# ANDERSON LOCALIZATION FOR RADIAL TREE-LIKE RANDOM QUANTUM GRAPHS

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Abstract. We prove that certain random models associated with radial, tree-like, rooted quantum graphs exhibit Anderson localization at all energies. The two main examples are the random length model (RLM) and the random Kirchhoff model (RKM). In the RLM, the lengths of each generation of edges form a family of independent, identically distributed random variables (iid). For the RKM, the iid random variables are associated with each generation of vertices and moderate the current flow through the vertex. We consider extensions to various families of decorated graphs and prove stability of localization with respect to decoration. In particular, we prove Anderson localization for the random necklace model.

## 1. Introduction

Quantum mechanics on metric graphs is a subject with a long history which can be traced back to the paper of Ruedenberg and Scherr [RSc53] on spectra of aromatic carbohydrate molecules elaborating an idea of L. Pauling. A new impetus came in the eighties from the need to describe semiconductor graph-type structures, cf. [EŠ89], and the interest to these problems driven both by mathematical curiosity and practical applications e.g. nano-technology, network theory, optics, chemistry and medicine is steadily growing.

Mathematically, many of these problems can be described by suitably definded Laplace operators on graphs. For example, relevant information of the corresponding model like transport properties of the medium may be infered by the spectrum of the Laplacian. There are basically two classes of operators on graphs: On a *combinatorial* or *discrete* graph, the Laplacian or Schrödinger operator is defined as a *difference* operator on function on the *vertices*. The edges here only play the role of an incidence relation. In contrast, on a *metric* graph, the basic operator acts on each edge as a one-dimensional Schrödinger-type operator with certain boundary conditions at each vertex assuring that the global operator is self-adjoint. The metric graph together with a self-adjoint differential operator is usually called a *quantum graph*. It is almost impossible to give a complete account to all relevant literature here. Instead, we refer to the introductive surveys [Ku08, Ku05, Ku04] as well as to the proceedings [EFKK08, BCFK06] and the references therein.

Since quantum graphs are supposed to model various real graph-like structures with the transverse size which is small but non-zero, one has to ask naturally how close are such system to an "ideal" graph in the limit of zero thickness. For anwers to this question we refer to the papers [KuZ01, RuS01, P06, EP07] and the references therein.

In this paper, we study families of infinite quantum graphs with some inherent randomness and prove that the spectra of the associated Schrödinger-type operators are almost surely pure point. In this manner, the radial random quantum graphs act

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as one-dimensional random Schrödinger operators exhibiting Anderson localization at all energies.

We consider quantum graphs consisting of a rooted infinite metric tree that are *radial.* A radial quantum graph is one for which all variables, such as the branching number, edge length, and vertex boundary conditions, depend only on the generation. The generation of a vertex is determined by the distance from the root vertex. A common example of a rooted infinite metric tree is the rooted Bethe lattice.

We study two main models of random quantum graphs for which the randomness is introduced in two ways. The *Random Length Model* (RLM) is a quantum graph for which the edge length $\ell_e$ is given, for example, by $\ell_e(\omega_e) = \ell_0 e^{\omega_e}$, where $\{\omega_e\}$ is a family of independent, identically distributed (*iid*) random variables. In a *radial RLM*, the family of *iid* random variables $\{\omega_e\}$ depends only on the generation, not on the individual edge. The *Random Kirchhoff Model* (RKM) is a quantum graph and a family $\{q(v)\}$ *iid* random variables associated with each vertex and entering into the Kirchhoff boundary conditions at each vertex. Roughly speaking, if $E_v$ is the set of edges entering the vertex $v$, the Kirchhoff boundary condition is

$$\sum_{e \in E_v} f_e'(v) = q(v) f(v). \tag{1.1}$$

where the precise formulation is given in (2.6)–(2.7). Physically, the current flow through the vertex is determined by the random coupling $q(v) = \omega_v$. A *radial RKM* is one for which the *iid* random variables $\{\omega_v\}$ depend only on the generation of the vertex. Under some conditions, we prove that the almost sure spectrum of both of these models is pure point with exponentially decaying eigenfunctions.

Random quantum graphs have been studied more extensively only in the last years. There are works concerning the existence and continuity properties of the integrated density of states (IDS) of various random graph models, see e.g. [KS04, GLV07a, GLV07b, HV07]. Localization has been proved e.g. in [HKK05, EHS07, KP08] where the considered models resemble the RKM or random potential model on the edges but where different methods are used.

There is one major article that we are aware of on random length models. An important contribution and the basis for our work on the *nonradial* RLM is given by Aizenman, Sims, and Warzel [ASW05]. These authors consider the nonradial RLM in the weak disorder limit. As for the radial RLM, the edge lengths $\ell_e$ are given by $\ell_e(\omega_e) = \ell_0 e^{\tau\omega_e}$, where $\{\omega_e\}$ is a family of independent, identically distributed random variables and $\tau$ is a measure of the disorder. They prove that as the disorder parameter $\tau \to 0$, there is some absolutely continuous spectrum near the absolutely continuous spectrum of the unperturbed model with probability one. As we prove that the radial RLM always exhibits only localization for any nonzero disorder, this shows that the assumption that the graph is radial is a strong one. One might expect that in the nonradial case and for moderate disorder there are localized states near the band edges of the unperturbed quantum graph, but the proof of this requires different methods. Proving localization for the radial case is a first step.

