, resp. on $C^\infty(M)$, the space of smooth functions up to the boundary. The corresponding operator will be denoted by Δ_M^{D} resp. Δ_M^{N}.

If M is compact the spectrum of Δ_M (with any boundary condition if $\partial M \neq \emptyset$) is purely discrete. We denote the corresponding eigenvalues by $\lambda_k(M)$ (resp. $\lambda_k^{\mathrm{D}}(M)$ or $\lambda_k^{\mathrm{N}}(M)$ in the Dirichlet or Neumann case) written in increasing order and repeated according to multiplicity. The *Min-max Principle* allows us to express the k-th eigenvalue of Δ_M in terms of the quadratic form q_M, i.e.,

$$\lambda_k(M) = \inf_L \sup_{u \in L, u \neq 0} \frac{q_M(u)}{\|u\|_M^2}, \tag{1}$$

where the infimum is taken over all k-dimensional subspaces L of the domain of the (closed) quadratic form q_M (see e.g. [D96]). Of course, the same is true for the Laplacians with boundary conditions.

A d-dimensional (non-compact) Riemannian manifold $\tilde{M}$ will be called Γ-*periodic* if $\Gamma = \mathbb{Z}^r$ acts properly discontinously, isometrically and cocompactly, i.e., the quotient $\tilde{M}/\Gamma$ is a d-dimensional compact Riemannian manifold such that the quotient map is a local isometry. Throughout this article we study manifolds of dimension $d \geq 2$.

A closed (compact) subset M of $\tilde{M}$ is called *period cell* if M is the closure of a fundamental domain D, i.e., $M = \overline{D}$, D is open and connected, D is disjoint from any translate γD for all $\gamma \in \Gamma$, $\gamma \neq 0$, and the union over all translates γM is equal to $\tilde{M}$.

Floquet theory allows us to analyse the spectrum of the Laplacian on $\tilde{M}$ by analysing the spectra of Laplacians with quasi-periodic boundary conditions on a period cell M. In order to do this, we define θ-periodic boundary conditions. Let θ be an element of the dual group $\hat{\Gamma} = \mathrm{Hom}(\Gamma, \mathbb{T}^1)$ of $\Gamma = \mathbb{Z}^r$, which is isomorphic to the r-dimensional torus $\mathbb{T}^r = \{\theta \in \mathbb{C}^r;\, |\theta_i| = 1 \text{ for all } i\}$. Denote by Δ_M^θ the operator corresponding to the quadratic form q_M defined on the space of smooth functions u on M satisfying

$$u(\gamma x) = \overline{\theta(\gamma)}\, u(x)$$

for all $x \in \partial M$ and all $\gamma \in \Gamma$ such that $\gamma x \in \partial M$. Again, Δ_M^θ has purely discrete spectrum denoted by $\lambda_k^\theta(M)$. The eigenvalues depend continuously on θ. From

Floquet theory we obtain

$$\operatorname{spec}\Delta_{\tilde{M}} = \bigcup_{\theta\in\hat{\Gamma}} \operatorname{spec}\Delta_M^\theta = \bigcup_{k\in\mathbb{N}} B_k(\tilde{M})$$

where $B_k = B_k(\tilde{M}) = \{\lambda_k^\theta(M);\, \theta \in \hat{\Gamma}\}$ is a compact interval, called *k-th band* (see e.g. [RS78], [D81]). In general, we do not know whether the intervals B_k overlap or not. But we can show the existence of gaps by proving that $\lambda_k^\theta(M)$ does not vary too much in θ.

3. Construction of a Periodic Manifold

Suppose that X is a compact Riemannian manifold of dimension $d \geq 2$ (for simplicity without boundary). We want to construct a $\mathbb{Z}^r$-periodic manifold. We choose $2r$ distinct points $x_1, \dots, x_{2r}$. For each point x_i, denote by B_ε^i the open geodesic ball around x_i of radius $\varepsilon > 0$. Suppose further that $B_{\varepsilon_0}^i$ are pairwise disjoint, where $\varepsilon_0 > 0$ denotes the injectivity radius of X. Denote by B_ε the union of all balls B_ε^i. Let $X_\varepsilon := X \setminus B_{2\varepsilon}$ for $0 < 2\varepsilon < \varepsilon_0$ with metric inherited from X.

