GAPS IN THE DIFFERENTIAL FORMS SPECTRUM ON CYCLIC COVERINGS

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ABSTRACT. We are interested in the spectrum of the Hodge-de Rham operator on a $\mathbb{Z}$-covering X over a compact manifold M of dimension $n+1$. Let Σ be a hypersurface in M which does not disconnect M and such that $M - \Sigma$ is a fundamental domain of the covering. If the cohomology group $H^{n/2}(\Sigma)$ is trivial, we can construct for each $N \in \mathbb{N}$ a metric $g = g_N$ on M, such that the Hodge-de Rham operator on the covering (X, g) has at least N gaps in its (essential) spectrum. If $H^{n/2}(\Sigma) \neq 0$, the same statement holds true for the Hodge-de Rham operators on p-forms provided $p \notin \{n/2, n/2 + 1\}$.

1. INTRODUCTION

A common feature of periodic operators is its *band-gap* nature of the spectrum. It is natural to ask how we can create gaps between the bands of the spectrum. Here we will extend the analysis done by the third named author in [20] to the Hodge-de Rham operator on *forms*. However, there are topological obstructions for the existence of gaps in the spectrum of the Hodge-de Rham operator. The following Theorem A is a direct consequence of [8, Theorem 0.1]:

Theorem A. *Let (M^{4k+1}, g) be a compact oriented Riemannian manifold. Assume that $\Sigma \subset M$ is an oriented hypersurface, with* non-zero signature *and not disconnecting M. Let $\mathbb{Z} \to \widetilde{M} \to M$ be the cyclic covering associated to Σ, then for any complete Riemannian metric on $\widetilde{M}$ (periodic or not) the spectrum of the Hodge-de Rham Laplacian on $\widetilde{M}$ is $[0, \infty[$.*

The result we present here has also a topological restriction:

Theorem B. *Assume that $\Sigma^n \subset M^{n+1}$ is a hypersurface in a compact manifold M and assume that Σ does not disconnect M. Let $\mathbb{Z} \to \widetilde{M} \to M$ be the cyclic covering associated to Σ.*

If $p \neq n/2$ and $p \neq n/2 + 1$, then there is a family of periodic Riemannian metrics g_ε on $\widetilde{M}$ such that the spectrum of the Hodge-de Rham Laplacian acting on p-forms has N_ε gaps with $\lim_{\varepsilon \to 0} N_\varepsilon = +\infty$.

If $p = n/2$ or $p = n/2 + 1$, the same conclusion holds provided that the $(n/2)$-Betti number of Σ vanishes, i.e., $b_{n/2}(\Sigma) = 0$.

Our result is obtained through a convergence result of the differential form spectrum which generalises the study of the first author and B. Colbois [3]. The family

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of metrics g_ε is defined on M as follows: outside a collar neighbourhood of Σ, the metric is independent of ε and on this collar neighbourhood of Σ the Riemannian manifold (M, g_ε) is isometric to the union of two copies of the truncated cone $([\varepsilon, 1] \times \Sigma, dr^2 + r^2 h)$, where h is a fixed Riemannian metric on Σ, and of a thin handle $[0, L] \times \Sigma$ endowed with the Riemannian metric $dr^2 + \varepsilon^2 h$.

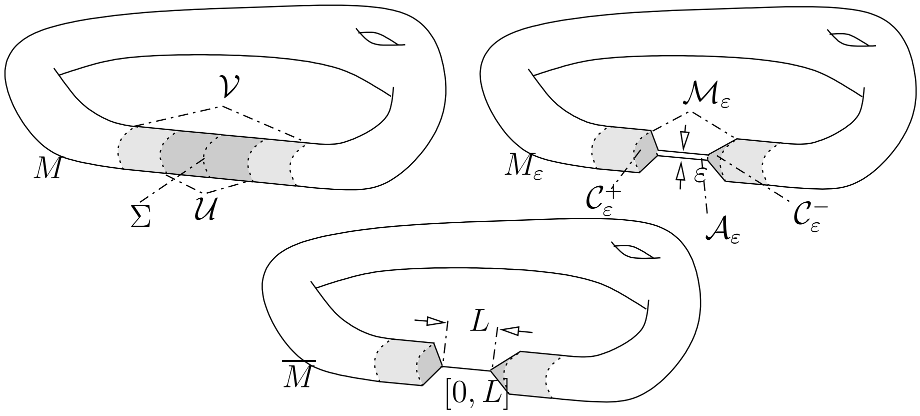

FIGURE 1. Construction of the manifold M_ε and the limit manifold $\overline{M}$. We start with a manifold M having product structure on $\mathcal{U}$. The cones on M_ε have length $1 - \varepsilon$, and the handle has length L and radius ε. The limit consists of the manifold $\overline{M}$ with two conical singularities, and the line segment $[0, L]$.

Geometrically, the manifold (M, g_ε) is converging in the Gromov-Hausdorff topology to the union of a manifold $(\overline{M}, \overline{g})$ with two conical singularities and of a segment of length L joining the two singularities. On $(\overline{M}, \overline{g})$, the operator $D := d + d^*$, a priori defined on the space of smooth forms with support in the regular part of $\overline{M}$, is not necessary essentially self adjoint. After the pioneering work [9] of J. Cheeger dealing with the Friedrichs extension $D_{\max} \circ D_{\min}$, the closed extensions of D have been studied carefully (see for instance [7], [16], [25] and [15]).

Denote by $\sigma^{\mathrm{D}} = \{ (\pi k/L)^2 \,;\, k = 1, 2, \dots \}$ the Dirichlet spectrum of the Laplacian on functions on the interval $[0, L]$ and similarly by $\sigma^{\mathrm{N}} := \sigma^{\mathrm{D}} \cup \{0\}$ the Neumann spectrum. Our main theorem is the following:

Theorem C. *Suppose, in the case when n is even, that the cohomology group $H^{n/2}(\Sigma) = 0$. The spectrum of the Hodge-de Rham operator acting on p-forms of the manifold (M, g_ε) converges to the spectrum σ_p of the limit problem, where σ_p is given as follows:*

- $\mathbf{p < (n+1)/2}$**:** *The limit spectrum σ_p is the union of the spectrum of the operator $D_{\max} \circ D_{\min}$ on p-forms on $\overline{M}$, the Neumann spectrum σ^{N} with multiplicity* $\dim H^{p-1}(\Sigma)$ *and the Dirichlet spectrum σ^{D} with multiplicity* $\dim H^p(\Sigma)$.
- $\mathbf{p > (n+1)/2}$**:** *The limit spectrum σ_p is the union of the spectrum of the operator $D_{\max} \circ D_{\min}$ on p-forms on $\overline{M}$, the Neumann spectrum σ^{N} with multiplicity* $\dim H^p(\Sigma)$ *and the Dirichlet spectrum σ^{D} with multiplicity* $\dim H^{p-1}(\Sigma)$,

$\mathbf{p} = (\mathbf{n}+\mathbf{1})/\mathbf{2}$: *The limit spectrum σ_p is the union of the spectrum of the operator $D_{\min} \circ D_{\max}$ on p-forms on $\overline{M}$, and the Neumann spectrum σ^{N} with multiplicity* $\dim H^p(\Sigma) \oplus \dim H^{p-1}(\Sigma)$.

Remarks. Our Theorem 12 gives also a convergence result in the case when n is even, the $(n/2)$-cohomology group of Σ is non-trivial and $p = n/2$ or $p = (n+1)/2$. In this case the limit spectrum is obtained by a coupled problem between the manifold $\overline{M}$ and the line segment. Consequently, the result of Theorem 12 does not help for the determination of the spectrum on the periodic manifold: The spectrum depends in fact on the spectral flow (see [4, p.93] for a definition) of the family of operators defined by the Floquet parameter.

We remark also that the presence of the handle influences the definition of the limit problem on the manifold $\overline{M}$, namely in the case $p = (n+1)/2$ where in fact the operator $D_{\min} \circ D_{\max}$ appears. If the handle is not present (i.e. $L = 0$), the index of the Gauß-Bonnet operator in this situation has been studied by R. Seeley in [25], and the convergence of the spectrum of the Hodge-de Rham operator acting on p-forms by P. Macdonald ([18]), and next by R. Mazzeo and J. Rowlett ([19, 23]). The result is that, *with the topological hypothesis $H^{n/2}(\Sigma) = 0$, this spectrum converges to the spectrum of the Friedrichs extension $D_{\max} \circ D_{\min}$ of the Hodge-de Rham operator on $\overline{M}$ for any degree p.* This fact can be recovered by our analysis.

Finally, our work has also an extension to the Dirac operator: there is an analogue of Theorem A due to J. Roe for the Dirac operator ([22]). On the other hand, if we consider a compact spin manifold M^{n+1} and an oriented hypersurface Σ with trivial $\widehat{A}$-invariant or trivial α-invariant, then the recent work of B. Ammann, M. Dahl and E. Humbert [1] provides a Riemannian metric h on Σ with no harmonic spinors. Then we can scale this metric so that its associated Dirac operator on Σ has no eigenvalue in a large symmetric interval. Then our construction also applies in this case, and gives, with J. Roe's results, the following

Theorem D. *Assume that $\Sigma^n \subset M^{n+1}$ is an oriented hypersurface in a compact spin manifold M, which does not disconnect M, and consider $\mathbb{Z} \to \widetilde{M} \to M$, the associated cyclic covering. Then there is a family g_ε of periodic Riemannian metrics on $\widetilde{M}$, whose Dirac operator has a large number of gaps in its spectrum* if and only if $\widehat{A}(\Sigma) = 0$, *in the case $n = 4k$, or $\alpha(\Sigma) = 0$, in the case $n = 8k+1$ or $n = 8k+2$.*

Recall that the spin cobordism α-invariant satisfies $\alpha(\Sigma) \in \mathbb{Z}/2\mathbb{Z}$. This last result can be compared with the recent one of D. Ruberman and N. Saveliev. Indeed they prove in [24, Theorem 2] that, the Dirac operator on a cyclic covering $\widetilde{M} \to M$ is invertible for a generic set of $\mathbb{Z}$-periodic metric , if and only if $\alpha_{n+1}(M) = 0$ and $\alpha_n(\Sigma) = 0$. The topological invariant $\alpha_n(X)$ for a closed manifold X of dimension n

is defined as an element of KO_n, and we have

$$\alpha_n(X) = \begin{cases} \widehat{A}(X), & \text{if } n = 8k, \\ \widehat{A}(X)/2, & \text{if } n = 8k+4, \\ \alpha(X), & \text{if } n \in \{8k+1, 8k+2\}, \\ 0, & \text{otherwise.} \end{cases}$$

They use also the construction of B. Ammann, M. Dahl and E. Humbert [1]. In particular, the results of Ruberman and Saveliev imply that generically, the first band of the spectrum of the Dirac operator does not touch 0; it is not a result about the presence of many gaps in the spectrum.

It is tempting to ask whether an equivalence as in Theorem D also holds for the Hodge-de Rham operator, but we have no guess about the validity of such an extension. We think that it is an interesting question and we intend to consider this question in a future work.

The paper is organised as follows: In the next section, we fix the geometric setting for the quotient manifold M, namely the family of metrics g_ε. In Section 3 we describe the Hodge-de Rham operator in natural coordinates on the collar neighbourhood of Σ. In Section 4 we provide basic estimates on a sequence of eigenforms used in the main convergence result, which will be presented in Section 5. In Section 6 we deduce the existence of spectral gaps and in Section 7 we discuss the possible appearance of small eigenvalues in the setting of Theorem C.

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2. The geometric set-up

In this section, we explain the construction of the deforming family of metrics g_ε. We assume that M is a compact manifold of dimension $n+1$ and that Σ is a compact hypersurface in M which does not disconnect the manifold (note that this hypothesis, needed for the construction of a *connected* periodic manifold, does not play any role in the proof of the preliminary Theorem C). We choose a metric g on M such that there exists a collar neighbourhood $\mathcal{U} = \left]-2,2\right[\times \Sigma$ of Σ where g is of the form $dt^2 + h$ for a (fixed) metric h on Σ.

For $\varepsilon \in \left]0,1\right]$, we construct a family of continuous, piecewise smooth metrics g_ε on M as follows:

- Outside the collar neighbourhood $\mathcal{V} := \left]-1,1\right[\times \Sigma \subset \mathcal{U}$, we do not change the metric, i.e. $g_\varepsilon = g$ on $M \setminus \mathcal{V}$.
- On the collar neighbourhood $\mathcal{V}$, the metric is chosen in such a way that $(\mathcal{V}, g_\varepsilon)$ is isometric to the union $\mathcal{M}_\varepsilon = \mathcal{C}_\varepsilon^- \cup \mathcal{A}_\varepsilon \cup \mathcal{C}_\varepsilon^+$, where $\mathcal{C}_\varepsilon^\pm$ are cones $\left]\varepsilon,1\right[\times \Sigma$ endowed with the metric $dt^2 + t^2 h$ and with distinct orientation, and where $\mathcal{A}_\varepsilon$ is the handle $\left]0,L\right[\times \Sigma$ endowed with the metric $dt^2 + \varepsilon^2 h$.

Using a coordinate τ on all three parts, $(\mathcal{M}_\varepsilon, g_\varepsilon)$ is isometric to $]-(L/2+1-\varepsilon), L/2+1-\varepsilon[\times\Sigma$, endowed with the warped product metric $d\tau^2+\rho_\varepsilon(\tau)^2 h$, where

$$\rho_\varepsilon(\tau)=\begin{cases}\varepsilon & \text{if } |\tau|\le L/2\\ |\tau|-L/2+\varepsilon & \text{if } |\tau|\ge L/2.\end{cases}$$

We denote by M_ε the new Riemannian manifold. We are interested in studying the limit, as $\varepsilon\to 0$, of the spectrum $\lambda_k^p(\varepsilon)=\lambda_k^p(g_\varepsilon)$, $k\ge 1$, of the Hodge-de Rham operator acting on p-forms defined on the manifold M_ε. We remark that g_ε is only continuous.

The Hodge-de Rham operator is defined in this case as follows (see [3] for more details). The manifold is the union of two smooth parts with boundary. For a manifold $M=M_1\cup M_2$, denote by D_1, D_2 the Gauß-Bonnet operator on each part. The quadratic form $q(\varphi)=\int_{M_1}|D_1(\varphi\restriction_{M_1})|^2+\int_{M_2}|D_2(\varphi\restriction_{M_2})|^2$ is well defined and closed on the domain

$$\operatorname{dom}(q)=\big\{\,\varphi=(\varphi_1,\varphi_2)\in\mathsf{H}^1(M_1)\times\mathsf{H}^1(M_2)\,;\,\varphi_1\restriction_{\partial M_1}\overset{\mathsf{L}_2}{=}\varphi_2\restriction_{\partial M_2}\,\big\},$$

and on this space the total Gauß-Bonnet operator is defined and selfadjoint. The Hodge-de Rham operator of M is then defined as the operator obtained by the polarization of the quadratic form q. This gives compatibility conditions between φ_1 and φ_2 on the commun boundary, these conditions are explained in detail in the next section.

Finally we remark that it is not a loss of generality to concider only continuous metrics: the family $(g_\varepsilon)_{\varepsilon>0}$ can be approched by a family of smooth Riemannian metrics $(g_{\varepsilon,\eta})_{\varepsilon>0}$ such that

$$\mathrm{e}^{-\eta}g_\varepsilon\le g_{\varepsilon,\eta}\le\mathrm{e}^{\eta}g_\varepsilon \tag{1}$$

for all ε.

Proof. Let f_η be a smooth, increasing function on $\mathbb{R}^+$ such that

$$f_\eta(r)=1\qquad\text{for } r\le 1\qquad\text{and}\qquad f_\eta(r)=r\qquad\text{for } r\ge 1+\eta.$$

Then the metric $g_{\varepsilon,\eta}=d\tau^2+f_{\varepsilon,\eta}(\tau)^2h$ on $\mathcal{M}_\varepsilon$ with $f_{\varepsilon,\eta}$ defined by

$$f_{\varepsilon,\eta}(\tau)=\begin{cases}\varepsilon & \text{if } |\tau|\le L/2,\\ \varepsilon f_\eta\Big(\frac{|\tau|-L/2+\varepsilon}{\varepsilon}\Big) & \text{if } |\tau|\ge L/2\end{cases}$$

satisfies the estimate (1). □

A result of Dodziuk [13, Prop. 3.3] implies now, that the corresponding eigenvalues satisfy

$$\mathrm{e}^{-(n+2p)\eta}\lambda_k^p(g_\varepsilon)\le\lambda_k^p(g_{\varepsilon,\eta})\le\mathrm{e}^{(n+2p)\eta}\lambda_k^p(g_\varepsilon).$$

Note that the result of Dodziuk also applies to our singular metrics, based on the Hodge decomposition and the fact that the spectrum away from 0 is given by exact forms. Hence, it is enough to prove our results only for a family of continuous (but piecewise smooth) metrics, and the convergence results extend also to a family of smooth metrics. The above definition of a family of non-smooth metrics will simplify some of our arguments in the next section. Namely, we can solve certain differential equations explicitly due to the special form of the metric on the cones and the handle.