As other applications of the methods developed here, we examine the random necklace model of Kostrykin and Schrader [KS04] (see Section 5.3), and various families of decorated graphs. The random necklace model consists of loops with perimeters given by iid random variables and joined by straight line segments of length one. Kostrykin and Schrader studied the integrated density of states and proved the positivity of the Lyapunov exponent for these models. We complete this study by proving Anderson localization for the random necklace model in Theorem 5.13. Graph decorations have been studied as a mechanism for introducing spectral gaps in the combinatorial [AS00] and quantum [Ku05] case. We consider decorated graphs obtained from the RLM or the RKM by adjoining compact graphs at each generation. We prove that such decorations preserve localization, although there is a discrete set of exceptional energies determined by the Dirichlet Laplacian on the compact decoration graphs.

The contents of this paper are as follows. In Section 2, we describe the basic family of radial metric trees and the corresponding operators. We refer to a tree plus the corresponding differential operator as a quantum graph. Using a symmetry reduction emphasized by Solomyak [Sol04], we reduce the problem on rooted radial trees to an effective half-line problem with certain singularities at the vertices. We present a generalized version of this symmetry reduction in Appendix A for completeness (cf. [SoS02] for the standard case). Transfer matrix methods can now be used to describe solutions to the generalized eigenvalue problem on the effective half-line. We conclude by computing the spectrum of the periodic problems and the deterministic spectrum of the random models. Section 3 is devoted to the proof of localization for the RLM and the RKM (cf. Theorem 3.19). The proof relies on the positivity of the Lyapunov exponent [Ish73, Kot86] and an extension of Kotani's spectral averaging method [Kot86]. The spectral averaging technique employed here is new as one must deal with *complex* matrices in $\mathrm{SL}_2(\mathbb{C})$ instead of real ones in $\mathrm{SL}_2(\mathbb{R})$. We consider general decorated graphs in Section 4. We define the permissible decoration graphs and construct radial tree-like quantum graphs corresponding to the RKM and RLM. By the symmetry reduction procedure, we obtain line-like quantum graphs in analogy to the reduction of the RKM and RLM, and construct their transfer matrices. In Section 5, we extend the arguments of Section 3 to these families of decorated graphs and prove localization (cf. Theorem 5.8, Thms. 5.10–5.12). We show how to prove localization for the random necklace model by extending the methods used here to the line, following the general arguments in Kotani and Simon [KS87].

There are many works on quantum graphs, cf. volume **14** of *Waves in Random Media* and two review papers of Kuchment [Ku04, Ku05]. Much of these works emphasize compact quantum graphs or compact quantum graphs with leads extending to infinity. Both of these classes of quantum graphs are different from those considered here. There are many results on unbounded quantum graphs that might be well-known to the experts but whose proofs we could not find in the literature. In the appendices, we systematically present these results. Appendix A present the proof of the symmetry reduction for generalized radial tree graphs. In Appendix B, we extend many known results concerning generalized eigenfunctions to quantum graphs. We apply these results to establish a functional calculus using the generalized eigenfunctions. Appendix C is devoted to the extension of these results to the line-like graphs obtained from decorated radial graphs by the symmetry reduction. Appendix D presents an application of the material on generalized eigenfunctions to the transfer matrices and Weyl-Titchmarsh functions associated with quantum graphs. The Dirichlet-to-Neumann map for quantum graphs is introduced and used to study the transfer matrix. The Dirichlet-to-Neumann map is particularly useful in the analysis of decorated graphs. The last Appendix E is devoted to the extension of spectral averaging needed in the proofs of localization.

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## 2. Radial quantum tree graphs and their reduction

In this section, we define the basic concept of quantum tree graphs. We specialize to the family of radial quantum tree graphs and state a theorem on the reduction of the full graph Hamiltonian to a countable family of half-line Hamiltonians, with singularities at a discrete set of points. In the ergodic case, such as the RKM and RLM, these half-line Hamiltonians are unitarily equivalent. Finally, we introduce the transfer matrices on the half-line models. Transfer matrices will play an important role in the spectral theory of the random models.