We now define the modified metric. For simplicity, we assume that the metric g is flat on B_{ε_0}, i.e., g is given in polar coordinates $(s,\sigma) \in]0,\varepsilon_0[\times\mathbb{S}^{d-1}$ around x_i by

$$g = \mathrm{d}s^2 + s^2\mathrm{d}\sigma^2,$$

where $\mathrm{d}\sigma^2$ denotes the standard metric on the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$. For a more general setting see [P00]. Let r_ε be a smooth monotone function with $r_\varepsilon(s) = \varepsilon$ in a neighbourhood of $s = 0$ and $r_\varepsilon(s) = s$ for $2\varepsilon \leq s \leq \varepsilon_0$. We denote the completion of $X \setminus \{x_1, \dots, x_{2r}\}$ together with the modified metric

$$g_\varepsilon^i := \mathrm{d}s^2 + r_\varepsilon(s)^2\mathrm{d}\sigma^2$$

near x^i by M_ε. Note that X_ε is embedded in M_ε and that the boundary of M_ε has $2r$ disjoint components Z_ε^i, each of them isometric to the sphere of radius ε. Let A_ε^i be the part of the manifold M_ε near x_i given in coordinates by $[0, 2\varepsilon] \times \mathbb{S}^{d-1}$. Denote by A_ε the union of all A_ε^i, $i = 1, \dots, 2r$.

Let γM_ε be an isometric copy of M_ε with identification $x \mapsto \gamma x$ for each $\gamma \in \Gamma$. We construct a new (noncompact) manifold $\tilde{M}_\varepsilon$ by identifying $\gamma Z_\varepsilon^{2i-1}$ with $e_i\gamma Z_\varepsilon^{2i}$ for each $\gamma \in \Gamma$ and $i = 1, \dots, r$. Here, e_i denotes the i-th generator $(0, \dots, 1, \dots, 0)$ of $\Gamma = \mathbb{Z}^r$. Since in a neighbourhood of Z_ε^i the manifold is isometric to a cylinder of radius ε, we can choose a smooth atlas and a smooth metric on the glued manifold $\tilde{M}_\varepsilon$. We therefore obtain a (non-compact) $\mathbb{Z}^r$-periodic manifold $\tilde{M}_\varepsilon$ and M_ε is a period cell for $\tilde{M}_\varepsilon$.

Now we are able to state the following theorem (Theorem 1.1 follows via Floquet Theory):

Theorem 3.1. *We have the convergence $\lambda_k^\theta(M_\varepsilon) \to \lambda_k(X)$ as $\varepsilon \to 0$ uniformly in $\theta \in \hat{\Gamma}$.*

Therefore, the k-th band $B_k(\tilde{M}_\varepsilon)$ reduces to the point $\{\lambda_k(X)\}$ as $\varepsilon \to 0$. Note that the convergence is *not* uniform in k (see the discussion in [CF81]). We therefore could not expect that an *infinite* number of gaps occur.

The proof of Theorem 3.1 is based on the following two lemmas. The idea is to compare the θ-periodic eigenvalues on M_ε with Dirichlet and Neumann eigenvalues on X_ε. The crucial point is, that the corresponding θ-periodic eigenfunctions on M_ε do not concentrate on A_ε, i.e., on the cylindrical ends. This will be shown in the following lemma:

Lemma 3.2. *There exists a positive function $\omega(\varepsilon)$ converging to 0 as $\varepsilon \to 0$ such that*

$$\int_{A_\varepsilon} |u|^2 \le \omega(\varepsilon) \int_{M_\varepsilon} \big(|u|^2 + |\mathrm{d}u|^2\big), \tag{2}$$

for all u in the domain of the quadratic form with θ-periodic boundary conditions on M_ε.