3. Description of the Hodge-de Rham operator on $\mathcal{M}_\varepsilon$

In this section we express the norm of a p-form, the Gauß-Bonnet, the Hodge-de Rham operator and its associated quadratic form in the new coordinates. On the cones $\mathcal{C}_\varepsilon^\pm$, we use the same parametrisation of the forms as the one introduced in [7] and [6], namely a p-form φ can be written as

$$\varphi = dt \wedge t^{-(n/2-p+1)} \beta_\pm + t^{-(n/2-p)} \alpha_\pm$$

and we set

$$U_\pm \varphi := \sigma_\pm := (\beta_\pm, \alpha_\pm) \in \mathsf{C}^\infty(]\varepsilon, 1[, \mathsf{C}^\infty(\Lambda^{p-1} T^* \Sigma) \oplus \mathsf{C}^\infty(\Lambda^p T^* \Sigma)).$$

Similarly, on the handle, we have

$$\varphi = dr \wedge \varepsilon^{-(n/2-p+1)} \beta + \varepsilon^{-(n/2-p)} \alpha$$

and we set

$$U \varphi := \sigma := (\beta, \alpha) \in \mathsf{C}^\infty(]0, L[, \mathsf{C}^\infty(\Lambda^{p-1} T^* \Sigma) \oplus \mathsf{C}^\infty(\Lambda^p T^* \Sigma)).$$

Since we included the factor ρ_ε of the (warped) product $g_\varepsilon = dt^2 + \rho_\varepsilon(t)^2 h$ in the definition of the transformation, it is now straightforward to see that $U_\pm$ extends to a unitary operator on the corresponding L^2-spaces and similarly for U. In particular, we have

$$\begin{aligned} \|\varphi\|^2_{\mathsf{L}^2(\Lambda^\bullet T^* \mathcal{M}_\varepsilon, g_\varepsilon)} &= \int_{\mathcal{M}_\varepsilon} |\varphi|^2_{g_\varepsilon} d\operatorname{vol}_{g_\varepsilon} \\ &= \sum_{s=\pm} \int_\varepsilon^1 \bigl[|\beta_s(t)|^2 + |\alpha_s(t)|^2 \bigr] dt + \int_0^L \bigl[|\beta(t)|^2 + |\alpha(t)|^2 \bigr] dt, \quad (2) \end{aligned}$$

where $|\cdot|$ denotes the L^2-norm on $\mathsf{L}^2(\Lambda^\bullet T^* \Sigma, h)$.

We can now transform the *Gauß-Bonnet operator* $D := d + d^*$, which in fact depends on ε as the metric does, using the transformations $U_\pm$ resp. U and obtain

$$UDU^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Bigl(\partial_t + \frac{1}{\varepsilon} \begin{pmatrix} 0 & -D_0 \\ -D_0 & 0 \end{pmatrix} \Bigr) \qquad \text{on the handle } \mathcal{A}_\varepsilon \text{ and}$$

$$U_\pm D U_\pm^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Bigl(\partial_t + \frac{1}{t} \begin{pmatrix} \frac{n}{2} - P & -D_0 \\ -D_0 & P - \frac{n}{2} \end{pmatrix} \Bigr) \qquad \text{on the cones } \mathcal{C}_\varepsilon^\pm,$$

where $D_0 = d_0 + d_0^*$ denotes the Gauß-Bonnet operator on the compact Riemannian manifold (Σ, h) and where P is the linear operator multiplying with the degree of the form. For further purposes, it will be useful to denote

$$A_0 = \begin{pmatrix} 0 & -D_0 \\ -D_0 & 0 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} \frac{n}{2} - P & -D_0 \\ -D_0 & P - \frac{n}{2} \end{pmatrix}$$

the parts, in the transformed Gauß-Bonnet operators, acting non-trivially in the transversal direction Σ.

In this representation, a piecewise smooth form φ is in the domain of D if and only if φ is in the H^1-space of each part and if the components of $U\varphi$ and $U_\pm\varphi$ satisfy the *compatibility* or *transmission conditions*

$$\begin{cases} \beta(L) = \beta_+(\varepsilon), & \beta(0) = -\beta_-(\varepsilon) \\ \alpha(L) = \alpha_+(\varepsilon), & \alpha(0) = \alpha_-(\varepsilon). \end{cases} \tag{3}$$

The *Hodge-de Rham operator* is now given by D^2. A simple calculation shows that on the handle $\mathcal{A}_\varepsilon$, we have

$$UD^2U^* = -\partial_t^2 + \frac{1}{\varepsilon^2}A_0^2 = -\partial_t^2 + \frac{1}{\varepsilon^2}\begin{pmatrix} \Delta_\Sigma & 0 \\ 0 & \Delta_\Sigma \end{pmatrix},$$

where $\Delta_\Sigma = D_0^2$ denotes the Hodge-de Rham operator of the Riemannian manifold (Σ, h). Similarly, on the cones $\mathcal{C}_\varepsilon^\pm$ we have the expression

$$U_\pm D^2 U_\pm^* = -\partial_t^2 + \frac{1}{t^2}(A + A^2),$$

where

$$A + A^2 = \begin{pmatrix} \Delta_\Sigma + \Big(\dfrac{n}{2} - P\Big)\Big(\dfrac{n}{2} - P + 1\Big) & -2d_0^* \\ -2d_0 & \Delta_\Sigma + \Big(\dfrac{n}{2} - P\Big)\Big(\dfrac{n}{2} - P - 1\Big) \end{pmatrix}. \tag{4}$$

The domain of D^2 consists of those forms φ in the domain of D such that $D\varphi$ is also in the domain of D. In particular, the domain of the transformed operator UD^2U^* consists of pairs of forms satisfying — in addition to (3) — the following *compatibility* or *transmission conditions* (of first order) on the derivatives:

$$\beta'(L) = \beta'_+(\varepsilon) + \frac{1}{\varepsilon}\Big(\frac{n}{2} - P\Big)\beta_+(\varepsilon), \qquad \beta'(0) = \beta'_-(\varepsilon) + \frac{1}{\varepsilon}\Big(\frac{n}{2} - P\Big)\beta_-(\varepsilon) \tag{5a}$$

$$\alpha'(L) = \alpha'_+(\varepsilon) - \frac{1}{\varepsilon}\Big(\frac{n}{2} - P\Big)\alpha_+(\varepsilon), \qquad \alpha'(0) = -\alpha'_-(\varepsilon) + \frac{1}{\varepsilon}\Big(\frac{n}{2} - P\Big)\alpha_-(\varepsilon). \tag{5b}$$

Let us now compute the expression

$$\|D\varphi\|^2_{\mathsf{L}^2(\Lambda^\bullet T^*\mathcal{M}_\varepsilon, g_\varepsilon)} = \int_{\mathcal{M}_\varepsilon} |D\varphi|^2_{g_\varepsilon} d\,\mathrm{vol}_{g_\varepsilon},$$

i.e., the quadratic form on $\mathcal{M}_\varepsilon$, for a form φ in the domain of D, and with support in $\mathcal{M}_\varepsilon$ in terms of

$$\sigma_\pm = \begin{pmatrix} \beta_\pm \\ \alpha_\pm \end{pmatrix} = U_\pm\varphi \qquad \text{and} \qquad \sigma = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = U\varphi$$

using the isometries $U_\pm$ and U.

Denote by $\langle\cdot,\cdot\rangle$ the scalar product in $\mathsf{L}^2(\Lambda^\bullet T^*\Sigma, h) \oplus \mathsf{L}^2(\Lambda^\bullet T^*\Sigma, h)$ and by $\mathcal{C}_\varepsilon$ one of the two cones, oriented by $dt \wedge d\,\mathrm{vol}_\Sigma$. The expression of the transformed quadratic form on the cone is then

$$\begin{aligned}
\int_{\mathcal{C}_\varepsilon} |D\varphi|^2_{g_\varepsilon} d\operatorname{vol}_{g_\varepsilon} &= \int_\varepsilon^1 \left| \Big(\partial_t + \frac{1}{t}A\Big)\sigma_\pm \right|^2 dt \\
&= \int_\varepsilon^1 \Big[|\sigma_\pm'|^2 + \frac{2}{t}\langle \sigma_\pm', A\sigma_\pm\rangle + \frac{1}{t^2}|A\sigma_\pm|^2 \Big] dt \\
&= \int_\varepsilon^1 \Big[|\sigma_\pm'|^2 + \partial_t\Big(\frac{1}{t}\langle \sigma_\pm, A\sigma_\pm\rangle\Big) + \frac{1}{t^2}\big(\langle \sigma_\pm, A\sigma_\pm\rangle + |A\sigma_\pm|^2\big) \Big] dt \\
&= \int_\varepsilon^1 \Big[|\sigma_\pm'|^2 + \frac{1}{t^2}\langle \sigma_\pm, (A + A^2)\sigma_\pm\rangle \Big] dt - \frac{1}{\varepsilon}\big\langle \sigma_\pm(\varepsilon), A\sigma_\pm(\varepsilon)\big\rangle.
\end{aligned}$$

Similarly, on the handle we have

$$\begin{aligned}
\int_{\mathcal{A}_\varepsilon} |D\varphi|^2_{g_\varepsilon} d\operatorname{vol}_{g_\varepsilon} &= \int_0^L \left| \Big(\partial_t + \frac{1}{\varepsilon}A_0\Big)\sigma \right|^2 dt \\
&= \int_0^L \Big[|\sigma'|^2 + \frac{1}{\varepsilon^2}|A_0\sigma|^2 + \frac{2}{\varepsilon}\langle \sigma', A_0\sigma\rangle \Big] dt \\
&= \int_0^L \Big[|\sigma'|^2 + \frac{1}{\varepsilon^2}|A_0\sigma|^2 \Big] dt \\
&\qquad + \frac{1}{\varepsilon}\Big(\big\langle \sigma(L), A_0\sigma(L)\big\rangle - \big\langle \sigma(0), A_0\sigma(0)\big\rangle \Big).
\end{aligned}$$

The total boundary term is

$$\begin{aligned}
\mathfrak{b}(\varphi,\varphi) = \Big(-\big\langle \sigma_+(\varepsilon), A\sigma_+(\varepsilon)\big\rangle - \big\langle \sigma_-(\varepsilon), A\sigma_-(\varepsilon)\big\rangle \\
+ \big\langle \sigma(L), A_0\sigma(L)\big\rangle - \big\langle \sigma(0), A_0\sigma(0)\big\rangle \Big).
\end{aligned}$$

Using the compatibility conditions (3) and the relation

$$A = A_0 - \begin{pmatrix} P - \frac{n}{2} & 0 \\ 0 & \frac{n}{2} - P \end{pmatrix},$$

we obtain for the boundary term $\mathfrak{b}(\varphi,\varphi)$ the following expression

$$\mathfrak{b}(\varphi,\varphi) = \sum_{s=\pm} \Big\langle \sigma_s(\varepsilon), \begin{pmatrix} P - \frac{n}{2} & 0 \\ 0 & \frac{n}{2} - P \end{pmatrix} \sigma_s(\varepsilon) \Big\rangle,$$

which does not contain derivatives any more. Finally, we can express the quadratic form associated to the Hodge-de Rham operator on $\mathcal{M}_\varepsilon$ as

$$\begin{aligned}
\int_{\mathcal{M}_\varepsilon} |D\varphi|^2 d\operatorname{vol}_{g_\varepsilon} = \sum_{s=\pm} \int_\varepsilon^1 \Big(|\sigma_s'|^2 + \frac{1}{t^2}\langle \sigma_s, (A + A^2)\sigma_s\rangle \Big) dt \\
+ \int_0^L \Big(|\sigma'|^2 + \frac{1}{\varepsilon^2}|A_0\sigma|^2 \Big) dt + \frac{1}{\varepsilon}\mathfrak{b}(\varphi,\varphi) \qquad (6)
\end{aligned}$$

for $p-$forms supported in $\mathcal{M}_\varepsilon$.

4. Asymptotic estimates

4.1. **Spectrum of the operator $A + A^2$.** The expression of $A + A^2$ was given in formula (4). We remark that the function

$$f(p) = \Big(\frac{n}{2} - p\Big)\Big(\frac{n}{2} - p - 1\Big)$$

has zeros for the values $n/2$ and $n/2 - 1$. In particular, for $p \in \mathbb{N}$ we have always $f(p) \geq 0$ if n is even and $f(p) \geq -1/4$ if n is odd. The value $-1/4$ is obtained only for $p = (n-1)/2$. Setting

$$a_p := \frac{n+1}{2} - p$$

we have the relation $f(p) = a_{p+1}^2 - 1/4$.

The following lemma is a direct consequence of the Hodge decomposition theorem for the compact manifold (Σ, h) and the expression of the operator $A + A^2$ on each of the subspaces given in the lemma:

Lemma 1. *The space $\mathsf{L}^2(\Lambda^{p-1}\Sigma) \oplus \mathsf{L}^2(\Lambda^p\Sigma)$ is the orthonormal sum of the following five spaces, and $A + A^2$ acts on these spaces as indicated:*

$$\begin{aligned}
&\mathcal{H}_1 = \{(\beta, 0);\ \Delta_\Sigma \beta = 0\,\}, && (A + A^2)(\beta, 0) = (f(p-2)\beta, 0),\\
&\mathcal{H}_2 = \{(0, \alpha);\ \Delta_\Sigma \alpha = 0\}, && (A + A^2)(0, \alpha) = (0, f(p)\alpha),\\
&\mathcal{H}_3 = \{(\beta, 0);\ \beta \text{ exact}\}, && (A + A^2)(\beta, 0) = \big((\Delta_\Sigma + f(p-2))\beta, 0\big),\\
&\mathcal{H}_4 = \{(0, \alpha);\ \alpha \text{ co-exact}\}, && (A + A^2)(0, \alpha) = \big(0, (\Delta_\Sigma + f(p))\alpha\big),\\
&\mathcal{H}_5 = \{(\beta, \alpha);\ \beta \text{ co-exact, } \alpha \text{ exact}\}, && (A + A^2)(\beta, \alpha) =\\
&&& = \big((\Delta_\Sigma + f(p-2))\beta - 2d_0^*\alpha, (\Delta_\Sigma + f(p))\alpha - 2d_0\beta\big).
\end{aligned}$$

In addition, this decomposition is preserved by $A + A^2$ and A_0^2.

We can now compute explicitly the eigenvalues of the operator $A^2 + A$ in terms of the spectrum of the Hodge-de Rham operator on Σ. Clearly, on the spaces $\mathcal{H}_i$, $i = 1, \dots, 4$, the operator $A^2 + A$ is already diagonalised provided α and β are eigenforms of Δ_Σ.