2.1. **Tree graphs.** A *discrete graph* $T$ is given by a triple $T \equiv (V, E, \partial)$, where $V = V(T)$ denotes the set of vertices, $E = E(T)$ the set of (directed) edges and the map $\partial \colon E \longrightarrow V \times V$ maps an edge $e$ onto its start/end point $\partial e = (\partial_- e, \partial_+ e)$. For two vertices $v, w \in V$ such that there is an edge $e \in E$ with $\partial e = (v, w)$ or $\partial e = (w, v)$ we write $v \sim w$. For each vertex $v \in V$ we set

$$E_v^\pm := \{\, e \in E \,|\, \partial_\pm e = v \,\} \qquad \text{and} \qquad E_v := E_v(T) := E_v^+ \uplus E_v^-, \tag{2.1}$$

i.e., $E_v^\pm = E_v^\pm(T)$ consists of all edges starting $(-)$, respectively, ending $(+)$ at $v$, and $E_v$ their *disjoint* union. Note that the *disjoint* union is necessary in order to allow self-loops, i.e., edges having the same starting and ending point so that the edge occurs in both $E_v^+$ and $E_v^-$, whereas we only want it to occur once in $E_v$. The *degree* $\deg v$ of a vertex $v$ is given by the number of edges emanating from $v$, i.e., $\deg v := |E_v|$.

A *path of length* $n$ from a vertex $v$ to a vertex $w$ is a sequence of vertices $v_0 = v, \dots, v_n = w$ such that $v_i \sim v_{i+1}$. The *discrete distance* $\delta(v, w)$ of $v$ and $w$ is the shortest length of a simple path joining $v$ and $w$.

A *tree graph* is a graph $T$ without (nontrivial) closed paths (i.e., every closed path has length 0). If we fix a vertex $o \in V(T)$ (the *root vertex*) we say that $T$ is *rooted at* $o$. *We will always assume that our tree graphs are rooted.*

On a rooted tree graph we can define the notion of the *generation* $\operatorname{gen} v$: Every vertex with $\delta(o, v) = n$ is said to be in *generation* $n$. All edges are supposed to be directed *away* from the root $o$, i.e. $\partial_- e = w$ and $\partial_+ e = v$ where $\operatorname{gen} w = n - 1 < \operatorname{gen} v = n$. The generation of an edge $e$ is then the generation of the subsequent vertex, i.e., $\operatorname{gen} e := \operatorname{gen} \partial_+ e = n$. The *branching number* of a vertex $v$ is the number of *succeeding* edges, i.e., $b(v) := \deg v - 1$.

A rooted tree graph is *radial* if the branching number $b(v)$ is a function of the generation only, i.e., there exists a sequence $(b_n)$ such that $b(v) = b_n$ for all $v \in V$ with $\operatorname{gen} v = n$ (cf. Figure 1).

A discrete tree graph $T \equiv (V, E, \partial)$ becomes a *metric* tree graph if there is a length function $\ell \colon E \longrightarrow (0, \infty)$ assigning a length $\ell_e$ to each edge $e \in E$. We identify each edge $e$ with the interval $(0, \ell_e)$ turning $T$ into a one-dimensional space with singularities at the vertices. In this way we can define a *continuous* distance function $d(x, y)$ for $x, y \in T$ so that $T$ becomes a metric space.

A metric tree graph is *radial* if it is a radial tree graph and the length function depends only on the generation, i.e., if there is a sequence $\{\ell_n\}_n$ such that $\ell_e = \ell_n$ for all edges $e$ in generation $n$. We assume that the lengths are bounded from below and from above by finite, positive constants $\ell_\pm > 0$, i.e.,

$$\ell_- \le \ell_n \le \ell_+ \tag{2.2}$$

for all $n \in \mathbb{N}$. *In the remaining parts of this and the next section (Sections 2 and 3), we will only consider radial metric tree graphs.* We will consider decorations of such graphs in Section 4.

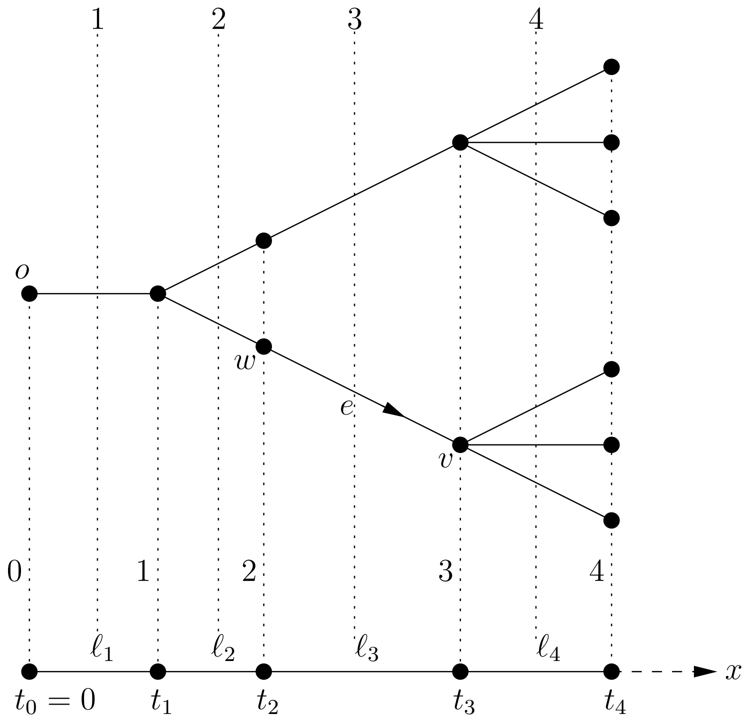


FIGURE 1. A radial tree graph with tree generations and branching numbers $b_0 = 1$, $b_1 = 2$, $b_2 = 1$, $b_3 = 3$; above the edge generation and below the vertex generation, e.g., the vertex $v$ and the edge $e$ are in generation 3. The bottom line is the corresponding half-line of the symmetry reduction.