Proof. Without loss of generality, we can assume that $u \in C^\infty(M_\varepsilon)$. Suppose furthermore that $u(\varepsilon_0, \sigma) = 0$ for all $\sigma \in \mathbb{S}^{d-1}$. First we show an L_2-estimate over $A^i_{\varepsilon,s} := \{s\} \times \mathbb{S}^{d-1} \subset A^i_\varepsilon$ with its induced metric $r_\varepsilon(s)^2 \mathrm{d}\sigma^2$.

Applying the Cauchy-Schwarz Inequality yields

$$|u(s,\sigma)|^2 = \Big| \int_s^{\varepsilon_0} \partial_t u(t,\sigma)\,\mathrm{d}t \Big|^2 \le \int_s^{\varepsilon_0} r_\varepsilon(t)^{1-d} \mathrm{d}t \int_s^{\varepsilon_0} |\partial_t u(t,\sigma)|^2 r_\varepsilon(t)^{d-1}\,\mathrm{d}t.$$

If we integrate over $\sigma \in \mathbb{S}^{d-1}$ we obtain

$$\begin{aligned} \int_{A^i_{\varepsilon,s}} |u|^2 &= \int_{\mathbb{S}^{d-1}} |u(s,\sigma)|^2 r_\varepsilon(s)^{d-1} \mathrm{d}\sigma \\ &\le r_\varepsilon(s)^{d-1} \int_s^{\varepsilon_0} r_\varepsilon(t)^{1-d} \mathrm{d}t \int_{M_\varepsilon} |\mathrm{d}u|^2. \end{aligned} \tag{3}$$

If $0 \le s \le 2\varepsilon$ we have $r(s)^{d-1} \le (2\varepsilon)^{d-1}$. Furthermore, the integral over t can be split into an integral over $s \le t \le 2\varepsilon$ and $2\varepsilon \le t \le \varepsilon_0$. The first integral can be estimated by ε^{2-d}, the second by $\int_{2\varepsilon}^{\varepsilon_0} t^{1-d} \mathrm{d}t$. Therefore we have an estimate of the order $O(\varepsilon)$ if $d \ge 3$ resp. $O(\varepsilon |\ln \varepsilon|)$ if $d = 2$. Finally, if we integrate the integral on the LHS of (3) over $s \in [0, 2\varepsilon]$ we obtain the desired Estimate (2). If $u(\varepsilon_0, \sigma) \ne 0$ we choose a cut-off function. $\square$

Remark 3.3. Note that $\omega(\varepsilon)$ only depends on the geometry of X near x_i, not on u or on θ. The argument in the proof is due to [A87].

The following lemma is proven in [CF78] resp. [A87].

Lemma 3.4. *We have $\lambda_k^{\mathrm{D}}(X_\varepsilon) \to \lambda_k(X)$ resp. $\lambda_k^{\mathrm{N}}(X_\varepsilon) \to \lambda_k(X)$.*

Now we show Theorem 3.1:

Proof. From the Min-max Principle (1) we conclude

$$\lambda_k^{\mathrm{D}}(X_\varepsilon) \geq \lambda_k^\theta(M_\varepsilon)$$

since the domains of the quadratic forms obey the opposite inclusions. In particular, $\lambda_k^\theta(M_\varepsilon)$ is bounded in θ and ε by some constant $c_k > 0$. To prove the opposite inequality we estimate

$$\begin{aligned}\frac{q_{X_\varepsilon}(u)}{\|u\|^2_{X_\varepsilon}} - \frac{q_{M_\varepsilon}(u)}{\|u\|^2_{M_\varepsilon}} &\leq \frac{1}{\|u\|^2_{X_\varepsilon}} \frac{q_{M_\varepsilon}(u)}{\|u\|^2_{M_\varepsilon}} \big(\|u\|^2_{M_\varepsilon} - \|u\|^2_{X_\varepsilon}\big) \\ &\leq \frac{\|u\|^2_{M_\varepsilon}}{\|u\|^2_{X_\varepsilon}} \lambda_k^\theta(M_\varepsilon)\, \omega(\varepsilon)\big(1 + \lambda_k^\theta(M_\varepsilon)\big) \leq \omega(\varepsilon) \frac{c_k(1+c_k)}{1 - \omega(\varepsilon)(1+c_k)} =: \delta_k(\varepsilon)\end{aligned}$$