If $(\beta, \alpha) \in \mathcal{H}_5$ is an eigenvector of $A + A^2$ for the eigenvalue λ then they satisfy the equations

$$\big(\Delta_\Sigma + f(p-2) - \lambda\big)\beta = 2d_0^*\alpha \tag{7}$$

$$\big(\Delta_\Sigma + f(p) - \lambda\big)\alpha = 2d_0\beta. \tag{8}$$

Applying d_0 to the first and $\big(\Delta_\Sigma + f(p-2) - \lambda\big)$ to the second equation, and substituting the β term, leads to the equation

$$4\Delta_\Sigma \alpha = \big(\Delta_\Sigma + f(p-2) - \lambda\big)\big(\Delta_\Sigma + f(p) - \lambda\big)\alpha \tag{9}$$

for α. If α is an exact eigenform with $\Delta_\Sigma \alpha = \mu^2 \alpha$ then λ is a solution of the second order polynomial equation

$$4\mu^2 = \big(\mu^2 + f(p-2) - \lambda\big)\big(\mu^2 + f(p) - \lambda\big). \tag{10}$$

A direct computation shows that the solutions of this equation are

$$\lambda_\pm(\mu^2) = \gamma_\pm(\mu^2)(\gamma_\pm(\mu^2)+1)$$

where

$$\gamma_\pm(\mu^2) = -\frac{1}{2} + \Big|\sqrt{\mu^2 + a_p^2} \pm 1\Big|. \tag{11}$$

Now if $(\alpha_k)_{k\in\mathbb{N}}$ is a complete family of exact eigenforms with corresponding eigenvalues $(\mu_k^2)_{k\in\mathbb{N}}$, then it is easily seen that the family

$$\Big(\frac{2}{{\mu_k}^2 + f(p-2) - \lambda_s(\mu_k^2)} d_0^*\alpha_k, \alpha_k\Big)_{k\in\mathbb{N}, s=\pm} \tag{12}$$

defines a basis of $\mathcal{H}_5$ of eigenvectors of $A + A^2$ with corresponding eigenvalues $\{\lambda_s(\mu_k^2)\}_{k\in\mathbb{N},s=\pm}$.

For further purpose, it will be very convenient to write the eigenvalues of $A+A^2$ in the form $\gamma(\gamma+1)$, as we have already done in the above calculation of the spectrum of $A + A^2$ on $\mathcal{H}_5$. The spectrum of the restriction of $A + A^2$ on $\mathcal{H}_3$ is given by $\gamma(\mu^2)(\gamma(\mu^2)+1)$, where

$$\begin{aligned}\gamma(\mu^2) &= -\frac{1}{2} + \sqrt{\mu^2 + \Big(\frac{n+3}{2} - p\Big)^2} \\ &= -\frac{1}{2} + \sqrt{\mu^2 + \Big(a_p + 1\Big)^2} \\ &= -\frac{1}{2} + \sqrt{\mu^2 + a_{p-1}^2}\end{aligned} \tag{13}$$

for μ^2 running over the exact spectrum of Δ_Σ acting on $(p-1)$-forms.

Similarly, the spectrum of $A + A^2$ restricted to $\mathcal{H}_4$ is given by $\gamma(\mu^2)(\gamma(\mu^2)+1)$, where

$$\begin{aligned}\gamma(\mu^2) &= -\frac{1}{2} + \sqrt{\mu^2 + \Big(\frac{n-1}{2} - p\Big)^2} \\ &= -\frac{1}{2} + \sqrt{\mu^2 + \Big(a_p - 1\Big)^2} \\ &= -\frac{1}{2} + \sqrt{\mu^2 + a_{p+1}^2}\end{aligned} \tag{14}$$

for μ^2 running over the co-exact spectrum of Δ_Σ acting on p-forms.

The spectrum of $A + A^2$ on $\mathcal{H}_1$, is $\gamma(\gamma+1)$ with multiplicity $b_{p-1}(\Sigma)$ where

$$\gamma = -\frac{1}{2} + |a_{p-1}| = -\frac{1}{2} + \Big|\frac{n+1}{2} - p + 1\Big|. \tag{15}$$

The spectrum of $A + A^2$ on $\mathcal{H}_2$, is $\gamma(\gamma+1)$ with multiplicity $b_p(\Sigma)$ where

$$\gamma = -\frac{1}{2} + |a_{p+1}| = -\frac{1}{2} + \Big|\frac{n+1}{2} - p - 1\Big|. \tag{16}$$

Remark 2. The decomposition given in Lemma 1 is also preserved by A_0^2. Therefore, the expression (6) of the quadratic form for a p-form supported in $\mathcal{M}_\varepsilon$ shows that the pointwise decomposition of a form is preserved by the quadratic form. Namely, if $\varphi = \sum_{1\le i\le 5} \varphi^i$ with $\varphi^i(t) \in \mathcal{H}_i$ for all t, then

$$\int |D\varphi|^2 = \sum_{1\le i\le 5} \int |D\varphi^i|^2.$$

For our asymptotic analysis below we need a spectral decomposition in a low and high eigenvalue part. Namely, we need the decomposition

$$\varphi^3 + \varphi^4 + \varphi^5 = \varphi_\Lambda + \varphi^\Lambda, \tag{17}$$

where $U_\pm\varphi_\Lambda$ and $U\varphi_\Lambda$ belong (pointwise) to the orthogonal sum of the eigenspace of A_0^2 associated to the eigenvalues smaller that Λ^2. Similarly, $U_\pm\varphi^\Lambda$ and $U\varphi^\Lambda$ belong (pointwise) to the orthogonal sum of the eigenspace of A_0^2 associated to the eigenvalues strictly larger than Λ^2.

4.2. **Study of a sequence of eigenforms.** We consider now a sequence ε_m converging to 0 such that there is a sequence λ_m of eigenvalues of the Hodge-de Rham operator Δ_{ε_m} on M_{ε_m} and converging to some λ. Let φ_m be the corresponding normalised p-eigenform. In the following we will write $\varepsilon = \varepsilon_m$. Thus

$$\Delta_\varepsilon \varphi_m = \lambda_m \varphi_m, \qquad \|\varphi_m\| = 1.$$

We want to understand the behaviour of φ_m when $m \to \infty$. Since this sequence is bounded in $\mathsf{H}^1_{\mathrm{loc}}$ and by elliptic regularity, after passing to a subsequence, we can assume that φ_m converges to φ on $M \setminus \mathcal{V}$ in the H^1-topology and also in $\mathsf{C}^\infty_{\mathrm{loc}}$. Similarly, we can also assume that φ_m converge to φ on each of the cones $\mathcal{C}^\pm_\eta$ for fixed $\eta > 0$ such that $\varepsilon_m \le \eta$. The main difficulty is to understand the behaviour of φ_m on $\mathcal{M}_\varepsilon$. For this purpose we introduce a smooth cut-off function χ with support in $\mathcal{M}_\varepsilon$, $0 \le \chi \le 1$, and such that $\chi = 1$ on $\mathcal{A}_\varepsilon \cup (\mathcal{C}^+_\varepsilon \setminus \mathcal{C}^+_{1/2}) \cup (\mathcal{C}^-_\varepsilon \setminus \mathcal{C}^-_{1/2})$.

On $\mathcal{M}_\varepsilon$ we have the decomposition

$$\varphi_m = \varphi_m^1 + \varphi_m^2 + \varphi_{m,\Lambda} + \varphi_m^\Lambda \tag{18}$$

of Lemma 1 and (17).

4.2.1. *Non-harmonic terms.* We study here the behaviour of the last two terms.

Lemma 3. *The high-energy component of the eigenforms can be estimated near the handle by*

$$\|\varphi_m^\Lambda\|^2_{\mathsf{L}^2(\mathcal{A}_\varepsilon)} \le C\frac{\varepsilon^2}{\Lambda^2} \qquad \text{and} \qquad \|\varphi_m^\Lambda\|^2_{\mathsf{L}^2(\mathcal{C}^\pm_\varepsilon \setminus \mathcal{C}^\pm_\eta)} \le C\frac{\eta^2}{\Lambda^2}$$

provided Λ *is large enough and* $\varepsilon = \varepsilon_m \le \eta$. *The estimate is uniform in* $m \to \infty$.

Proof. Let $\sigma_m = U\chi\varphi_m^\Lambda$ and $\sigma_{\pm,m} = U_\pm\chi\varphi_m^\Lambda$, we have that

$$\begin{aligned}\|D\chi\varphi_m^\Lambda\|^2_{\mathsf{L}^2} &= \int_{\mathcal{M}_\varepsilon} |d\chi|^2|\varphi_m^\Lambda|^2| + \int_{\mathcal{M}_\varepsilon} \chi^2|D\varphi_m^\Lambda|^2 \\ &\le |d\chi|^2_\infty + |\chi|^2_\infty \int_{\mathcal{M}_\varepsilon} |D\varphi_m^\Lambda|^2 \le |d\chi|^2_\infty + |\chi|^2_\infty \lambda_m = C_\chi(\lambda_m)\end{aligned}$$

is uniformly bounded, but on the other hand

$$\begin{aligned}\|D\chi\varphi_m^\Lambda\|^2_{\mathsf{L}^2} \ge \sum_{s=\pm} \int_\varepsilon^{1/2} \Big[|\sigma'_{\pm,m}|^2 + \frac{\langle (A^2+A)\sigma_{\pm,m}, \sigma_{\pm,m}\rangle}{t^2}\Big] dt \\ + \int_0^L \Big[|\sigma'_m|^2 + \frac{|A_0\sigma_m|^2}{\varepsilon^2}\Big] dt - \frac{n+1}{2}\big[|\sigma_m(0)|^2 + \sigma_m(L)|^2\big]. \quad (19)\end{aligned}$$

The boundary term can be estimated using the following optimal inequality

$$\int_0^L \Big[|v'(t)|^2 + \frac{\Lambda^2}{\varepsilon^2}|v(t)|^2\Big] dt \ge \frac{\Lambda}{\varepsilon}\tanh\Big(\frac{\Lambda L}{2\varepsilon}\Big)\big[|v(0)|^2 + |v(L)|^2\big],$$

which is true for all $v \in \mathsf{H}^1([0,L])$. Namely, if we choose $\Lambda > 0$ sufficiently large such that

$$\frac{\Lambda}{\varepsilon}\tanh\Big(\frac{\Lambda L}{2\varepsilon}\Big) \ge (n+1)$$

for all $\varepsilon \in\,]0,1]$, then the boundary term can be estimated in terms of the last integral in (19). In addition, the spectrum of the restriction of the operator $A^2 + A$ to the orthogonal sum of the eigenspaces of A_0^2 associated to the eigenvalues strictly larger that Λ^2, is bounded from below by $\Lambda^2/2$. Consequently, we obtain

$$\begin{aligned}C_\chi(\lambda_m) &\ge \sum_{s=\pm}\int_\varepsilon^{1/2} \Big[|\sigma'_{\pm,m}|^2 + \frac{\Lambda^2|\sigma_{\pm,m}|^2}{2t^2}\Big] dt + \frac{1}{2}\int_0^L \Big[|\sigma'_m|^2 + \frac{|A_0\sigma_m|^2}{\varepsilon^2}\Big] dt \\ &\ge \sum_{s=\pm}\int_\varepsilon^{1/2} \frac{\Lambda^2|\sigma_{\pm,m}|^2}{2t^2} dt + \int_0^L \frac{\Lambda^2|\sigma_m|^2}{2\varepsilon^2} dt \\ &\ge \sum_{s=\pm}\int_\varepsilon^{\eta} \frac{\Lambda^2|\sigma_{\pm,m}|^2}{2\eta^2} dt + \int_0^L \frac{\Lambda^2|\sigma_m|^2}{2\varepsilon^2} dt\end{aligned}$$

and we are done with $C = 4C_\chi(\lambda)$ since $\lambda_m \to \lambda$. □

The next lemma says that also the non-harmonic low energy part of φ_m goes to zero on the handle when $m \to \infty$.

Lemma 4.

$$\lim_{m\to\infty} \|\varphi_{m,\Lambda}\|^2_{\mathsf{L}^2(\mathcal{A}_\varepsilon)} = 0.$$

Proof. Let $U_s\chi\varphi_m = \sigma_{s,m}$ for $s = \emptyset, +, -$. Since $\varphi_{m,\Lambda}$ is a finite sum of forms, which transversally are (non-harmonic) eigenforms of $A^2 + A$ and A_0^2, we can assume that $A_0^2\sigma_{s,m} = \mu^2\sigma_{s,m}$ with $\mu \neq 0$, and $(A^2 + A)\sigma_{s,m} = \gamma(\gamma + 1)\sigma_{s,m}$ where $\gamma \geq -1/2$ depends on μ as in (11), (13) and (14).

On the handle, i.e., on $[0, L]$, σ_m satisfies the (form-valued) equation

$$-\sigma_m''(t) + \frac{\mu^2}{\varepsilon^2}\sigma_m(t) = \lambda_m\sigma_m(t).$$

Consequently, if $\delta_m = \sqrt{\mu^2/\varepsilon^2 - \lambda_m}$, we can write

$$\sigma_m(t) = \frac{1}{\sqrt{\varepsilon}}\big[a_m \mathrm{e}^{-\delta_m t} + b_m \mathrm{e}^{-\delta_m(L-t)}\big],$$

where the coefficients a_m, b_m are pairs of forms on Σ.

It is not hard to check that there is a constant C independent of m such that

$$C^{-1}(|a_m|^2 + |b_m|^2) \leq \int_0^L |\sigma_m(t)|^2 dt \leq C(|a_m|^2 + |b_m|^2)$$

where $|\cdot|$ denotes the L^2-norm of pairs of forms on Σ.

Our aim is to show that a_m and b_m converge to 0. To do so, we need also the behaviour of solutions on the cones. Namely, on $[\varepsilon, 1/2]$, the transformed eigenform $\sigma_{\pm,m}$ solves the equation

$$-\sigma_{\pm,m}''(t) + \frac{\gamma(\gamma+1)}{t^2}\sigma_{\pm,m}(t) = \lambda_m\sigma_{\pm,m}.$$

Hence we can express the solution of the equation in terms of Bessel's functions. As a result, there are entire functions F_γ and G_γ with $F_\gamma(0) = G_\gamma(0) = 1$, such that the solutions are linear combinations of

$$f_\gamma(t) = t^{\gamma+1}F_\gamma(\lambda_m t^2)$$
$$g_\gamma(t) = \begin{cases} t^{-\gamma}G_\gamma(\lambda_m t^2) & \text{if } \gamma + 1/2 \notin \mathbb{N} \\ t^{-\gamma}G_\gamma(\lambda_m t^2) + a\log(t)f_\gamma(t) & \text{if } \gamma + 1/2 \in \mathbb{N}. \end{cases}$$

Namely, there exist pairs of forms $c_{\pm,m}$, $d_{\pm,m}$ on Σ (independent of t), such that

$$\sigma_{\pm,m}(t) = c_{\pm,m}f_\gamma(t) + d_{\pm,m}g_\gamma(t) \tag{20}$$

In both cases, we obtain the estimate

$$\begin{aligned} C^{-1}\big(|c_{\pm,m}|^2 + h_\gamma(\varepsilon)|d_{\pm,m}|^2\big) \leq \int_\varepsilon^{1/2} |\sigma_{\pm,m}(t)|^2 dt \\ \leq C\big(|c_{\pm,m}|^2 + h_\gamma(\varepsilon)|d_{\pm,m}|^2\big) \end{aligned} \tag{21}$$

where

$$h_\gamma(\varepsilon) \sim \begin{cases} \varepsilon^{-2\gamma+1} & \text{if } \gamma > 1/2 \\ |\log(\varepsilon)| & \text{if } \gamma = 1/2 \\ 1 & \text{if } \gamma < 1/2. \end{cases}$$

We now use the transmission conditions (3) to combine the solutions on the handle and the cones. Let

$$J = \begin{pmatrix} -\operatorname{id} & 0 \\ 0 & \operatorname{id} \end{pmatrix}. \tag{22}$$

Then the transmission conditions (3) read as

$$\frac{1}{\sqrt{\varepsilon}}\big[a_m \mathrm{e}^{-\delta_m L} + b_m\big] = \sigma_{+,m}(\varepsilon) \tag{23}$$

and

$$\frac{1}{\sqrt{\varepsilon}}\big[a_m + b_m \mathrm{e}^{-\delta_m L}\big] = J\sigma_{-,m}(\varepsilon). \tag{24}$$

Since a_m and b_m belong to a compact set (namely, to a ball of the finite-dimensional space of eigenforms of A_0^2 below Λ^2), we can assume (after passing to a subsequence) that

$$\lim_{m\to\infty} a_m = a_\infty \qquad \text{and} \qquad \lim_{m\to\infty} b_m = b_\infty.$$

Recall that the main point is to show that $a_\infty = b_\infty = 0$.

The transmission conditions (23) and (24), the behaviour of $\delta_m \sim \mu/\varepsilon$ as ε goes to 0, and the fact that the sequence $c_{\pm,m}$ is bounded (cf. (21)), imply that

$$\lim_{m\to\infty} d_{+,m}\varepsilon^{-\gamma+1/2} = b_\infty \tag{25}$$

and

$$\lim_{m\to\infty} d_{-,m}\varepsilon^{-\gamma+1/2} = Ja_\infty. \tag{26}$$

We conclude from these last equalities together with (21) that $a_\infty = b_\infty = 0$ in the case $\gamma < \frac{1}{2}$. A similar argument holds in the case $\gamma = \frac{1}{2}$.