2.2. **Radial Quantum Tree Graphs.** We associate a Hilbert space $\mathsf{L}_2(T)$ with a general metric tree graph by setting[1] $\mathsf{L}_2(T) := \bigoplus_{e \in E} \mathsf{L}_2(e)$, with norm given by

$$\|f\|^2 := \|f\|_T^2 := \sum_{e \in E} \int_e |f_e(x)|^2 \, \mathrm{d}x. \tag{2.3}$$

For *radial* functions, i.e., functions depending only on $d(o,x)$, this norm takes a simple form. Let $f_n$ denote the restriction of the edge function $f_e$ to one of the edges at generation $n$. We then have

$$\|f\|^2 = \sum_{n=1}^{\infty} \widetilde{b}_n \int_0^{\ell_n} |f_n(x)|^2 \, \mathrm{d}x, \tag{2.4}$$

where $f_n = f_e$ for an edge $e$ at generation $n$ and where $\widetilde{b}_n$ is the number of edges at generation $n$ and is a function of the branching numbers $\{b_n\}_n$. For a radial tree graph with branching number $b_n = b$ ($n \geq 1$) and $b_0 = 1$, often referred to as a Bethe lattice, we have $\widetilde{b}_n = b^{n-1}$.

We next define our main operator on metric trees that make these trees into *quantum trees*. The *Dirichlet Hamiltonian* $H = H(T, q)$, with strength $q \colon V \longrightarrow \mathbb{R}$, is defined by

$$(Hf)_e = -f_e'' \tag{2.5}$$

[1]Here and in the sequel, $\bigoplus_n \mathcal{H}_n$ always means the Hilbert space of all square-integrable sequences $\{f_n\}$, i.e., the *closure* of the algebraic direct sum.

on each edge for functions $f \in \operatorname{dom} H$ satisfying $f \in \bigoplus_{e\in E} \mathsf{H}^2(e)$ and satisfying two conditions. First, the functions are continuous at each vertex,

$$f_{e_1}(v) = f_{e_2}(v), \qquad \forall e_1, e_2 \in E_v. \tag{2.6}$$

We will write $f(v)$ for the unique value. Second, the functions satisfy the Kirchhoff boundary conditions at each vertex,

$$\sum_{j=1}^{b(v)} f'_{e_j}(\partial_- e_j) - f'_{e_0}(\partial_+ e_0) = q(v) f(v), \tag{2.7}$$

for all vertices $v \in V \setminus \{o\}$, where $e_0$ is the edge preceding $v$ and $e_j$ label the $b(v)$ subsequent edges at the vertex $v$. For the root vertex we impose a Dirichlet boundary condition, i.e.,

$$f(o) = 0. \tag{2.8}$$

Without loss of generality, we suppose that there is only one edge emanating from the root vertex, i.e., $b_0 = 1$, since otherwise, the (radial) Dirichlet Hamiltonian $H$ decouples into $b_0$ many operators on the edge subtrees of $o$.

We assume that $q$ is a radial function, i.e., there is a sequence $\{q_n\}_n$ such that $q(v) = q_n$ for all vertices $v$ at generation $n$. In this case, we also say that the Hamiltonian $H$ is *radial*. In addition, we assume that there are constants $q_\pm \in \mathbb{R}$ such that

$$q_- \le q_n \le q_+ \tag{2.9}$$

for all $n$.

The *free* Hamiltonian or *Kirchhoff Laplacian* $\Delta_T$ on $T$ is the Hamiltonian without the potential $q$ at the vertices, i.e., $\Delta_T := H(T, 0)$.

In summary, a *radial quantum tree graph* is a metric graph with an operator $H(T, q)$ satisfying (2.5)–(2.8). It is determined by the branching numbers $\{b_n\}_n$, the edge lengths $\{\ell_n\}_n$, and potentials $\{q_n\}_n$ that depend only on the generation.

2.3. **Reduction of Radial Quantum Tree Graphs.** For simplicity, we assume in this section, that $b_n = b$ for all $n \ge 1$. We will show in a more abstract setting that under the assumptions (2.2) and (2.9), the operator $H$ is essentially self-adjoint on the space of functions $f \in \operatorname{dom} H$ with *compact* support (cf. Lemma C.10) and that $H$ is relatively form-bounded with respect to $\Delta_T$ with relative bound 0 (cf. Lemma C.8).