for $u \in L$, where L denotes the space generated by the first k eigenvalues of $\Delta^\theta_{M_\varepsilon}$. Note that $\delta_k(\varepsilon) \to 0$ as $\varepsilon \to 0$ by Lemma 3.2 which we have used twice. Since $\delta_k(\varepsilon)$ is independent of $u \in L$ the Min-max Principle implies

$$\lambda_k^{\mathrm{N}}(X_\varepsilon) - \delta_k(\varepsilon) \leq \lambda_k^\theta(M_\varepsilon). \tag{4}$$

Note that $L \restriction_{X_\varepsilon}$ is still k-dimensional (by Lemma 3.2). Together with Lemma 3.4 we have proven Theorem 3.1. □

4. Eigenvalues in Gaps

In this section we discuss a simple example how to produce eigenvalues in a spectral gap by locally perturbing the metric. Suppose that $\tilde{M}_\varepsilon$ is a periodic metric as in the previous section with period cell M_ε. Let $(\Gamma^n)_n$ be an exhaustive sequence, i.e., a monotone sequence with $\bigcup_n \Gamma^n = \Gamma$. Denote by M_ε^n the union of all γM_ε with $\gamma \in \Gamma^n$. Furthermore, we assume that M_ε^n and $M_\varepsilon^n \setminus M_\varepsilon^{n_0}$ are connected.

Let $(\rho_\tau)_\tau$ be a family of smooth, strictly positive functions on $\tilde{M}_\varepsilon$ such that $\tau \mapsto \rho_\tau$ is continuous with respect to the C^1-topology. Suppose further that

$$\rho_0 = 1 \qquad \text{on } \tilde{M}_\varepsilon \tag{5}$$

$$\rho_\tau = 1 \qquad \text{on } \tilde{M}_\varepsilon \setminus M_\varepsilon^{n_0} \qquad \text{for all } \tau \tag{6}$$

$$\rho_\tau = \mathrm{e}^\tau \qquad \text{on the period cell } M_\varepsilon \qquad \text{for all } \tau. \tag{7}$$

Finally we denote by $\tilde{M}_\varepsilon(\tau)$ the manifold $\tilde{M}_\varepsilon$ together with the metric $\rho_\tau^2 \tilde{g}_\varepsilon$ if $\tilde{g}_\varepsilon$ denotes the metric of $\tilde{M}_\varepsilon$. Similar notations are understood in the same way. Note that all domains $\operatorname{dom} q_{M(\tau)}$ and Hilbert spaces $L_2(M(\tau))$ are the same as vector spaces if τ varies. We choose Dirichlet boundary conditions on M_ε^n in order to have the inclusion $\operatorname{dom} q_{M_\varepsilon^n} \subset \operatorname{dom} q_{M_\varepsilon^{n'}} \subset \operatorname{dom} q_{\tilde{M}_\varepsilon}$ for the domains of the (closed) quadratic forms if $n \leq n'$.

First, we guarantee that no eigenvalue of the approximating problem lies in the gap; the boundary of M_ε resp. M_ε^n is so small such that boundary conditions almost have no influence on the eigenvalues:

Lemma 4.1. *If $\lambda_k(X) < \lambda_{k+1}(X)$ then there exist numbers a, b such that $\lambda_k(X) < a < b < \lambda_{k+1}(X)$ and such that the interval $I = (a, b)$ is a common gap, i.e.,*

$$I \cap \operatorname{spec} \Delta_{\tilde{M}_\varepsilon} = \emptyset \quad \text{and} \quad I \cap \operatorname{spec} \Delta^{\mathrm{D}}_{M^n_\varepsilon} = \emptyset \tag{8}$$

for all $\varepsilon > 0$ small enough.

The lemma follows from the Dirichlet-Neumann bracketing and the Min-max Principle (see [RS78] or [P00]). Note that $\lambda^{\mathrm{D}}_k(M_\varepsilon), \lambda^{\mathrm{N}}_k(M_\varepsilon) \to \lambda_k(X)$ as in Theorem 3.1 with the same error estimate (4).