It remains to consider the case $\gamma > \frac{1}{2}$. From the transmission condition of first order (5) we obtain

$$\frac{\delta_m}{\sqrt{\varepsilon}}\big[-a_m \mathrm{e}^{-\delta_m L} + b_m\big] = \sigma'_{+,m}(\varepsilon) - \frac{1}{\varepsilon}\Big(\frac{n}{2} - P\Big)J\sigma_{+,m}(\varepsilon) \tag{27}$$

for $\sigma_{+,m}$ and

$$\frac{\delta_m}{\sqrt{\varepsilon}}\big[-a_m + b_m \mathrm{e}^{-\delta_m L}\big] = -J\sigma'_{-,m}(\varepsilon) + \frac{1}{\varepsilon}\Big(\frac{n}{2} - P\Big)\sigma_{-,m}(\varepsilon) \tag{28}$$

for $\sigma_{-,m}$.

In the remaining part of the proof, we want to show that $b_\infty = 0$ using (27). The argument for $a_\infty = 0$ follows similarly from (28). Namely, from (27), we obtain the additional information

$$b_\infty = \lim_{m\to\infty} \frac{\sqrt{\varepsilon}}{\delta_m}\Big(-\gamma\varepsilon^{-\gamma-1}d_{+,m} - \Big(\frac{n}{2} - P\Big)\varepsilon^{-\gamma-1}Jd_{+,m}\Big)$$

for b_∞. This equality combined with (25) give the necessary condition on b_∞, namely

$$(\mu+\gamma)b_\infty = \begin{pmatrix} \frac{n}{2} - p + 1 & 0 \\ 0 & p - \frac{n}{2} \end{pmatrix} b_\infty.$$

Using now the result of the following sublemma, we conclude that $b_\infty = 0$. Indeed we restrict ourself to the case where $\gamma > 1/2$, so the last case $\gamma = 1/2 - \mu$ is not possible with $\mu \geq 0$. $\square$

Sublemma 1. *Let b be an eigenform of $A(A+1)$ with eigenvalue $\gamma(\gamma+1)$ relative to a non-zero eigenvalue μ^2 of Δ_Σ and denote*

$$N_{\gamma,\mu,p} = \gamma + \mu + \Big(\frac{n}{2} - P\Big)J = \gamma + \mu - \begin{pmatrix} \frac{n}{2} - p + 1 & 0 \\ 0 & p - \frac{n}{2} \end{pmatrix}.$$

The operator $N_{\gamma,\mu,p}$ restricted to $\mathcal{H}_j$ is identically 0 iff $j = 5$, $\gamma = \gamma_-(\mu^2)$ and $p = (n+1)/2$. In this case, $\mu \in]0,1]$ and $\gamma = 1/2 - \mu$.

In all other cases, i.e., if $b \in \mathcal{H}_j$, $j = 3, 4$, or $b \in \mathcal{H}_5$ and $\gamma = \gamma_+(\mu^2)$ or $\gamma = \gamma_-(\mu^2)$ but $p \neq (n+1)/2$ or $\mu > 1$, then $N_{\gamma,\mu,p}(b) = 0$ implies $b = 0$.

Proof. We distinguish the three cases $b \in \mathcal{H}_3$, $b \in \mathcal{H}_4$ and $b \in \mathcal{H}_5$.

If $b \in \mathcal{H}_3$ then we have

$$b = \begin{pmatrix} b_1 \\ 0 \end{pmatrix}$$

and

$$\gamma = -\frac{1}{2} + \sqrt{\mu^2 + \Big(\frac{n+3}{2} - p\Big)^2}.$$

So $N_{\gamma,\mu,p}(b) = 0$ means

$$\Big(\mu + \sqrt{\mu^2 + \Big(\frac{n+3}{2} - p\Big)^2}\Big) b_1 = \Big(\frac{n+3}{2} - p\Big) b_1.$$

Since $\mu > 0$, it follows that $b_1 = 0$.

If $b \in \mathcal{H}_4$ then we have

$$b = \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$$

and

$$\gamma = -\frac{1}{2} + \sqrt{\mu^2 + \Big(\frac{n-1}{2} - p\Big)^2}.$$

Consequently, we obtain if $N_{\gamma,\mu,p}(b) = 0$,

$$\Big(\mu + \sqrt{\mu^2 + \Big(\frac{n-1}{2} - p\Big)^2}\Big) b_2 = \Big(p - \frac{n-1}{2}\Big) b_2,$$

hence $b_2 = 0$ since again, $\mu > 0$.

It remains to treat the case where $b \in \mathcal{H}_5$. Here, we have

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Moreover, we know that $b_1 = 0$ if and only if $b_2 = 0$ due to the expression (12) for the eigenforms. In addition,

$$\gamma_\pm = -\frac{1}{2} + \Big| \sqrt{\mu^2 + \Big(\frac{n+1}{2} - p\Big)^2} \pm 1 \Big|.$$

Then, $N_{\gamma,\mu,p}(b)=0$ means

$$\Big(\mu+\Big|\sqrt{\mu^2+\Big(\frac{n+1}{2}-p\Big)^2}\pm 1\Big|\Big)b_1=\Big(\frac{n+3}{2}-p\Big)b_1$$

and also

$$\Big(\mu+\Big|\sqrt{\mu^2+\Big(\frac{n+1}{2}-p\Big)^2}\pm 1\Big|\Big)b_2=\Big(p-\frac{n-1}{2}\Big)b_2.$$

If $b\neq 0$, we obtain first, that $\Big(p-\frac{n-1}{2}\Big)=\Big(\frac{n+3}{2}-p\Big)$, or $p=\frac{n+1}{2}$ (i.e. $a_p=0$) and secondly, that $\mu+|\mu\pm 1|=1$.

So assume that $p=\frac{n+1}{2}$. Since $\mu>0$, we have $\mu+|\mu+1|>1$ and as a consequence

$$N_{\gamma_+(\mu^2),\mu,\frac{n+1}{2}}(b)=0 \quad\Rightarrow\quad b=0.$$

As well, if $\mu>1$, then $\mu+|\mu-1|=2\mu-1>1$, so we have also

$$\mu>1 \qquad\text{and}\qquad N_{\gamma_-(\mu^2),\mu,\frac{n+1}{2}}(b)=0 \quad\Rightarrow\quad b=0.$$

In the remaining case $\mu\in\,]0,1]$, we obtain $\mu+|\mu-1|=1$ (and $\gamma_-(\mu^2)=1/2-\mu$), and $N_{\gamma_-(\mu^2),\mu,(n+1)/2}=0$. ☐

We study now the behaviour of the low energy forms $\varphi_{m,\Lambda}$ on the cones $\mathcal{C}_\varepsilon^\pm$:

LEMMA 5. *On $\mathcal{C}_\varepsilon^\pm$, we have*

$$U_\pm\chi\varphi_{m,\Lambda}=u_m+v_m$$

where

$$\lim_{m\to\infty}\|v_m\|_{\mathsf{L}^2(\mathcal{C}_\varepsilon^\pm)}=0$$

and u_m is given as follows:

(i) *If $p\neq(n+1)/2$ or if there is no eigenvalue of Δ_Σ for exact p-forms in the interval $]0,1[$, then*

$$u_m=\sum_\gamma c_\gamma(m)t^{\gamma+1}F_\gamma(\lambda_m t^2)\sigma_\gamma$$

where the sum is finite over $\gamma\in[-\frac12,\infty[$, and the sequence $(c_\gamma(m))_m$ is bounded. Moreover, $F_\gamma\in\mathsf{C}^\infty([0,\infty[)$ and $F_\gamma(0)=1$. Finally, $(\sigma_\gamma)_\gamma$ is independent of t and is an orthonormal family in $\mathsf{L}^2(\Lambda^{p-1}T^*\Sigma)\oplus\mathsf{L}^2(\Lambda^pT^*\Sigma)$.

(ii) *If $p=(n+1)/2$ and if there is an eigenvalue μ^2 of Δ_Σ for exact p-forms in the interval $]0,1[$, then we have*

$$u_m=\sum_\gamma c_\gamma(m)t^{\gamma+1}F_\gamma(\lambda_m t^2)\sigma_\gamma+\sum_\mu c_\mu(m)t^{\mu-1/2}G_\gamma(\lambda_m t^2)\sigma_\mu,$$

where the first sum satisfies the same properties as in (i) but we only have $\gamma\in\{-\frac12\}\cup[\frac12,\infty[$. In addition, the second sum is finite and runs over $\mu\in\,]0,1[$ such that μ^2 is an exact eigenvalue of Δ_Σ, and the sequence $(c_\mu(m))_m$ is bounded. In addition, $G_\gamma\in\mathsf{C}^\infty([0,\infty[)$ with $G_\gamma(0)=1$. Moreover, the family $\{U_\pm^{-1}\sigma_\gamma\}_\gamma\cup\{U_\pm^{-1}\sigma_\mu\}_\mu$ is orthonormal.

Proof. We continue with the same notation as in Lemma 4. We will only work on the behaviour on $\mathcal{C}_\varepsilon^+$ since the other is similar. We assume that $U_+(\varphi_{m,\Lambda}) = \sigma_m$ is a common eigenvector of both A_0^2 and $A^2 + A$ for each t, i.e.,

$$A_0^2\sigma_m = \mu^2\sigma_m \qquad \text{and} \qquad (A^2 + A)\sigma_m = \gamma(\gamma+1)\sigma_m,$$

where we have dropped the subscript $+$. The expression of σ_m is given in (20).

From (21), (25) and Lemma 4 we conclude that for $\gamma > \frac{1}{2}$ we have

$$\|d_m g_\gamma\|^2 \simeq h_\gamma(\varepsilon)|d_m|^2 = o(1),$$

if m tends to ∞.

We concentrate now on the case where $\gamma \in [-\frac{1}{2}, \frac{1}{2}]$. Equations (27) and (23) imply, by elimination of b_m, that for a certain constant c we have

$$\delta_m\sigma_m(\varepsilon) - \sigma_m'(\varepsilon) + \frac{1}{\varepsilon}\Big(\frac{n}{2} - P\Big)J\sigma_m(\varepsilon) = O\left(\mathrm{e}^{-c/\varepsilon}\right). \tag{29}$$

But from (20) we conclude that for $\gamma \in]-\frac{1}{2}, \frac{1}{2}[$ we have

$$\begin{aligned}\sigma_m'(\varepsilon) = c_m\varepsilon^\gamma(\gamma+1)F_\gamma(\lambda_m\varepsilon^2) + 2c_m\varepsilon^{\gamma+2}\lambda_m F_\gamma'(\lambda_m\varepsilon^2)\\ - \gamma d_m\varepsilon^{-\gamma-1}G_\gamma(\lambda_m\varepsilon^2) + 2d_m\varepsilon^{-\gamma+1}\lambda_m G_\gamma'(\lambda_m\varepsilon^2).\end{aligned}$$

and if $\gamma = \pm 1/2$ we obtain

$$\begin{aligned}\sigma_m'(\varepsilon) = \varepsilon^\gamma\big((c_m + ad_m\log\varepsilon)(\gamma+1) + d_m a\big)F_\gamma(\lambda_m\varepsilon^2)\\ + 2\varepsilon^{\gamma+2}(c_m + d_m a\log\varepsilon)\lambda_m F_\gamma'(\lambda_m\varepsilon^2)\\ - \gamma d_m\varepsilon^{-\gamma-1}G_\gamma(\lambda_m\varepsilon^2) + 2d_m\varepsilon^{-\gamma+1}\lambda_m G_\gamma'(\lambda_m\varepsilon^2).\end{aligned}$$

Note that δ_m also depends on ε, namely, $\delta_m(\varepsilon) = \mu/\varepsilon + O(\varepsilon)$. Therefore, equation (29) together with the fact that the sequences $\{c_m\}_m$ and $\{d_m\}_m$ are bounded and the previous expressions for $\sigma_m'(\varepsilon)$ leads to

$$\Big(\gamma + \mu + \Big(\frac{n}{2} - P\Big)J\Big)\varepsilon^{-\gamma-1}G_\gamma(\lambda\varepsilon^2)d_m = \begin{cases} O(\varepsilon^\gamma) & \text{if } \gamma \neq \frac{1}{2} \\ O(\varepsilon^{\frac{1}{2}}|\log\varepsilon|) & \text{if } \gamma = \frac{1}{2}\end{cases}$$

for $\gamma \in]-\frac{1}{2}, \frac{1}{2}]$ and

$$\Big(\gamma + \mu + \Big(\frac{n}{2} - P\Big)J\Big)\varepsilon^{-\frac{1}{2}}(\log\varepsilon)aF_\gamma(\lambda\varepsilon^2)d_m = O(\varepsilon^{-\frac{1}{2}})$$

for $\gamma = -1/2$. Using the operator $N_{\gamma,\mu,p}$ introduced in Sublemma 1 we have obtained

$$N_{\gamma,\mu,p}(d_m) = \begin{cases} O(\varepsilon^{1+2\gamma}) & \text{if } \gamma > -1/2 \text{ and } \gamma \neq 1/2 \\ O(\varepsilon^2|\log\varepsilon|) & \text{if } \gamma = 1/2 \\ O(|\log\varepsilon|^{-1}) & \text{if } \gamma = -1/2.\end{cases}$$

Hence, if the operator $N_{\gamma,\mu,p}$ is invertible, we have the same type of estimates for d_m itself and $\lim_{m\to\infty} h_\gamma(\varepsilon)|d_m|^2 = 0$ or

$$\lim_{m\to\infty}\|\sigma_m - c_m f_\gamma\| = 0.$$

The result of Sublemma 1 shows that the operator $N_{\gamma,\mu,p}$ is invertible except in the case where d_m is in $\mathcal{H}_5$, $p = (n+1)/2$ and $\gamma = \gamma_-(\mu^2) = 1/2 - \mu$. The last equality imposes $\mu \in]0, 1]$. In particular, we have $\gamma \in [-\frac{1}{2}, \frac{1}{2}[$. Returning to Equation (29) we conclude then that

$$\sigma'_m(\varepsilon) + \frac{\frac{1}{2} - \mu}{\varepsilon}\sigma_m(\varepsilon) + O(\varepsilon)\sigma_m(\varepsilon) = O(\mathrm{e}^{-c/\varepsilon})$$

or equivalently

$$\varepsilon^{-\gamma}\frac{d}{dt}\Big(t^\gamma\sigma_m(t)\Big)\Big|_{t=\varepsilon} + O(\varepsilon)\sigma_m(\varepsilon) = O(\mathrm{e}^{-c/\varepsilon}).$$

Hence, if $\gamma = -1/2$, we obtain

$$d_m = O(\varepsilon^2 \log \varepsilon)$$

and as before $\lim_{m\to\infty} d_m = 0$ and $\lim_{m\to\infty} \|\sigma_m - c_m f_\gamma\| = 0$.
If $\gamma \in]-\frac{1}{2}, \frac{1}{2}[$, we have

$$\varepsilon^{-\gamma}\frac{d}{dt}\Big(t^\gamma\sigma_m(t)\Big)\Big|_{t=\varepsilon} = c_m(2\gamma+1)\varepsilon^\gamma F_\gamma(\lambda_m\varepsilon^2) + c_m 2\varepsilon^{\gamma+2}\lambda_m F'_\gamma(\lambda_m\varepsilon^2) + \varepsilon^{-\gamma+1}2\lambda_m G'_\gamma(\lambda_m\varepsilon^2)d_m.$$

and therefore

$$c_m = O(\varepsilon^{-2\gamma+1}), \qquad \text{i.e.} \qquad \lim_{m\to\infty} c_m = 0.$$

□

4.2.2. *Harmonic terms.* It remains now to describe the behaviour of the harmonic components φ^1_m and φ^2_m. We restrict our analysis to the space $\mathcal{H}_1$, since $\mathcal{H}_1$ and $\mathcal{H}_2$ are dual by the Hodge-$*$ operator.