The distance from the root vertex $o$ to a vertex of generation $n$, for $n \ge 1$, is denoted $t_n = \sum_{k=1}^n \ell_k$. We set $t_0 = 0$. The main reason why the analysis of *radial* Dirichlet Hamiltonians is much easier than the general case is the following symmetry reduction (cf. [NS00, SoS02, Sol04]). For completeness, we will give a proof in Appendix A, also in a more general setting. The points $t_k$ play the role of vertices. We denote by $f(t_k\pm) := \lim_{s\to t_k\pm} f(s)$.

**Theorem 2.1.** *The radial Hamiltonian $H$ on a radial quantum tree graph is unitarily equivalent to $H_1 \oplus \bigoplus_{n=2}^\infty (\oplus b^{n-2}(b-1)) H_n$, where $(\oplus m) H_n$ means the m-fold copy of $H_n$. The operator $H_n$ is the self-adjoint operator on $\mathsf{L}_2([t_{n-1}, \infty))$ given by $H_n f = -f''$ away from the points $t_k$ and with boundary conditions*

$$f(t_k-) = b^{-1/2} f(t_k+), \tag{2.10a}$$

$$f'(t_k-) + q_k f(t_k-) = b^{1/2} f'(t_k+) \tag{2.10b}$$

*for all $k \ge n$ and*

$$f(t_{n-1}+) = 0. \tag{2.10c}$$

We will refer to the reduced quantum graph, the half-line $[t_{n-1}, \infty)$, with boundary conditions at the vertices, as a *line-like quantum graph*. This is particularly useful in the discussion of decorated graphs, and we discuss this further in Section 4.2 and Definition 4.7.

Theorem 2.1 is particularly useful in the ergodic case, cf. Section 3. In this case, the operators $H_n$ are all simply related. First, ergodicity implies that each $H_n(\omega)$ has almost sure spectrum. Secondly, we have the relation $H_n(\tau_{n-1}\omega) = H_1(\omega)$, for any configuration $\omega$. Since the shift operator $U_k \colon \mathsf{L}_2([t_{n-1}, \infty)) \longrightarrow \mathsf{L}_2([t_{n+k-1}, \infty))$ is unitary, the operators are related as $U_{n-1}^{-1} H_n(\omega) U_{n-1} = H_1(\omega)$ and the operators are unitarily equivalent. Hence, the almost sure spectral components are *independent* of $n$, and it suffices to prove almost sure pure point spectrum for $H_1$, for example.

*Remark* 2.2. Note that the functions $f$ on $\mathsf{L}_2([t_{n-1}, \infty))$ are obtained from functions on the tree graph satisfying certain invariance conditions together with a exponential *weight* function reminiscent the fact that there are $b^{n-1}$ contributions from the edges at generation $n$. For example, the constant function $\mathbb{1}$ on the tree graph (not lying in either the domain of the Dirichlet Hamiltonian nor in $\mathsf{L}_2(T)$) is transformed in the step function $f(x) = b^{k/2}$ for $t_k < x < t_{k+1}$. In particular, $f$ increases exponentially.

On the other hand, suppose that $f_n$ is an eigenfunction of $H_n$, for $n \geq 2$, on $\mathsf{L}_2([t_{n-1}, \infty))$, with eigenvalue $\lambda$. We construct an eigenfunction $\tilde{f}_n$ of $H$ on $\mathsf{L}_2(T)$ with eigenvalue $\lambda$ as follows. The function $\tilde{f}_n$ will be supported on a subtree associated to any vertex of generation $(n-1)$ on the tree and equal zero outside of this subtree. The eigenvalue $\lambda$ will have a multiplicity at least equal to the number of vertices at generation $(n-1)$. Fixing $b = 2$ for simplicity, and a subtree of $T$ with vertex $o_{n-1}$, we construct $\tilde{f}_n$ at the first generation of the subtree by setting $\tilde{f}_n = (1/\sqrt{2}) f_n\restriction_{[t_{n-1}, t_n]}$ on one edge, and minus this value on the other. At the $m$th generation of the subtree, we use the weight $2^{-m/2}$ and $f_n$ restricted to $[t_{m+n-1}, t_{m+n}]$ to construct the value of $\tilde{f}_n$ on the edges with coefficients assigned according to the $b$th roots of unit. It is easy so see that

$$\|\tilde{f}_n\|_T^2 = \sum_{m \geq 1} \frac{1}{2^m} \|f_n\restriction_{[t_{m+n-1}, t_{m+n}]}\|^2. \tag{2.11}$$

In particular, if the eigenfunction $f_n$ of $H_n$ decays exponentially, that is, if $e^{\gamma d(0,x)} f_n \in \mathsf{L}_2([t_{n-1}, \infty))$, if follows from the fact that the distance function is a radial function and (2.11), that $\mathrm{e}^{\gamma d(o,x)} \tilde{f}_n \in \mathsf{L}_2(T)$.

2.4. **Transfer Matrices and Generalized Eigenfunctions of the Reduced Operator $H_1$.** We want to characterize the growth rate of the generalized eigenfunctions $f$ of $H_n$. We will consider $H_1$ explicitly since in the ergodic case the symmetry reduction in Theorem 2.1 shows that $H_1$ is unitarily equivalent to $H_n$.