From now on we fix $\varepsilon > 0$ and $I = (a, b)$ such that (8) is satisfied. We omit the index ε, e.g., $M = M_\varepsilon$ or $\tilde{M} = \tilde{M}_\varepsilon$. Furthermore, we choose $\lambda \in I$.

Next, we use the Decomposition Principle (see [DL79]) to prove that the essential spectrum remains invariant under the perturbation:

Theorem 4.2. *We have* $\operatorname{ess\,spec} \Delta_{\tilde{M}} = \operatorname{ess\,spec} \Delta_{\tilde{M}(\tau)}$ *for all $\tau \geq 0$.*

In particular, $\Delta_{\tilde{M}}$ and $\Delta_{\tilde{M}(\tau)}$ have the same spectral gap. In a spectral gap of the unperturbed Laplacian, the perturbed Laplacian can only have discrete eigenvalues (possibly accumulating at the band edges). It is essential here that the perturbation is localized on a compact set.

Now we prove that eigenfunctions of the approximating problem converge to eigenfunctions of the full problem:

Theorem 4.3. *Suppose that $\tau_n \to \tau$ and that*

$$\Delta^{\mathrm{D}}_{M^n(\tau_n)} u_n = \lambda u_n, \qquad \|u_n\|_{M^n} = 1.$$

Then there exists a function u in the domain of $\Delta_{\tilde{M}(\tau)}$ such that $u_n \to u$ weakly in $L_2(\tilde{M})$ and strongly in $L_{2,\mathrm{loc}}(\tilde{M})$. Furthermore, $u \neq 0$ and

$$\Delta_{\tilde{M}(\tau)} u = \lambda u \tag{9}$$

To prove the theorem we need the following two lemmas. The next lemma can be shown straight forward:

Lemma 4.4. *For each $\tau, \tau' \geq 0$, the (squared) norms $\|\cdot\|^2_{M(\tau)}$ and $\|\cdot\|^2_{M(\tau')}$ are equivalent. In particular, the constants depend continuously on τ and τ'. The same is true for the quadratic forms $q_{M(\tau)}$ and $q_{M(\tau')}$.*

From the last lemma and the Rellich-Kondrachov Compactness Theorem we conclude the following lemma:

Lemma 4.5. *Let u_n be the approximating eigenvalue functions of Theorem 4.3. Then there exists a subsequence of (u_n) (also denoted by (u_n)) such that $u_n \to u$ weakly in $L_2(\tilde{M})$ and strongly in $L_{2,\mathrm{loc}}(\tilde{M})$. Furthermore, Equation (9) is valid.*

Now we prove Theorem 4.3. We only have to show that $u \neq 0$ which is the main difficulty.

Proof. Suppose that $u = 0$. Since λ lies in a spectral gap, we have

$$\|(\Delta^{\mathrm{D}}_{M^n} - \lambda)u_n\|_{M^n} \geq (b-a)\|u_n\|_{M^n} \geq \mathrm{const} > 0. \tag{10}$$

by the spectral calculus. On the other hand, we estimate

$$\begin{aligned} &\|(\Delta^{\mathrm{D}}_{M^n} - \lambda)u_n\|_{M^n} \\ &\qquad \leq \|(\Delta^{\mathrm{D}}_{M^n} - \Delta^{\mathrm{D}}_{M^n(\tau_n)})u_n\|_{M^n \setminus M^{n_0}} + \|\Delta^{\mathrm{D}}_{M^n} u_n\|_{M^{n_0}} + \lambda\|u_n\|_{M^{n_0}} \end{aligned} \tag{11}$$

for $n \geq n_0$ where the first term on the RHS is equal to 0 (note that the perturbation of the metric is localized on M^{n_0}). The second term can be estimated to

$$\|\Delta^{\mathrm{D}}_{M^n} u_n\|_{M^{n_0}} \leq \mathrm{const}\big(\|u_n\|_{M^{n_1}} + \|\Delta^{\mathrm{D}}_{M^n(\tau_n)} u_n\|_{M^{n_1}}\big) \leq \mathrm{const}' \, \|u_n\|_{M^{n_1}}$$

for some appropriate $n_1 > n_0$ and all $n \geq n_1$ by regularity theory. Since (u_n) converges strongly to $u = 0$ in $L_{2,\mathrm{loc}}(\tilde{M})$, the LHS of (11) converges to 0 which contradicts (10). ☐