Again, we let $(\beta_{s,m}, 0) = U_s\chi\varphi^1_m$ for $s = \emptyset, +, -$ be the transformed pair of forms corresponding to the the handle and the cones, respectively. We know that on the handle, β_m satisfies the equation

$$-\beta''_m = \lambda_m\beta_m \qquad \text{on } [0, L], \tag{30}$$

whereas on the cones, $\beta_{\pm,m}$ fulfills

$$-\beta''_{\pm,m} + \frac{\nu(\nu+1)}{t^2}\beta_{\pm,m} = \lambda_m\beta_{\pm,m}, \qquad \text{with } \nu = n/2 - p + 1. \tag{31}$$

If $\nu \neq 0$ we put $\gamma = -\frac{1}{2} + \left|\frac{n+3}{2} - p\right| = -\frac{1}{2} + \left|\nu + \frac{1}{2}\right|$ as in (13) with $\mu = 0$. The transmission conditions (3) and (5) now reads as

$$\beta_m(L) = \beta_{+,m}(\varepsilon), \qquad \beta_m(0) = -\beta_{-,m}(\varepsilon) \tag{32}$$

and

$$\beta'_m\Big(\frac{L \pm L}{2}\Big) = \beta'_{\pm,m}(\varepsilon) + \frac{\nu}{\varepsilon}\beta_{\pm,m}(\varepsilon) = \varepsilon^{-\nu}\frac{d}{dt}\Big(t^\nu\beta_{\pm,\varepsilon}(t)\Big)\Big|_{t=\varepsilon}. \tag{33}$$

Since the L^2-norm of $\beta_{s,m}$ is bounded, it follows from equation (30) and the transmission conditions (32) and (33), that $\beta_m(0)$, $\beta_m(L)$, $\beta'_m(0)$, $\beta'_m(L)$ and $\beta_{\pm,m}(\varepsilon)$ are all bounded sequences. Hence after passing to a subsequence, we can assume that these sequences converge. Moreover, from the quadratic form expression (6), we also know that there is a uniform constant C such that

$$\sum_{s=\pm}\Big(\int_\varepsilon^1 \Big(|\beta'_{s,m}|^2+\frac{\nu(\nu+1)}{t^2}|\beta_{s,m}|^2\Big)dt-\frac{1}{\varepsilon}\nu|\beta_{s,m}(\varepsilon)|^2\Big)+\int_0^L|\beta'_m|^2dt\le C. \tag{34}$$

We express the solutions of (31) as in (20):

$$\beta_{\pm,m}(t)=c_{\pm,m}f_\gamma(t)+d_{\pm,m}g_\gamma(t).$$

As a consequence of the estimate (21) on the L^2-norm of $\beta_{\pm,m}$, we obtain

$$c_{\pm,m}=O(1) \qquad \text{and} \qquad d_{\pm,m}=\begin{cases}O\big(\varepsilon^{\gamma-1/2}\big) & \text{if } \gamma>1/2,\\ O\big(|\log\varepsilon|^{-1/2}\big) & \text{if } \gamma=1/2,\\ O(1) & \text{if } \gamma<1/2.\end{cases} \tag{35}$$

Again, after passing to a subsequence, we can assume that the sequences $\{c_{\pm,m}\}_m$ converge to $c_{\pm,\infty}$.

But now from the transmission condition (32) we know that $\beta_{\pm,m}(\varepsilon)$ and $c_{\pm,m}$ are bounded, and as a consequence, $(d_m g_\gamma(\varepsilon))_m$ is also bounded so

$$\gamma\ge 0 \qquad \text{implies} \qquad d_{\pm,m}=O(\varepsilon^\gamma). \tag{36}$$

In particular, we have

Corollary 6. *If $\gamma>0$, i.e., $\nu\notin\{-1,-1/2,0\}$, then*

$$\|\beta_{\pm,m}-c_{\pm,\infty}f_\gamma\|_{\mathsf{L}^2}\to 0.$$

Proof. By the estimate (21), there exists a constant $C>0$ such that

$$\|\beta_{\pm,m}-c_{\pm,\infty}f_\gamma\|_{\mathsf{L}^2}\le C\sqrt{h_\gamma(\varepsilon)}|d_{\pm,m}|.$$

By the preceding remark and (21), we arrive at

$$\|\beta_{\pm,m}-c_{\pm,\infty}f_\gamma\|_{\mathsf{L}^2}=\begin{cases}O(\sqrt{\varepsilon}) & \text{if } \gamma>1/2\\ O(\sqrt{\varepsilon|\log\varepsilon|}) & \text{if } \gamma=1/2.\end{cases}$$

□

We study now the limit boundary conditions.

Lemma 7. *If $p-1\ge\frac{n+1}{2}$ or $\nu\le -1/2$ (and therefore $\gamma=-1-\nu$), then we obtain, at the limit, the Dirichlet boundary conditions:*

$$\lim_{m\to\infty}\beta_m(0)=0 \qquad \textit{and} \qquad \lim_{m\to\infty}\beta_m(L)=0.$$

Proof. If $\nu<-1/2$ then $\nu(\nu+1)\ge 0$ and the estimate of the quadratic form (34) gives

$$\beta_\varepsilon(0)=-\beta_{-,\varepsilon}(\varepsilon)=O(\sqrt{\varepsilon}), \qquad \beta_\varepsilon(L)=\beta_{+,\varepsilon}(\varepsilon)=O(\sqrt{\varepsilon}).$$

Now suppose that $\nu=-1/2$. In this case the estimate (34) gives

$$\int_\varepsilon^1|\beta'_{\pm,m}(t)|^2dt-\int_\varepsilon^1\frac{1}{4t^2}|\beta_{\pm,m}(t)|^2dt+\frac{1}{2\varepsilon}|\beta_{\pm,m}(\varepsilon)|^2\le C.$$

However, after integration by parts, the left hand side of this inequality is

$$\int_\varepsilon^1 t\Big|\frac{d}{dt}\Big(t^{-1/2}\beta_{\pm,m}(t)\Big)\Big|^2 dt.$$

Let $v \in \mathsf{C}_{\mathrm{c}}^\infty([\varepsilon,1[)$ and let $\varphi(t) = \sqrt{|\log t|}\,v(t)$, then we have

$$\begin{aligned}\int_\varepsilon^1 t|\varphi'(t)|^2 dt &= \int_\varepsilon^1 |v(t)|^2\frac{dt}{4t|\log t|} + \int_\varepsilon^1 |v'(t)|^2|\log(t)|dt - \int_\varepsilon^1 v(t)v'(t)\,dt\\ &\geq -\int_\varepsilon^1 v(t)v'(t)\,dt\\ &= \frac{1}{2}|v(\varepsilon)|^2.\end{aligned}$$

Applying the former estimate with $v(t) = (t|\log t|)^{-1/2}\beta(t)$, we obtain

$$\beta_{\pm,m}(\varepsilon) = O\big(\sqrt{\varepsilon|\log\varepsilon|}\big)$$

which proves the claim. □

We focus now on the case where $\nu > 0$ and consequently $\gamma = \nu$.

Lemma 8. *If $p - 1 \leq \frac{n-1}{2}$, or $\nu \geq \frac{1}{2}$ (and therefore $\gamma = \nu$), then we obtain, at the limit, the Neumann boundary conditions*

$$\lim_{m\to\infty}\beta'_m(0) = 0 \qquad \textit{and} \qquad \lim_{m\to\infty}\beta'_m(L) = 0.$$

Proof. The first order transmission conditions (33) imply that we have to look at the limit of the sequence formed by

$$\varepsilon^{-\nu}\frac{d}{dt}\Big(t^\nu\beta_{\pm,\varepsilon}(t)\Big)\Big|_{t=\varepsilon}.$$

But the limit of this sequence is

$$\begin{cases}\lim_{m\to\infty} d_{\pm,m}\varepsilon^{1-\nu}\lambda_m G'_\gamma(\lambda_m\varepsilon^2) & \text{if } \nu \geq 1,\\ \lim_{m\to\infty} 2ad_{\pm,m}\varepsilon^{1/2}(\log\varepsilon)F_\gamma(\lambda_m\varepsilon^2) & \text{if } \nu = 1/2.\end{cases}$$

Now following (36), $d_{\pm,m} = O(\varepsilon^\nu)$, and we obtain finally

$$\Big|\beta'_m\Big(\frac{L\pm L}{2}\Big)\Big| = \begin{cases}O(\varepsilon) & \text{if } \nu > 1/2,\\ O(\varepsilon|\log\varepsilon|) & \text{if } \nu = 1/2\end{cases}$$

and the result follows. □

Corollary 9. *If $H^{n/2}(\Sigma) = 0$, then we have*

$$U(\varphi_{m,1}) = (c_{\pm,m}f_\gamma, 0) + r_m$$

on $\mathcal{C}_\varepsilon^\pm$, where

$$\lim_{m\to\infty}\|r_m\|_{\mathsf{L}^2} = 0,$$

the sequence $c_{\pm,m}$ converges to $c_{\pm,\infty}$ and f_γ is given by (20) with $\gamma = -\frac{1}{2} + \big|\frac{n+3}{2} - p\big|$.

Proof. With the preceding notations we have to show that

$$\|\beta_{\pm,m} - c_{\pm,\infty} f_\gamma\|_{\mathsf{L}^2} \to 0.$$

Recall that $\gamma = -\frac{1}{2} + \left|\nu + \frac{1}{2}\right|$ with $\nu = n/2 - p + 1$. Corollary 6 fulfills the case $\gamma > 0$. If $\gamma \leq 0$ then, by the estimate (21), $\|\beta_{\pm,m} - c_{\pm,\infty} f_\gamma\|_{\mathsf{L}^2}$ is controlled by $|d_{\pm,m}|$. It remains to show that $\lim_{m\to\infty} d_{\pm,m} = 0$ if $\gamma \leq 0$ (and $\nu \neq 0$ by hypothesis).

The case $\gamma = 0$ corresponds only to $\nu = -1$. The boundedness of the quadratic form gives then $\beta_{\pm,m}(\varepsilon) = O(\sqrt{\varepsilon})$. But this implies, by the expression of the solutions of (31), that

$$d_{\pm,m} = O(\sqrt{\varepsilon}).$$

The case $\gamma = -1/2$ corresponds to $\nu = -1/2$. In this case we have already seen that

$$\beta_{\pm,m}(\varepsilon) = O\big(\sqrt{\varepsilon|\log\varepsilon|}\big).$$

The expression of the solutions of (31) for $\gamma = -1/2$ implies that

$$d_{\pm,m} = O\left(\frac{1}{\sqrt{|\log\varepsilon|}}\right)$$

and the result follows. □

Remark 10. If the cohomology group $H^{n/2}(\Sigma)$ is non-trivial, what happens in the case $\nu = 0$, i.e., for forms of degree $p = n/2 + 1$? The quadratic form (34) becomes

$$\sum_{s=\pm} \int_\varepsilon^1 |\beta'_{s,m}|^2 + \int_0^L |\beta'_m|^2.$$

Actually, we are just on intervals and the transmission condition gives the limit situation. From the sequence $\{\beta_m\}_m$, which is bounded in H^1 on the global interval, one can extract a sequence which converges to an eigenform on $\overline{M}$ with eigenvalue λ, and the boundary values $\beta_\pm(0)$ and $\beta'_\pm(0)$ must satisfy the transmission conditions

$$\beta_-(0) = -\beta(0), \qquad \beta_+(0) = \beta(L); \qquad \beta'_-(0) = \beta'(0), \qquad \beta'_+(0) = \beta'(L) \tag{37}$$

for β satisfying $-\beta'' = \lambda\beta$ on $[0, L]$.

For instance if we come from the situation where $M = \mathbb{R}/\mathbb{Z} \times \Sigma$ is a 3-torus and $\Sigma = \mathbb{R}^2/\mathbb{Z}^2$ a generating torus, the limit problem described here is *not* decoupling.

5. The limit problem

We first recall the results of [7] and [16] concerning the closed extensions of the operator $D = d + d^*$ on the manifold with conical singularities $\overline{M}$. They are classified by the spectrum of its *Mellin symbol*, which is here the operator with parameter $A + z$. In our case, we need two copies of $A + z$, since we have two conical singularities. Recall that A is the operator defined in (3) by

$$A = \begin{pmatrix} \dfrac{n}{2} - P & -D_0 \\ -D_0 & P - \dfrac{n}{2} \end{pmatrix}.$$

If $\operatorname{spec}(A) \cap \ ]-\frac{1}{2}, \frac{1}{2}[$ is empty then $D_{\max} = D_{\min}$. In particular, D is essentially selfadjoint on the space of smooth functions with compact support *away* from the conical singularities. Otherwise, the quotient $\operatorname{dom}(D_{\max}) / \operatorname{dom}(D_{\min})$ is isomorphic to

$$B_+ \oplus B_- \qquad \text{where} \qquad B_\pm := \bigoplus_{\gamma \in]-\frac{1}{2}, \frac{1}{2}[} \operatorname{Ker}(A - \gamma).$$

More precisely, by Lemma 3.2 of [7], there is a surjective linear map

$$\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_- \colon \operatorname{dom}(D_{\max}) \to B_+ \oplus B_-$$

with $\ker \mathcal{L} = \operatorname{dom}(D_{\min})$. Furthermore, we have the estimate

$$\|u_\pm(t) - t^{-A} \mathcal{L}_\pm(\varphi)\|_{\mathsf{L}^2(\Sigma)} \le C(\varphi) |t \log t|^{1/2}$$

for $\varphi \in \operatorname{dom}(D_{\max})$ and $u_\pm = U_\pm(\varphi)$, where $U_\pm$ is defined in Section 3.

Now to any subspace $W \subset B_+ \oplus B_-$, we associate the operator D_W with domain $\operatorname{dom}(D_W) := \mathcal{L}^{-1}(W)$. As a result of [7], all closed extensions of $D_{\min}$ are obtained by this way. Remark that each D_W defines a selfadjoint extension $(D_W)^* \circ D_W$ of the Hodge-Laplace operator, and we have $(D_W)^* = D_{\mathbb{I}(W^\perp)}$, where

$$\mathbb{I} = \begin{pmatrix} 0 & \operatorname{id} \\ -\operatorname{id} & 0 \end{pmatrix}, \qquad \text{ie.,} \quad \mathbb{I}(\beta, \alpha) = (\alpha, -\beta).$$

This extension is associated to the quadratic form $\varphi \mapsto \|D\varphi\|^2$ with domain $\operatorname{dom}(D_W)$. We have already computed the spectrum of the operator $A^2 + A$ restricted to the spaces $\mathcal{H}_1, \dots, \mathcal{H}_5$ in Section 4.1. It is expressed for each space $\mathcal{H}_i$ in the form $\gamma(\gamma + 1)$ with $\gamma \ge -1/2$, where γ is given in (11) and (13)–(16).

Hence the spectrum of A is among the values $\gamma_\pm, -1-\gamma_\pm$ where $\gamma_\pm$ is given by (11), and the $\gamma, -1-\gamma$, for the γ appearing in (13)–(16).[1] We have to take care of the fact that the spaces are not all stable under the action of A. Indeed $\mathcal{H}_1$ and $\mathcal{H}_2$ are stable by the action of A and consequently the spectrum of A contains $\frac{n}{2} - p + 1$ with multiplicity $b_{p-1}(\Sigma)$ and $p - \frac{n}{2}$ with multiplicity $b_p(\Sigma)$ where p runs over $0, ..., n$, $\mathcal{H}_5$ also is stable by A, but $\mathcal{H}_3$ and $\mathcal{H}_4$ are not. Nevertheless we remark that A satisfies the relation $A \circ \mathbb{I} = -\mathbb{I} \circ A$, and, if $Au = \gamma u$ with $u \in \mathcal{H}_5$, then $A(\mathbb{I}u) = -\gamma \mathbb{I}u$ with $\mathbb{I}u$ living in the $\mathcal{H}_3 \oplus \mathcal{H}_4$ components of *other* degrees. Then, considering all the degrees together, $\sum_p (\mathcal{H}_3 \oplus \mathcal{H}_4)$ is stable under the action of A and its spectrum on this component is the opposite of its spectrum on $\mathcal{H}_5$. Thus the spectrum of A is determined by its restriction on $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_5$.