We study functions $f \colon [0, \infty) \longrightarrow \mathbb{C}$ satisfying $-f'' = \lambda f$ away from the vertices $t_k$ and (2.10) at the vertices $t_k$. We assume here that $\lambda \neq 0$ (the case $\lambda = 0$ can be treated similarly, but it is unimportant for our purposes). If we know that $H_1 \geq 0$ (e.g., if $q \geq 0$), we may assume here $\lambda > 0$. In the definition of the Weyl-Titchmarsh functions (see Section D.2) we also need generalized eigenfunctions for *complex* $z = \lambda + \mathrm{i}\varepsilon$, $\varepsilon > 0$. In concrete examples, it is often more convenient to use $\mu = \sqrt{|\lambda|}$ (or in the complex case, $w = \sqrt{z}$, the branch with $\operatorname{Im} w > 0$) as parameter. We will switch between these two parameters without mentioning.

A basic fact that we use often is that the existence of a generalized eigenfunction of $H_1$ solving $H_1 f = \lambda f$ is equivalent with the existence of a nontrivial solution of a discrete map $\vec{F}_\lambda \colon \mathbb{N} \longrightarrow \mathbb{C}^2$ since, on the open interval $(t_{n-1}, t_n)$, the eigenfunction must have the form

$$f(x) = f(t_{n-1}^+)\cos(\sqrt{\lambda}x) + \frac{1}{\sqrt{\lambda}} f'(t_{n-1}^+)\sin(\sqrt{\lambda}x), \quad \text{for } x \in (t_{n-1}, t_n), \tag{2.12}$$

for $\lambda > 0$ (and similar for the other cases). The infinite family of coefficients

$$\{f(t_{n-1}^+), f'(t_{n-1}^+)\}_n$$

is determined iteratively by the map $\vec{F}_\lambda$ defined below and the boundary conditions (2.10).

The discrete map $\vec{F}_\lambda$ is defined using the transfer matrix as follows. The *transfer matrix* $T_\lambda(n)$ is given by

$$T_\lambda(n) = D(b)S(q_n)R_{\sqrt{\lambda}}(\sqrt{\lambda}\ell_n) \tag{2.13}$$

where the factors of the transfer matrix are the matrices

$$S(\kappa) := \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix}, \qquad D(b) := \begin{pmatrix} b^{1/2} & 0 \\ 0 & b^{-1/2} \end{pmatrix} \tag{2.14a}$$

$$R_\mu(\varphi) := \begin{pmatrix} \cos\varphi & \dfrac{\sin\varphi}{\mu} \\ -\mu\sin\varphi & \cos\varphi \end{pmatrix}. \tag{2.14b}$$

These are the standard matrices of shearing, dilation, and (elliptic) rotation, respectively. Note that $|\operatorname{tr} S(\kappa)| = 2$, $|\operatorname{tr} D(b)| > 2$ and $|\operatorname{tr} R_\mu(\varphi)| < 2$ (for real $\mu$ and $\varphi$). A matrix $A \in \mathrm{SL}_2(\mathbb{R})$ is called *parabolic, hyperbolic*, respectively, *elliptic*, if $|\operatorname{tr} A| = 2$, $|\operatorname{tr} A| > 2$, respectively, $|\operatorname{tr} A| < 2$. For $\lambda < 0$ we set $\mu := \sqrt{|\lambda|}$ and we obtain the hyperbolic "rotation" matrix

$$R^{\mathrm{h}}_\mu(\varphi) := R_{\mathrm{i}\mu}(\mathrm{i}\varphi) = \begin{pmatrix} \cosh\varphi & \frac{1}{\mu}\sinh\varphi \\ \mu\sinh\varphi & \cosh\varphi \end{pmatrix}.$$

Given a vector $\vec{\alpha}_0 \in \mathbb{C}^2$, we obtain another vector $\vec{\alpha}_n$ by

$$\vec{\alpha}_n = T_\lambda(n)T_\lambda(n-1)\dots T_\lambda(1)\vec{\alpha}_0. \tag{2.15}$$

We define the map $\vec{F}_\lambda \colon \mathbb{N} \longrightarrow \mathbb{C}^2$ at site $n$ as the product of transfer matrices acting on $\vec{\alpha}_0$,

$$\vec{F}_\lambda(n) = \vec{\alpha}_n = T_\lambda(n)T_\lambda(n-1)\dots T_\lambda(1)\vec{\alpha}_0. \tag{2.16}$$

The map $\vec{F}_\lambda$ depends on the energy $\lambda \in \mathbb{R}$ and the initial vector $\vec{\alpha}_0$. We note that $\vec{F}_\lambda$ satisfies the condition

$$\vec{F}_\lambda(n) = T_\lambda(n)\vec{F}_\lambda(n-1), \quad \text{for } n \geq 1. \tag{2.17}$$

Given an initial condition $\vec{\alpha}_0$ and the corresponding sequence of coefficients $\vec{\alpha}_n$ obtained as in (2.16), we can construct a generalized eigenfunction $f$ for $H_1$ with eigenvalue $\lambda$, as in (2.12), by using the vector $\vec{\alpha}_n$ for the coefficients $\{f(t_{n-1}^+), f'(t_{n-1}^+)\}$.