In order to show the existence of eigenfunctions of the approximating problem we define the *eigenvalue counting function*

$$\mathcal{N}_{\tau_0,\tau}(Q(\cdot), \lambda) := \sum_{\tau_0 \leq \tau' \leq \tau} \dim\ker(Q(\tau') - \lambda).$$

This function counts the number of eigenvalues λ (with multiplicity) of the family $(Q(\tau'))_{\tau_0 \leq \tau' \leq \tau}$. Note the difference to the eigenvalue counting function of a single operator $Q \geq 0$ counting the number of eigenvalues below λ, i.e.,

$$\dim_\lambda(Q) := \sum_{0 \leq \lambda' \leq \lambda} \dim\ker(Q - \lambda').$$

The next lemma follows from the fact that the eigenvalue branches $\tau \mapsto \lambda^{\mathrm{D}}_k(M^n(\tau))$ are continuous and that the number of eigenvalue branches coming from above is a lower bound for the number how often the eigenvalue branches cross the level λ. Note that the eigenvalue branches could oscillate several times around λ.

Lemma 4.6. *We have*

$$\mathcal{N}_{\tau_0,\tau}(\Delta_{M^n(\cdot)}, \lambda) \geq \dim_\lambda(\Delta^{\mathrm{D}}_{M^n}(\tau)) - \dim_\lambda(\Delta^{\mathrm{D}}_{M^n}(\tau_0)).$$

The proof of the following lemma is essentially the same as the proof of Lemma 4.1:

Lemma 4.7. *For $n \geq n_0$ we have*

$$\dim_\lambda(\Delta^{\mathrm{D}}_{M^n}(\tau)) - \dim_\lambda(\Delta^{\mathrm{D}}_{M^n}(\tau_0)) = \dim_\lambda(\Delta^{\mathrm{D}}_{M^{n_0}}(\tau)) - \dim_\lambda(\Delta^{\mathrm{D}}_{M^{n_0}}(\tau_0)).$$

Finally we prove the existence of approximating eigenfunctions. Together with Theorem 4.2 and Theorem 4.3 we conclude Theorem 1.2.

Theorem 4.8. *There exist an infinite number of sequences (τ_n) and (u_n) such that $\tau_n \to \hat{\tau}$ as $n \to \infty$ and such that u_n is an eigenfunction of the Dirichlet-Laplacian on $M^n(\tau_n)$ with eigenvalue λ.*

Proof. The Min-max Principle yields

$$0 \leq \lambda_k^{\mathrm{D}}(M^{n_0}(\tau)) \leq \lambda_k^{\mathrm{D}}(M(\tau)) = \mathrm{e}^{-2\tau}\lambda_k^{\mathrm{D}}(M) \to 0$$

and therefore $\dim_\lambda(\Delta^{\mathrm{D}}_{M^{n_0}(\tau)}) \to \infty$ as $\tau \to \infty$. From the last two lemmas we conclude $\mathcal{N}_{\tau_0,\tau}(\Delta^{\mathrm{D}}_{M^n(\cdot)}, \lambda) \to \infty$ uniformly in $n \in \mathbb{N}$ as $\tau \to \infty$. If the counting number increases by 1 at the parameter τ we can choose a sequence (τ_n), $\tau_0 \leq \tau_n \leq \tau$ converging to some number $\hat{\tau}$. To this sequence corresponds a sequence of eigenvalues (u_n). In the next step we let τ_0 be the old value of τ. We raise τ until the counting number increases again by 1 and so forth. □

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Institut für Reine und Angewandte Mathematik, Rheinisch-Westfälische Technische Hochschule Aachen, Templergraben 55, 52062 Aachen, Germany

E-mail address: post@iram.rwth-aachen.de