For our concern we have the following result:

Lemma 11. *Let n be odd. Then the eigenvalues γ of A restricted to $\mathcal{H}_5$ with $\gamma \in \ ]-\frac{1}{2}, \frac{1}{2}[$ are precisely the values $\gamma = \gamma_-(\mu^2)$ entering in the description of the spectrum of $A(A+1)$ with $a_p = 0$, i.e., $p = (n+1)/2$, and $\mu \in]0, 1[$, thus $\gamma_-(\mu^2) = \mu - \frac{1}{2}$.*

[1] Using [7], we can calculate explicitely the spectrum of A. In fact, $\operatorname{spec}(A)$ consists of the values $\gamma = \pm\frac{1}{2} \pm \sqrt{\mu^2 + a_{p+1}^2}$, where μ^2 runs over the spectrum of Δ_Σ acting on co-closed p-forms.

Proof. Let $\sigma = (\beta, \alpha) \in \mathcal{H}_5$ and denote the degree of α by p. Then A is given by

$$A = \begin{pmatrix} p - 1 - \frac{n}{2} & -d_0^* \\ -d_0 & \frac{n}{2} - p \end{pmatrix},$$

so that $A\sigma = \gamma\sigma$ is equivalent to

$$\Big(p - 1 - \frac{n}{2} - \gamma\Big)\beta = d_0^*\alpha \qquad \text{and} \qquad \Big(\frac{n}{2} - p - \gamma\Big)\alpha = d_0\beta, \quad \text{i.e.,}$$

$$\Delta_\Sigma\alpha = \Big(\big(\frac{1}{2} + \gamma\big)^2 - a_p^2\Big)\alpha \qquad \text{and} \qquad \Delta_\Sigma\beta = \Big(\big(\frac{1}{2} + \gamma\big)^2 - a_p^2\Big)\beta.$$

In particular, the latter equalities mean that there exists

$$\mu^2 \in \operatorname{spec}(\Delta^p_{\Sigma,\mathrm{c}}) \cap \operatorname{spec}(\Delta^{p-1}_{\Sigma,\mathrm{cc}}) \qquad \text{with} \qquad \gamma = -\frac{1}{2} \pm \sqrt{\mu^2 + a_p^2}$$

where $\Delta_{\Sigma,\mathrm{c}}$ resp. $\Delta_{\Sigma,\mathrm{cc}}$ denotes the Laplacian acting on closed resp. co-closed, forms. Now, $\gamma \in \,]-\frac{1}{2}, \frac{1}{2}[$ implies $\gamma = -\frac{1}{2} + \sqrt{\mu^2 + a_p^2}$ with $a_p = 0$ and $\mu^2 \in \,]0,1[$ or with $a_p = \pm\frac{1}{2}$ and $\mu^2 \in [0, \frac{3}{4}[$. Since we assumed that n is odd, the unique possibility is $a_p = 0$, i.e., $p = (n+1)/2$, and therefore $\gamma = \mu - \frac{1}{2}$ since $\mu^2 \in \,]0,1[$. Reciproquely, if $a_p = 0$ and $\Delta^p_{\Sigma,\mathrm{c}}\alpha = \mu^2\alpha$ with $\mu^2 \in \,]0,1[$, then $A(\beta,\alpha) = (\mu - \frac{1}{2})(\beta, \alpha)$ with $\beta = -d_0^*\alpha/\mu$, and also $\Delta^{p-1}_{\Sigma,\mathrm{cc}}\beta = \mu^2\beta$. Then $\sigma = \sigma_\mu$, with the notations of Lemma 5. $\square$

In fact we have proved more, namely: $\operatorname{spec}(A) \cap \,]-\frac{1}{2}, \frac{1}{2}[\, = \emptyset$ if and only if

- the spectrum of Δ_Σ on exact $(n+1)/2$-forms (or co-exact $(n-1)/2$-forms) does not meet the interval $]0,1[$, for n odd,
- the spectrum of Δ_Σ on $n/2$-forms does not meet the interval $[0, \frac{3}{4}[$, for n even.

Indeed, in the last alternative, for $a_p = \pm\frac{1}{2}$, i.e., $p \in \{\frac{n}{2}, \frac{n}{2}+1\}$, if $\Delta^p_{\Sigma,\mathrm{c}}\alpha = \mu^2\alpha$ with $\mu \neq 0$, then $\gamma = -\frac{1}{2} + \sqrt{\mu^2 + \frac{1}{4}} > 0$ and again $A(\beta,\alpha) = \gamma(\beta,\alpha)$ with $\beta = -\frac{d_0^*\alpha}{\frac{1}{2} \pm \frac{1}{2} + \gamma}$, and also $\Delta^{p-1}_{\Sigma,\mathrm{cc}}\beta = \mu^2\beta$; we remark that either α is an exact $n/2$-form, or β is a co-exact $n/2$-form. The case $\mu = 0$ corresponds in fact to components in $\mathcal{H}_1$ for $p = \frac{n}{2} + 1$ and $\mathcal{H}_2$ for $p = \frac{n}{2}$. They have already been described and correspond to $n/2$-harmonic forms.

We can now describe the extensions of the Laplacian obtained for the limit operator; they depend on p:

- If $p \notin \{(n+1)/2, n/2, n/2+1\}$ or if $p \in \{n/2, n/2+1\}$ and $b_{n/2}(\Sigma) = 0$ or if $p = (n+1)/2$ and the spectrum of Δ_Σ on exact $(n+1)/2$-forms does not meet the interval $]0,1[$, then on the manifold part $\overline{M}$ the limit operator is the Friedrichs extension of the Hodge-Laplace operator, that is $D_{\max} \circ D_{\min}$ restricted to $\mathsf{L}^2(\Lambda^p T^*M)$. It is the Friedrichs extension of the Laplacian defined by the quadratic form $\sigma \mapsto \|D\sigma\|^2$ with domain $\operatorname{dom}(D_{\min})$.
- If $p = (n+1)/2$ and Δ_Σ has exact $(n+1)/2$-eigenvalues in the interval $]0,1[$, then the limit operator is (on the manifold part) $D_{\min} \circ D_{\max}$ restricted to $\mathsf{L}^2(\Lambda^{\frac{n+1}{2}} T^*M)$. It is the Friedrichs extension of the Laplacian defined by the quadratic form $\varphi \mapsto \|D\varphi\|^2$ with domain $\operatorname{dom}(D_{\max})$.

- In the case when $H^{n/2}(\Sigma) \neq \{0\}$, and $p = \frac{n}{2}$ or $p = \frac{n}{2} + 1$, the limit operator does not come from a selfadjoint extension of the Hodge-Laplace operator for the conical manifold $\overline{M}$ but from a selfadjoint extension of an operator acting on
$$\mathsf{C}^\infty_c(\Lambda^p T^*\overline{M} \setminus S, g_0) \oplus \mathsf{C}^\infty_c(]0, L[, \mathcal{H}^{p-1}(\Sigma) \oplus \mathcal{H}^p(\Sigma)) \ ,$$
where $\mathcal{H}^p(\Sigma)$ denotes the space of harmonic p-forms on Σ and S is the singular part of $\overline{M}$, that is two points corresponding to the shrunken manifold Σ at the tip of each cone. This operator acts as the Laplacian on the first component and by $-d^2/dt^2$ on the last component.
  - Suppose that $p = n/2$, then the limit operator is associated to the quadratic form
$$(\varphi, \sigma) \mapsto q(\varphi, \sigma) := \int_{\overline{M}} \big(|d\varphi|^2 + |d^*\varphi|^2\big)\, d\operatorname{vol} + \int_0^L |\sigma'(t)| dt$$
with the domain $\operatorname{dom}(q)$ where $(\varphi, \sigma) \in \operatorname{dom}(q)$ if and only if the following conditions are satisfied:
$$\begin{aligned} &\varphi \in \mathsf{L}^2(\Lambda^{n/2} T^*\overline{M}) \cap \operatorname{dom}(D_{\max}) \\ &\mathcal{L}_\pm(\varphi) = (0, \alpha_\pm) \in \{0\} \oplus \mathcal{H}^{n/2}(\Sigma) \subset \ker A \\ &\sigma = (\beta, \alpha) \in \mathsf{H}^1\big([0, L], \mathcal{H}^{n/2-1}(\Sigma) \oplus \mathcal{H}^{n/2}(\Sigma)\big) \\ &\alpha_- = \alpha(0) \text{ and } \alpha_+ = \alpha(L) \end{aligned} \tag{38}$$
  - Suppose that $p = n/2 + 1$, then the limit operator is associated to the quadratic form
$$(\varphi, \sigma) \mapsto q(\varphi, \sigma) := \int_{\overline{M}} \big(|d\varphi|^2 + |d^*\varphi|^2\big)\, d\operatorname{vol} + \int_0^L |\sigma'(t)| dt$$
with the domain $\operatorname{dom}(q)$ where $(\varphi, \sigma) \in \operatorname{dom}(q)$ if and only if the following conditions are satisfied:
$$\begin{aligned} &\varphi \in \mathsf{L}^2(\Lambda^{n/2+1} T^*\overline{M}) \cap \operatorname{dom}(D_{\max}) \\ &\mathcal{L}_\pm(\varphi) = (\beta_\pm, 0) \in \mathcal{H}^{n/2}(\Sigma) \oplus \{0\} \subset \ker A \\ &\sigma = (\beta, \alpha) \in \mathsf{H}^1\big([0, L], \mathcal{H}^{n/2-1}(\Sigma) \oplus \mathcal{H}^{n/2}(\Sigma)\big) \\ &\beta_- = -\beta(0) \text{ and } \beta_+ = \beta(L). \end{aligned} \tag{39}$$

Proof of Theorem C. We are now able to prove our main convergence result, namely Theorem C. More generally, we show the following:

THEOREM 12. *If we drop the condition $H^{n/2}(\Sigma) = 0$, the convergence results are the same as in Theorem C except for the degrees $p = n/2$ and $p = n/2 + 1$ where the spectrum of the Hodge-de Rham operator of the manifold M_ε acting on these p-forms converges to the spectrum of the limit problem described in* (38)–(39).

Proof. By duality it is sufficient to consider $p < n/2+1$. Let $\{\mu_N\}$, $N \geq 1$, be the sequence of the eigenvalues, counted with multiplicity, of the limit operator as described by the theorem in this degree.

Upper bound. We show first that $\limsup_{\varepsilon\to 0} \lambda_N^p(\varepsilon) \leq \mu_N$ by transplanting the corresponding eigenforms on M_ε. The formula is then just a consequence of the minimax formula. Let us describe how the different type of eigenforms are transplanted.

Eigenforms in $\mathrm{dom}(D_{\min})$ *on* $\overline{M}$. These are the easiest because if $\varphi \in \mathrm{dom}(D_{\min})$ then by definition, we find a sequence $\varphi_l \in \mathsf{C}_c^\infty(\Lambda^\bullet T^*\overline{M} \setminus S)$ such that

$$\lim_{l\to\infty} \|\varphi_l - \varphi\| + \|D\varphi_l - D\varphi\| = 0.$$

These φ_l are transplanted easily on the manifold M_{ε_l}.

Eigenforms of $D_{\max} \cap \mathsf{L}^2(\Lambda^{(n+1)/2}T^*\overline{M})$ *on* $\overline{M}$. Any such form φ can be written as

$$\varphi = \varphi_0 + \varphi_1 = \varphi_0 + \varphi_+ + \varphi_-$$

where $\varphi_0 \in \mathrm{dom}(D_{\min})$ and φ_+, φ_- have support in $\mathcal{C}_0^\pm$ and

$$U_\pm(\varphi_\pm) = \sum_\gamma t^{-\gamma} c_{\pm,\gamma}\sigma_\gamma$$

where $c_{\pm,\gamma} \in \mathbb{C}$ and each $\sigma_\gamma \in \mathcal{H}_5$ satisfies $A\sigma_\gamma = \gamma\sigma_\gamma$ for a $\gamma \in \,]-1/2, 1/2[$ associated to μ_γ an exact p-eigenvalue of Δ_Σ. We only need to explain how we construct the transplantation $\varphi_{1,\varepsilon}$ of φ_1 on M_ε.

On $M_\varepsilon \setminus \mathcal{A}_\varepsilon$ we let $\varphi_{1,\varepsilon} = \varphi_1$ and on the handle $\mathcal{A}_\varepsilon$, we define $\varphi_{1,\varepsilon}$ by

$$U(\varphi_{1,\varepsilon}) = \sum_\gamma \varepsilon^{-\gamma}\Big(c_{+,\gamma}\sigma_\gamma\chi_0(L-t)\mathrm{e}^{-\frac{\mu_\gamma}{\varepsilon}(L-t)} + Jc_{-,\gamma}\sigma_\gamma\chi_0(t)\mathrm{e}^{-\frac{\mu_\gamma}{\varepsilon}(t)}\Big),$$

where χ_0 is a cut-off function χ_0 which satisfies $\chi_0(t) = 1$ for $0 < t < L/4$ and $\chi_0(t) = 0$ for $L/3 < t$.

It is an easy calculation to show that $\int_{\mathcal{A}_\varepsilon}\big(|D\varphi_{1,\varepsilon}|^2 + |\varphi_{1,\varepsilon}|^2)\big) = O(\varepsilon^{1-2\delta})$, for a certain $\delta \in \,]-1/2, 1/2[$.

Eigenforms of the interval with harmonic values in Σ. If we express the forms in terms of σ, as described at the beginning of Section 3, the Dirichlet spectrum of the interval corresponds to a form like $(0, f(t)\alpha)$ with α a p-form harmonic on Σ and f an eigenfunction for the Dirichlet Laplacian on the interval, it can be prolongated by 0. The Neumann spectrum of the interval corresponds to a form like $(f(t)\beta, 0)$ with β a $(p-1)$-form harmonic on Σ and f an eigenfunction for the Neumann Laplacian on the interval (or its dual by the Hodge-$*$ operator in the case $p = (n+1)/2$) it can be prolongated by

$$\sigma_+ = (\varepsilon^{n/2+1-p}\xi(t)f(L)t^{p-1-n/2}\beta, 0)$$

where ξ is a fixed cut-off function, $0 \leq \xi \leq 1$, $\xi = 1$ on $[0, 1/4]$ and $\xi = 0$ on $[3/4, 1]$, and with the same type of expression on the other end. The q-norm of the prolongation given here is of order $O(\sqrt{\varepsilon})$, except in the case $p = (n+1)/2$ where we obtain $O(\sqrt{\varepsilon|\log\varepsilon|})$ (the calculus is the same as in [3, Eq. (2.1)]).

Special case $H^{n/2}(\Sigma) \neq 0$. In this case, the eigenforms of degree $p = n/2$ belonging to the limit problem, can be transplanted as follows. Let $(\varphi, \sigma) \in \operatorname{dom}(q)$; we know that, as before, $\varphi = \varphi_0 + \varphi_1 = \varphi_0 + \varphi_+ + \varphi_-$ where $\varphi_0 \in \operatorname{dom}(D_{\min})$ and φ_+, φ_- have support in $\mathcal{C}_0^\pm$ and $U_\pm(\varphi_\pm) = (0, \alpha_\pm)$ where $\alpha \in \mathcal{H}^{n/2}(\Sigma)$ is constant on $[0, 1/2]$ and $\sigma = (\beta, \alpha) \in H^1([0, L], \mathcal{H}^{n/2-1}(\Sigma) \oplus \mathcal{H}^{n/2}(\Sigma))$ satisfies $\alpha(0) = \alpha_-$ and $\alpha(L) = \alpha_+$.

We extend β as before for the Neumann spectrum of the interval, and because $U_\pm(\varphi_\pm)$ is constant on $[0, 1/2]$ it is easy to transplant (φ_1, σ) on M_ε for $\varepsilon \leq 1/2$.

Conclusion. Now, for any rank N, let $\varphi_1, \dots, \varphi_N$ be an orthonormal basis of the total eigenspace E_N of the limit problem, corresponding to the N first eigenvalues. For any $\varepsilon > 0$ we define a linear operator T_ε from E_N to the domain of the quadratic form on M_ε by demanding that $T_\varepsilon(\varphi_j)$ is the transplanted form as described above. The preceding estimates show that $\langle T_\varepsilon(\varphi), T_\varepsilon(\psi)\rangle = \langle \varphi, \psi\rangle + o(1)$ and also that $q(T_\varepsilon(\varphi), T_\varepsilon(\psi)) = q(\varphi, \psi) + o(1)$. Evaluating the Rayleigh-Ritz quotient on the image $T_\varepsilon(E_N)$ gives then, with the minimax formula,

$$\lambda_N^p(\varepsilon) \leq \mu_N + o(1).$$

Lower bound. To show the other inequality, namely $\liminf_{\varepsilon\to 0} \lambda_N^p(\varepsilon) \geq \mu_N$, we use the estimates provided in Section 4. The eigenvalues inequality is then a consequence of the minimax principle applied to the limit of a subsequence for an orthonormal family of the N first eigenforms of M_ε.