Conversely, suppose we have a generalized eigenfunction $f$ of $H_1$ satisfying $H_1 f = \lambda f$, and Dirichlet boundary conditions $f(0+) = 0$. Then, for each $n \geq 1$, it is easy to check that

$$\vec{F}_\lambda(n) := \begin{pmatrix} f(t_n+) \\ f'(t_n+) \end{pmatrix}, \tag{2.18}$$

with the initial condition

$$\vec{F}_\lambda(0) = \begin{pmatrix} 0 \\ f'(0+) \end{pmatrix}. \tag{2.19}$$

We can interpret the *transfer* or *monodromy* matrix $T_\lambda(n)$ as follows: Starting with the vector $\vec{F}_\lambda(n-1)$ at the vertex $t_{n-1}$ we evolve the free eigenvalue equation on the edge until the vertex $t_n$ (rotation matrix). The shearing matrix corresponds to the delta-potential at $t_n$ and finally, the dilation matrix encodes the jump condition at $t_n$ due to the branching number. Note that $T_\lambda(n)$ is an unimodular matrix, i.e., $\det T_\lambda(n) = 1$.

We also need a control of the $\mathsf{L}_2$-norm of an (a priori) generalized eigenfunction of $H_1$ in terms of the sequence $\vec{F}_\lambda(n)$. We write $\vec{F}_\lambda(n) = (F_\lambda(n), F'_\lambda(n))^{\mathrm{tr}}$, for the components of $\vec{F}_\lambda(n)$, and define a norm $|\vec{F}_\lambda|^2_\lambda := |F_\lambda|^2 + \frac{1}{|\lambda|}|F'_\lambda|^2$. Then, for $\lambda > 0$, it follows from (2.12) and (2.18) that we have

$$\|f\|^2_{\mathsf{L}_2(t_{n-1},t_n)} + \frac{1}{\lambda}\|f'\|^2_{\mathsf{L}_2(t_{n-1},t_n)} = \int_{t_{n-1}}^{t_n} |R_\mu(\mu x)\vec{F}_\lambda(n-1)|^2_\lambda \,\mathrm{d}x \le \ell_+ |\vec{F}_\lambda(n-1)|^2_\lambda, \quad (2.20\mathrm{a})$$

due to (2.2). Note that $R_\mu(\varphi)$ (for real $\varphi$) is orthogonal with respect to this norm. In addition, for $\lambda < 0$ and $\mu := \sqrt{|\lambda|}$, we have

$$\|f\|^2_{\mathsf{L}_2(t_{n-1},t_n)} + \frac{1}{|\lambda|}\|f'\|^2_{\mathsf{L}_2(t_{n-1},t_n)} = \int_{t_{n-1}}^{t_n} |R_{\mathrm{i}\mu}(\mathrm{i}\mu x)\vec{F}_\lambda(n-1)|^2_\lambda \,\mathrm{d}x \le 2\mathrm{e}^{2\mu\ell_+} |\vec{F}_\lambda(n-1)|^2_\lambda. \quad (2.20\mathrm{b})$$

In particular, if $\{\vec{F}_\lambda(n)\}_n \in \ell_2(\mathbb{N}, \mathbb{C}^2)$, then the associated generalized eigenfunction $f$ and its derivative $f'$ are indeed square-integrable, i.e., $f, f' \in \mathsf{L}_2(\mathbb{R}_+)$. Since there is also a lower bound on $\ell_e$, we also have the converse statement; in particular, a generalized eigenfunction $f$ is in $\mathsf{L}_2(\mathbb{R}_+)$ if and only if $\vec{F}_\lambda$ is in $\ell_2(\mathbb{N}, \mathbb{C}^2)$.

2.5. **The Spectrum of a Quantum Graph for the Free and Periodic Problem.** We first consider the simple periodic problem obtained when all the parameters are constant, i.e., when the transfer matrices $T_\lambda = T_\lambda(n)$ are independent of $n$. In this case, it follows from Theorem 2.1 that all the reduced Hamiltonians $H_n$ are unitarily equivalent.