We give the argument first for one eigenvalue. For simplicity, we assume that $H^{n/2}(\Sigma) = 0$. The same proof, with a slight modification of the arguments, also works in the case when $H^{n/2}(\Sigma)$ is non-trivial.

We consider a subsequence $\lambda_m = \lambda_N^p(\varepsilon_m)$ such that

$$\lim_{m\to\infty} \lambda_m = \liminf_{\varepsilon\to 0} \lambda_N^p(\varepsilon) = \lambda$$

and denote the corresponding normalised eigenforms by φ_m, namely, (in the following we write $\varepsilon = \varepsilon_m$)

$$D^2\varphi_m = \lambda_m\varphi_m \qquad \text{and} \qquad \|\varphi_m\|_{\mathsf{L}^2} = 1.$$

For $\varepsilon > 0$ small enough, we will construct a form

$$\psi_m \in \mathsf{L}^2(\Lambda^p T^*\overline{M}) \oplus \mathsf{L}^2([0, L], \mathcal{H}^{p-1}(\Sigma) \oplus \mathcal{H}^p(\Sigma))$$

which is in the domain of the quadratic form of the associated limit operator. Here again, $\mathcal{H}^p(\Sigma)$ denotes the space of harmonic p-forms on (Σ, h). Recall that we have denoted the spectrum of the limit operator by $\{\mu_N\}_N$. Moreover, the correspondence $\varphi_m \mapsto \psi_m$ will be an almost isometry. We begin to define ψ_m (or more precisely $U\psi_m$ on $[0, L]$). From Lemmas 3 and 4 we conclude that on the handle

$$\varphi_m = h_m + k_m$$

where $\lim_{m\to\infty} \|k_m\|_{\mathsf{L}^2} = 0$, $D^2 h_m = \lambda_m h_m$ and h_m is transversally harmonic.

Moreover, by Lemmas 7 and 8, we can decompose $h_m = h_m^{\mathrm{D}} + h_m^{\mathrm{N}}$ where

$$Uh_m^{\mathrm{D}}\Big(\frac{L \pm L}{2}\Big) = O\big(\sqrt{\varepsilon|\log\varepsilon|}\big)$$

and

$$\frac{d}{dt}(Uh_m^{\mathrm{N}})\Big(\frac{L \pm L}{2}\Big) = O(\varepsilon|\log\varepsilon|).$$

Since h_m satisfies the eigenvalue equation, we conclude that $u_m^{\mathrm{D}} = Uh_m^{\mathrm{D}}$ and $u_m^{\mathrm{N}} = Uh_m^{\mathrm{N}}$ both satisfy the equation

$$-u'' = \lambda_m u,$$

hence there is a constant (independent of m) such that

$$|u_m^{\mathrm{D}}(t)| + \left|u_m^{\mathrm{D}\prime}(t)\right| \le C\|u_m^{\mathrm{D}}\|_{\mathsf{L}^2} \quad \text{and} \quad |u_m^{\mathrm{N}}(t)| + \left|u_m^{\mathrm{N}\prime}(t)\right| \le C\|u_m^{\mathrm{N}}\|_{\mathsf{L}^2}$$

for all $t \in [0, L]$. We will modify u_m^{D} in order to satisfy the Dirichlet boundary condition: for $\eta = \sqrt{\varepsilon|\log\varepsilon|}$ we define

$$\chi_m(t) = \begin{cases} t/\eta & \text{if } t \in [0, \eta], \\ 1 & \text{if } t \in [\eta, L-\eta], \\ (L-t)/\eta & \text{if } t \in [L-\eta, L] \end{cases}$$

and we define ψ_m via

$$U\psi_m = \chi_m u_m^{\mathrm{D}} + u_m^{\mathrm{N}}.$$

We have

$$\|\varphi_m - \psi_m\|_{\mathsf{L}^2}^2 = \|k_m\|_{\mathsf{L}^2}^2 + \|(1-\chi_m)u_m^{\mathrm{D}}\|_{\mathsf{L}^2}^2$$

hence

$$\lim_{m\to\infty} \|\varphi_m - \psi_m\|_{\mathsf{L}^2} = 0.$$

Moreover

$$\int_0^L |(U\psi_m)'(t)|^2 dt = \int_0^L |(\chi_m u_m^{\mathrm{D}})'(t)|^2 dt + \int_0^L |u_m^{\mathrm{N}\prime}(t)|^2 dt.$$

But

$$\begin{aligned}\int_0^L |(\chi_m u_m^{\mathrm{D}})'(t)|^2 dt &= \int_0^L |\chi_m'(t)|^2 |u_m^{\mathrm{D}}(t)|^2 dt + \int_0^L \Big\langle \frac{d}{dt}(\chi_m^2 u_m^{\mathrm{D}}), \frac{d}{dt}u_m^{\mathrm{D}}\Big\rangle dt \\ &= \int_0^L |\chi_m'(t)|^2 |u_m^{\mathrm{D}}(t)|^2 dt + \lambda_m \int_0^L |\chi_m u_m^{\mathrm{D}}(t)|^2 dt\end{aligned}$$

and

$$\int_0^L |\chi_m'(t)|^2 |u_m^{\mathrm{D}}(t)|^2 dt = O(\sqrt{\varepsilon|\log\varepsilon|}).$$

Similarly, we have

$$\begin{aligned}\int_0^L |u_m^{\mathrm{N}\prime}(t)|^2 dt &= \lambda_m \int_0^L |u_m^{\mathrm{N}}(t)|^2 dt + \big[u_m^{\mathrm{N}} u_m^{\mathrm{N}\prime}\big]_0^L \\ &= \lambda_m \int_0^L |u_m^{\mathrm{N}}(t)|^2 dt + O(\varepsilon|\log\varepsilon|).\end{aligned}$$

Now on $\overline{M} \setminus (\mathcal{C}_0^+ \cup \mathcal{C}_0^-)$ we set

$$\psi_m = \varphi_m$$

and on $\mathcal{C}_\varepsilon^\pm$, we know that

$$\varphi_m = U^*(u_{\pm,m} + v_{\pm,m}) + \varphi_{\pm,m}^\Lambda$$

where $u_{\pm,m}$ is described in Lemma 5 and the corresponding assertion on the harmonics parts in Corollary 9. In particular, $u_{\pm,m}$ has a well defined extension $\overline{u}_{\pm,m}$ which is in the domain of the limit operator. Moreover, we know that

$$\lim_{m\to\infty} \|v_{\pm,m}\|_{\mathsf{L}^2(\mathcal{C}_\varepsilon^\pm)} = 0$$

and for a certain constant C we have

$$\|\varphi_{\pm,m}^\Lambda\|^2_{\mathsf{L}^2(\mathcal{C}_\varepsilon^\pm \setminus \mathcal{C}_\eta^\pm)} \le C\eta^2$$

for each $\eta > \varepsilon$. Moreover

$$UD^2U^* u_{\pm,m} = \lambda_m u_{\pm,m}, \quad UD^2U^* v_{\pm,m} = \lambda_m v_{\pm,m} \quad \text{and} \quad D^2\varphi_{\pm,m}^\Lambda = \lambda_m \varphi_{\pm,m}^\Lambda.$$

We consider two cut off functions

$$\xi_0(t) = \begin{cases} 1 & \text{if } t \ge 1/2, \\ 4t-1 & \text{if } t \in [1/4, 1/2], \\ 0 & \text{if } t \le 1/4 \end{cases}$$

and, with $\varepsilon = \varepsilon_m$,

$$\xi_m(t) = \begin{cases} 1 & \text{if } t \ge 2\sqrt{\varepsilon}, \\ \dfrac{\log(2\varepsilon) - \log(t)}{\log(\sqrt{\varepsilon})} & \text{if } t \in [2\varepsilon, 2\sqrt{\varepsilon}], \\ 0 & \text{if } t \le 2\varepsilon. \end{cases}$$

On $\mathcal{C}_0^\pm$, we define

$$\psi_m = U^*(\overline{u}_{\pm,m} + \xi_0 v_{\pm,m}) + \xi_m \varphi_{\pm,m}^\Lambda.$$

There exists a $\delta > 0$ such that

$$\|\psi_m\|_{\mathsf{L}^2(\mathcal{C}_0^\pm \setminus \mathcal{C}_\varepsilon^\pm)} = O(\varepsilon^\delta).$$

Moreover

$$\|\psi_m - \varphi_m\|_{\mathsf{L}^2(\overline{M})} \le O(\varepsilon^\delta) + \sum_{s=\pm} \Big[\|v_{s,m}\|_{\mathsf{L}^2(\mathcal{C}_\varepsilon^s)} + \|\varphi_{s,m}^\Lambda\|_{\mathsf{L}^2(\mathcal{C}_\varepsilon^s \setminus \mathcal{C}_{2\sqrt{\varepsilon}}^s)} \Big].$$

Hence

$$\lim_{m\to\infty} \|\psi_m - \varphi_m\|_{\mathsf{L}^2(\overline{M})} = 0$$

and the correspondence $\varphi_m \mapsto \psi_m$ is almost isometric.

We now deal with the quadratic form expression. Namely, we want to show that

$$\|(d+d^*)\psi_m\|^2_{\mathsf{L}^2(\overline{M})} \le \lambda_m \|\psi_m\|^2_{\mathsf{L}^2(\overline{M})} + o(1). \tag{40}$$

After an integration by part,we get

$$\|(d+d^*)\psi_m\|^2_{\mathsf{L}^2(\overline{M}\setminus(\mathcal{C}_0^+ \cup \mathcal{C}_0^-))} = \lambda_m \|\psi_m\|^2_{\mathsf{L}^2(\overline{M}\setminus(\mathcal{C}_0^0 \cup \mathcal{C}_0^-))} + BT \tag{41}$$

where BT is a certain boundary integral over the regular part of $\partial\mathcal{C}_0^+ \cup \partial\mathcal{C}_0^-$. Indeed the behaviour of $\overline{u}_{s,m}$ implies that

$$\|(d+d^*)\overline{u}_{\pm,m}\|^2_{\mathsf{L}^2(\mathcal{C}_0^\pm)} = \lambda_m \|\overline{u}_{\pm,m}\|^2_{\mathsf{L}^2(\mathcal{C}_0^\pm)} + BT_{\pm,u} \tag{42}$$

where $BT_{\pm,u}$ is a certain boundary integral over $\partial\mathcal{C}_0^\pm$. Similarly, we have

$$\begin{aligned}\|(d+d^*)(\xi_0 v_{\pm,m})\|^2_{\mathsf{L}^2(\mathcal{C}_0^\pm)} &= \int_{\mathcal{C}_0^\pm} |d\xi_0|^2 |v_{\pm,m}|^2 d\,\mathrm{vol} + \big\langle (d+d^*)(\xi_0^2 v_{\pm,m}), (d+d^*) v_{\pm,m}\big\rangle \\ &= \int_{\mathcal{C}_0^\pm} |d\xi_0|^2 |v_{\pm,m}|^2 d\,\mathrm{vol} + \lambda_m \int_{\mathcal{C}_0^\pm} |\xi_0 v_{\pm,m}|^2 d\,\mathrm{vol} + BT_{\pm,v} \\ &= \lambda_m \int_{\mathcal{C}_0^\pm} |\xi_0 v_{\pm,m}|^2 d\,\mathrm{vol} + o(1) + BT_{\pm,v} \end{aligned} \tag{43}$$

where again $BT_{\pm,v}$ is a certain boundary integral over $\partial\mathcal{C}_0^\pm$. Similarly, we get

$$\begin{aligned}&\|(d+d^*)(\xi_m \varphi^\Lambda_{\pm,m})\|^2_{\mathsf{L}^2(\mathcal{C}_0^\pm)} \\ &\quad= \int_{\mathcal{C}_0^\pm} |d\xi_m|^2 |\varphi^\Lambda_{\pm,m}|^2 d\,\mathrm{vol} + \big\langle (d+d^*)(\xi_m^2 \varphi^\Lambda_{\pm,m}), (d+d^*)\varphi^\Lambda_{\pm,m}\big\rangle \\ &\quad= \int_{\mathcal{C}_0^\pm} |d\xi_m|^2 |\varphi^\Lambda_{\pm,m}|^2 d\,\mathrm{vol} + \lambda_m \int_{\mathcal{C}_0^\pm} |\xi_m \varphi^\Lambda_{\pm,m}|^2 d\,\mathrm{vol} + BT_{\pm,\Lambda} \end{aligned} \tag{44}$$

where $BT_{\pm,\Lambda}$ is a certain boundary integral over $\partial\mathcal{C}_0^\pm$. Furthermore, we set $M(r) = \|\varphi^\Lambda_{\pm,m}\|^2_{\mathsf{L}^2(\mathcal{C}_\varepsilon^\pm \setminus \mathcal{C}_r^\pm)}$. By Lemma 3, $M(r)$ is of order $O(\frac{r^2}{\Lambda^2})$, and we have

$$\begin{aligned}\int_{\mathcal{C}_0^\pm} |d\xi_m|^2 |\varphi^\Lambda_{\pm,m}|^2 d\,\mathrm{vol} &= \frac{4}{|\log\varepsilon|^2} \int_{2\varepsilon}^{2\sqrt{\varepsilon}} \frac{1}{r^2}\, dM(r) \\ &= \frac{4}{|\log\varepsilon|^2} \left[\frac{M(2\sqrt{\varepsilon})}{4\varepsilon} - \frac{M(2\varepsilon)}{4\varepsilon^2} + 2\int_{2\varepsilon}^{2\sqrt{\varepsilon}} \frac{M(r)}{r^3} dr \right] \\ &= O\Big(\frac{1}{|\log\varepsilon|}\Big). \end{aligned}$$

We also have

$$BT + BT_{+,u} + BT_{-,u} + BT_{+,v} + BT_{-,v} + BT_{+,\Lambda} + BT_{-,\Lambda} = 0$$

and the square of the L^2-norm of $(d+d^*)\psi_m$ on $\overline{M}$ is the sum of (41)–(44). Hence we obtain (40).

The argument for the first N eigenvalues is as follows: Let $\varphi_m^k := \varphi_m^k(\varepsilon_m)$, $k = 1, \ldots, N$, be an orthonormal family of eigenforms for the eigenvalues $\lambda_k(\varepsilon_m)$ (we drop here the index p) such that $\lambda_1(\varepsilon_m) \le \cdots \le \lambda_N(\varepsilon_m)$ and $\lim_{m\to\infty} \lambda_N(\varepsilon_m) = \liminf_{\varepsilon\to 0} \lambda_N(\varepsilon)$ for $\lim_{m\to\infty} \varepsilon_m = 0$. We have just seen that to each φ_m^k we have associated a ψ_m^k in the domain of the limit quadratic form. Then the fact that the map $\varphi_m \to \psi_m$ is almost an isometry, shows that

$$|\langle \psi_m^k(\varepsilon_m), \psi_m^l(\varepsilon_m)\rangle - \delta(k,l)| = o(1)$$

as m tends to infinity for all k,l, where $\delta(k,l)$ denotes the Kronecker symbol.

Now if we calculate the Rayleigh-Ritz quotient for an element ψ of the vector space with base $\{\psi_m^k,\ k=1,\dots,n\}$, it follows from the two preceding estimates and (40) applied for each ψ_m^k that

$$\|(d+d^*)\psi\|^2 \le (\lambda_N(\varepsilon_m)+o(1))\|\psi\|^2.$$

The conclusion follows then from the minimax formula, namely $\mu_N \le \lambda_N(\varepsilon_m)+o(1)$ for all $m\in\mathbb{N}$ and at the limit: $\mu_N \le \liminf_{\varepsilon\to 0}\lambda_N(\varepsilon)$. □

6. Covering manifolds

In this section we explain how the convergence argument of the preceding section can be used also for a covering manifold in order to show the existence of spectral gaps. Let us first describe the covering manifold and the *Floquet decomposition* of a periodic operator on the covering.

Let X be an $(n+1)$-dimensional non-trivial covering manifold, with quotient M and covering group $\mathbb{Z}$. This covering defines a non-trivial element $c\in H^1(M,\mathbb{Z})$. To each element of $H^1(M,\mathbb{Z})$ corresponds a homotopy class of functions $f_c : M \to \mathbb{S}^1$ and if $c\neq 0$ then f_c is surjective. It can be chosen smooth, so we know, by the Sard's theorem, that f_c has a regular value y. Therefore, $\Sigma = f_c^{-1}(y)$ is a hypersurface of M such that $F := M\setminus\Sigma$ is a fundamental domain for X. Let $\{g_\varepsilon\}_\varepsilon$ be the family of metrics on M constructed in Section 2. We denote the lift of g_ε onto X also by g_ε.

Let $\chi\in\hat{\mathbb{Z}}$ be a character of the group $\mathbb{Z}$, i.e., a group homomorphism $\chi\colon \mathbb{Z}\longrightarrow\mathbb{S}^1$. Clearly, such a homomorphism is given by $\chi(\gamma)=\mathrm{e}^{\mathrm{i}\gamma\theta}$ for some $\theta\in[0,2\pi]$. We will identify χ and θ in the sequel.

We can associate a complex line bundle $E_\theta^0 \to M$ to the $\mathbb{Z}$-covering $X\to M$ since the covering $X\to M$ is a principal bundle with discrete fibre $\mathbb{Z}$. Similarly, we denote by $E_\theta^p \to M$ the bundle associated to the $\mathbb{Z}$-covering $\Lambda^pT^*X \to \Lambda^pT^*M$. A smooth section ω in E_θ^p can be considered as a smooth section in Λ^pT^*X satisfying the so-called *equivariance condition*

$$\omega(\gamma+x)=\mathrm{e}^{\mathrm{i}\gamma\theta}\omega(x) \tag{45}$$

for $x\in X$ and $\gamma\in\mathbb{Z}$ where we write the action of $\mathbb{Z}$ on X additively. Clearly, such a section is determined on a fundamental domain $F\subset X$. The L^2-space of θ-equivariant sections with respect to the metric g will be denoted by $\mathsf{L}^2(E_\theta^p,g_\varepsilon)$. Since L^2-functions do not "feel" the condition (45) on a fundamental domain, $\mathsf{L}^2(E_\theta^p,g_\varepsilon)$ is unitarily equivalent to $\mathsf{L}^2(\Lambda^pT^*F,g_\varepsilon)$, independently of θ.

Using Floquet theory (see e.g. [21, XIII.16]), the L^2-space of forms on (X,g_ε) can be transformed into

$$\mathsf{L}^2(\Lambda^pT^*X,g_\varepsilon)\cong\int_{\hat{\mathbb{Z}}}\mathsf{L}^2(E_\theta^p,g_\varepsilon)\,d\theta. \tag{46}$$

The Gauß-Bonnet operator D acting on (X,g_ε) can be decomposed under this direct integral representation as

$$D\cong\int_{\hat{\mathbb{Z}}}D_\theta\,d\theta \tag{47}$$

where the domain of D_θ consists of those forms ω having a θ-equivariant continuation in $\mathsf{H}^1_{\mathrm{loc}}(X)$. For our purposes, it will be convenient to use the fundamental domain corresponding to $F = M_\varepsilon \setminus \{2\} \times \Sigma$, i.e., we cut along the right end of the collar neighbourhood $\mathcal{U} = \left]-2,2\right[\times \Sigma$. The domain of D is then given by forms ω, such that their components are piecewise in H^1 and satisfy the boundary conditions

$$\omega_- = \mathrm{e}^{\mathrm{i}\theta}\omega_+ \tag{48}$$

where ω_- denotes the limit of ω on $\{2\} \times \Sigma \subset \overline{\mathcal{U}}$ and ω_+ the limit from the opposite side $M \setminus \mathcal{U}$.

The Hodge-de Rham operator $\Delta^p_\varepsilon = D^2$ acting on p-forms on (X, g_ε) decomposes similarly, where the domain of $\Delta^p_{\varepsilon,\theta} = D^2_\theta$ consists of those forms ω such that their components are piecewise in H^2 and satisfy additionally to (48) the first order boundary conditions

$$\omega'_- = -\mathrm{e}^{\mathrm{i}\theta}\omega'_+, \tag{49}$$

where ω'_- denotes the outward normal derivative of ω on $\{2\} \times \Sigma \subset \overline{\mathcal{U}}$ and similarly, ω'_+ the outward normal derivative from the opposite side.

The spectrum of the Hodge-de Rham operator $\Delta^p_{\varepsilon,\theta}$ is purely discrete and will be denoted by $\lambda^p_{k,\theta}(\varepsilon)$, ordered in increasing order and repeated according to the multiplicity. From the direct integral representation (and the continuous dependence on θ) it follows that the spectrum of the Hodge-de Rham operator Δ^p_ε on X is given as

$$\operatorname{spec} \Delta^p_\varepsilon = \bigcup_{k \in \mathbb{N}} B^p_k(\varepsilon) \qquad \text{where} \qquad B^p_k(\varepsilon) = \{\, \lambda^p_{k,\theta}(\varepsilon)\,;\, \theta \in [0, 2\pi] \,\} \tag{50}$$

are compact intervals, called *bands*.

Our convergence result Theorem C holds also for the θ-equivariant eigenvalues $\lambda^p_{k,\theta}(\varepsilon)$. Although we have shown this convergence only for $\theta = 0$, all arguments remain the same noting that the arguments are local in $\mathcal{V}$ or rely on elliptic regularity elsewhere. Let $\Lambda > 0$, then by continuity of the map $\theta \mapsto \lambda^p_{k,\theta}(\varepsilon)$, we know that there is some $\theta^\pm_\varepsilon$ such that

$$B^p_k(\varepsilon) \cap [0, \Lambda] = [\lambda^p_{k,\theta^-_\varepsilon}(\varepsilon), \lambda^p_{k,\theta^+_\varepsilon}(\varepsilon)]$$

provided $\lambda^p_k(0) < \Lambda$ and $\varepsilon > 0$ small enough. Applying the preceding convergence result to $\lambda^p_{k,\theta^-_\varepsilon}(\varepsilon)$ and $\lambda^p_{k,\theta^-_\varepsilon}(\varepsilon)$, we obtain that

$$\lim_{\varepsilon \to 0} \lambda^p_{k,\theta^\pm_\varepsilon}(\varepsilon) = \lambda^p_k(0)$$

where $\lambda^p_k(0)$ denotes the spectrum of the limit operator on p-forms.

Hence the limit does no longer depend on the Floquet parameter θ. This means, that the bands $B^p_k(\varepsilon)$ shrink to a point $\{\lambda^p_k(0)\}$, where $\lambda^p_k(0)$ denotes the spectrum of the limit operator.

We therefore have shown our main result (remind that $n+1$ is the dimension of X):

THEOREM 13. *Assume that n is odd or $H^{n/2}(\Sigma)$ is trivial. Given $N \in \mathbb{N}$ there is a metric $g = g_N$ such that the Hodge-de Rham operator on the $\mathbb{Z}$-covering (X, g_N) has at least N gaps in its (essential) spectrum.*

If n is even and $H^{n/2}(\Sigma) \neq 0$ then the result remains true for the Hodge-de Rham operator acting on p-forms providing that $p \neq n/2$ and $p \neq n/2 + 1$.

Proof of Theorem D. Let us now have a look at the Dirac operator on a spin manifold M. It is a consequence of [22] that the spectrum of the Dirac operator on the periodic manifold is the whole real line if $\alpha_n(\Sigma) \neq 0$. For the other implication, the same calculations as before, but with simpler expressions. Let us sketch the ideas here. If $\alpha_n(\Sigma) = 0$ then, by the result of [1], there exists a metric h on Σ such that the corresponding Dirac operator has no harmonic spinor. We endow M with a metric such that its restriction to Σ coincides with h. Let $\Lambda > 0$ be such that the spectrum of the Dirac operator D_0 on Σ does not intersect the interval $[-\Lambda, \Lambda]$. By a scale of the metric h we can always suppose that Λ is large enough such that the Dirac operator D is essentially self adjoint on the limit manifold $\overline{M}$ (see Section 5).

The precise behaviour of the Dirac operator on cones can be found in [11]. If $(\mathcal{M}_\varepsilon, g_\varepsilon)$ is isometric to $I_\varepsilon \times \Sigma$ endowed with the warped product metric $d\tau^2 + f_\varepsilon(\tau)^2 h$ where $I_\varepsilon =]-(L/2+1-\varepsilon), L/2+1-\varepsilon[$, then the Dirac operator on $\mathcal{M}_\varepsilon$ is unitarily equivalent to

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \left(\partial_\tau + \frac{1}{f_\varepsilon(\tau)} \begin{pmatrix} 0 & -D_0 \\ -D_0 & 0 \end{pmatrix} \right)$$

on $\mathcal{M}_\varepsilon$ using the isometry $U \colon \mathsf{L}^2(\mathcal{M}_\varepsilon, g_\varepsilon) \longrightarrow \mathsf{L}^2(I_\varepsilon, \mathsf{L}^2(\Sigma, h))$ as in Section 3. Here, f_ε can be chosen either continuous and piecewise smooth as before, or smooth on the whole interval by the argument described in Section 2. Anyway, we can redo the previous calculus with $A = A_0$, and there is no more boundary term in the expression of the quadratic form (6) or (19).

For $\varepsilon_m \to 0$, let φ_m be a family of eigenspinors on M_{ε_m} corresponding to the eigenvalues $\lambda_{\varepsilon_m} \to \lambda$. Due to our choice of h and Λ, the decomposition (18) of the eigenspinor φ_m on $\mathcal{M}_{\varepsilon_m}$ is reduced to the last term, and Lemma 3 applies directly to φ_m: There exists a constant $C > 0$ such that

$$\|\varphi_m\|^2_{\mathsf{L}^2(\mathcal{A}_\varepsilon)} \le C\frac{\varepsilon^2}{\Lambda^2} \qquad \text{and} \qquad \|\varphi_m\|^2_{\mathsf{L}^2(\mathcal{C}^\pm_\varepsilon \setminus \mathcal{C}^\pm_\eta)} \le C\frac{\eta^2}{\Lambda^2}$$

as soon as $\varepsilon_m \le \eta$. Thus, the L_2-norm of the eigenspinors on the handle converges to 0. Moreover, the limit spectrum will consist only on the spectrum of the Dirac operator with *minimal* domain $D_{\min}$ on $\overline{M}$. The proof of Theorem C can now be followed verbatim: for the 'upper bound', the proof is reduced to the easiest part, namely eigenspinors in $\operatorname{dom}(D_{\min})$, and for the 'lower bound' we use the cut-off function $\xi_m(t)$ on the cones defined there.

The limit spectrum is the same for the operator involving the Floquet parameter. Finally, the result of Theorem D follows.

7. Harmonic forms and small eigenvalues

Returning to the situation of Section 5, we can ask for the multiplicity of the zero eigenvalue, which is given by the cohomology. The calculation made there shows that "small eigenvalues" can occur, *i.e.* $\lambda_\varepsilon \neq 0$ such that $\lim_{\varepsilon\to 0}\lambda_\varepsilon = 0$.

We suppose here that if n is even then $H^{n/2}(\Sigma) = 0$ and if n is odd that Δ_Σ has no eigenvalue in $]0,1[$ then the only limit operator involved is $D_{\max}\circ D_{\min}$, and we know by the works of Cheeger that the kernel of $D_{\max}\circ D_{\min}$ coincides with the intersection cohomology of the manifold with conical singularities. Let N be the number of small, or null eigenvalues. By the precedent result we know that

$$\begin{aligned}
N &= \dim IH^p(\overline{M}) + \dim H^{p-1}(\Sigma) && \text{for } p < (n+1)/2,\\
N &= \dim IH^p(\overline{M}) + \dim H^{p}(\Sigma) && \text{for } p > (n+1)/2,\\
N &= \dim IH^p(\overline{M}) + \dim H^{p-1}(\Sigma) + \dim H^{p}(\Sigma) && \text{for } p = (n+1)/2.
\end{aligned}$$

The manifold M_ε is covered by the two open sets $U_0 = M \setminus (]-1,1[\times\Sigma)$ and the collar $U =]-2,2[\times\Sigma$. The Mayer-Vietoris argument gives then a long exact sequence

$$\cdots \to H^q_{\mathrm{c}}(U) \xrightarrow{j} H^q(M_\varepsilon) \xrightarrow{r} H^q(U_0) \to H^{q+1}_{\mathrm{c}}(U) \to \dots$$

But since U is a cylinder, $H^q_{\mathrm{c}}(U) = H^{q-1}(\Sigma)$ for all q. On the other hand $H^q(U_0) = IH^q(\overline{M})$ for $q \leq n/2$, the long exact sequence gives then that

$$\dim H^q(M_\varepsilon) \leq \dim IH^q(\overline{M}) + \dim H^{q-1}(\Sigma)$$

and the equality is obtained if and only if r in surjective and j is injective.

So, for $p < (n+1)/2$ there are small eigenvalues as soon as j is not injective or r is not surjective. In particular for $p = 0$ the three spaces $IH^0(\overline{M})$, $H^0(U_0)$ and $H^0(M_\varepsilon)$ are isomorphic to $\mathbb{R}$ and there is no small eigenvalue.

For $p > (n+1)/2$ we use that $IH^q(\overline{M}) = H^q_{\mathrm{c}}(U_0)$ for $q \geq 1 + n/2$ so we look at the long exact sequence

$$\cdots \to H^q_{\mathrm{c}}(U_0) \xrightarrow{j} H^q(M_\varepsilon) \xrightarrow{r} H^q(U) \to H^{q+1}_{\mathrm{c}}(U_0) \to \dots$$

and use the identity $H^q(U) = H^q(\Sigma)$.

For $p = (n+1)/2$ we have to look at the more complicate diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{\frac{n-1}{2}}(\Sigma) \arrow[r, "\delta"] \arrow[d, "\iota"] \arrow[dr, phantom, "\circlearrowleft"] & H_{\mathrm{c}}^{\frac{n+1}{2}}(U_0) \arrow[r, "j"] \arrow[d, equal] & H^{\frac{n+1}{2}}(M_\varepsilon) \arrow[r, "r"] & H^{\frac{n+1}{2}}(\Sigma) \arrow[r] & H_{\mathrm{c}}^{\frac{n+3}{2}}(U_0) \arrow[r] \arrow[d, equal] & \dots \\
\cdots \arrow[r] & H^{\frac{n-1}{2}}(\Sigma_- \cup \Sigma_+) \arrow[r, "\overline{\delta}"] & H_{\mathrm{c}}^{\frac{n+1}{2}}(U_0) \arrow[r, two heads] & IH^{\frac{n+1}{2}}(\overline{M}) \arrow[r] & 0 \arrow[r] & H_{\mathrm{c}}^{\frac{n+3}{2}}(U_0) \arrow[r] & \dots
\end{tikzcd}$$

Here $\iota(\omega) = (\omega,\omega) \in H^{\frac{n-1}{2}}(\Sigma_-\cup\Sigma_+) = \big(H^{\frac{n-1}{2}}(\Sigma)\big)^2$. The long exact sequence gives then

$$\begin{aligned}
\dim H^{\frac{n+1}{2}}(M_\varepsilon) &\leq \dim H^{\frac{n+1}{2}}(\Sigma) + \dim H_{\mathrm{c}}^{\frac{n+1}{2}}(U_0) - \dim \mathrm{Rg}(\delta)\\
&\leq \dim H^{\frac{n+1}{2}}(\Sigma) + \dim IH^{\frac{n+1}{2}}(\overline{M}) + \dim \mathrm{Rg}(\overline{\delta}) - \dim \mathrm{Rg}(\delta).
\end{aligned}$$

But $\dim \mathrm{Rg}(\overline{\delta}) - \dim \mathrm{Rg}(\delta) \leq \dim H^{\frac{n-1}{2}}(\Sigma)$ and the equality

$$\dim H^{\frac{n+1}{2}}(M_\varepsilon) = \dim H^{\frac{n+1}{2}}(\Sigma) + \dim H^{\frac{n-1}{2}}(\Sigma) + \dim IH^{\frac{n+1}{2}}(\overline{M})$$

holds if and only if r is surjective and $\dim \mathrm{Rg}(\overline{\delta}) = \dim Rg(\delta) + \dim H^{\frac{n-1}{2}}(\Sigma)$, this last relation means that $\ker \overline{\delta} \subset \iota(H^{\frac{n-1}{2}}(\Sigma))$.

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