**Theorem 2.3.** *Suppose that the transfer matrices are independent of $n$ and that $\lambda \mapsto \operatorname{tr} T_\lambda$ is nonconstant. Then, the spectrum of $H$ consists only of essential spectrum. The spectrum is given by the set $\Sigma_{\mathrm{ac}}$ of $\lambda \in \mathbb{R}$ for which $T_\lambda$ is elliptic or parabolic (i.e., $|\operatorname{tr} T_\lambda| \le 2$) and the set $\Sigma_{\mathrm{pp}}$ of all energies $\lambda$ such that $(0,1)^{\mathrm{tr}}$ is an eigenvector of $T_\lambda$ with eigenvalue $\tau$ such that $|\tau| < 1$. The spectrum is purely absolutely continuous on $\Sigma_{\mathrm{ac}}$ and pure point on $\Sigma_{\mathrm{pp}}$.*

*Proof.* In the periodic case, $H$ is unitarily equivalent to infinitely many copies of $H_1$ by Theorem 2.1. We let $\widetilde{H}_1$ be the periodic operator on $\mathsf{L}_2(\mathbb{R})$ with $\widetilde{H}_1 f = -f''$ on each edge and with boundary conditions (2.10a)–(2.10b) on $t_k > 0$ ($k > 0$) and similarly for $t_{-k} = -t_k < 0$ ($k \ge 0$) with $b^{1/2}$ replaced by $b^{-1/2}$. Let $H_{1,-}$ be the same operator as $H_1$, but on $\mathsf{L}_2(\mathbb{R}_-)$ (again, replacing $b^{1/2}$ by $b^{-1/2}$ in the boundary conditions, and with Dirichlet boundary condition at 0). Then $H_{1,-} \oplus H_1$ is a rank one perturbation of $\widetilde{H}_1$, in particular, the absolutely continuous spectrum is the same. But the latter can be calculated by Floquet theory (cf. [RS78, Sec. XIII.16]) and consists of the set of $\lambda = \mu^2$ for which there exists $\theta \in [0, \pi)$ such that $\operatorname{tr} T_\lambda = 2\cos\theta$. The latter equation determines the dispersion relation; since $\operatorname{tr} T_{\mu^2}$ is analytic (cf. (D.9) and (D.12)) and nonconstant, the spectrum $\Sigma_{\mathrm{ac}}$ is purely absolutely continuous (cf. [RS78, Thm. XIII.86]). Note that $\Sigma_{\mathrm{ac}}$ and $\Sigma_{\mathrm{pp}}$ are always disjoint, since for parabolic or elliptic matrices, all eigenvalues $\tau$ satisfy $|\tau| = 1$.

The additional eigenvalues of $H_1$ are of multiplicity 1 (and therefore of infinite multiplicity for $H$) and occur, if $T_\lambda(0,1)^{\mathrm{tr}} = \tau(0,1)^{\mathrm{tr}}$ with $|\tau| < 1$. $\square$

*Remark* 2.4. In Lemma D.6 (iii) we can express the eigenvalue $\tau$ in terms of the Dirichlet eigenfunction $\varphi_k$ provided $\lambda = \lambda_k$ is a simple eigenvalue of the Dirichlet problem and $\varphi_k^\dagger(o_i) \neq 0$ for both boundary points $i = 0$ and $i = 1$ (for the notation we refer to Section 4). Then $|\tau| < 1$ if and only if $|\varphi_k^\dagger(o_1)| \leq \sqrt{b}|\varphi_k^\dagger(o_0)|$.

Our two primary models, the RKM and the RLM, were described in the introduction and are presented in detail in Section 3. We apply Theorem 2.3 to compute the spectrum of the periodic version of the RKM when the vertex potential strength is a constant $q$, independent of $n$, and of the periodic version of the RLM when the edge length is a constant $\ell$. We will use these results to compute the deterministic spectra of these models in Theorem 3.4.

The spectrum of the periodic RLM is simply the spectrum of the free Hamiltonian $\Delta_{T(\ell)}$ on a rooted, regular, radial tree $T(\ell)$ with a fixed branching number $b \geq 1$ and constant edge length $\ell$. Let us define $\theta \equiv \arccos(2(b + b^{-1})^{-1})$. The identification of the spectrum is well-known (e.g. using Theorem 2.3 and (3.3a) or [Cat97]) and we refer to [SoS02] for a nice discussion. Carlson [Car97] proved that the spectrum is purely absolutely continuous away from the points $\{\, \pi^2 k^2/\ell^2 \,|\, k \in \mathbb{N} \,\}$.

**Theorem 2.5.** *The spectrum of the free Hamiltonian $\Delta_{T(\ell)}$ on a regular radial tree $T(\ell)$, with branching number $b \geq 1$ and constant edge length $\ell$ is a union of bands and points:*

$$\sigma(\Delta_{T(\ell)}) = \bigcup_{k=1}^{\infty} \Big( \frac{1}{\ell^2} B_k \cup \Big\{ \frac{\pi^2 k^2}{\ell^2} \Big\} \Big), \quad \textit{where} \quad B_k = \big[(\pi(k-1)+\theta)^2, (\pi k - \theta)^2\big], \tag{2.21}$$

*and is purely absolutely continuous on $\bigcup_k \frac{1}{\ell^2} B_k$. If $b > 1$, all gaps are open.*

Note that when $b = 1$, $\theta = 0$, and the spectrum (2.21) reduces to the known spectrum of the free Laplacian on the half-line with Dirichlet boundary conditions at zero. In this case, $\pi^2 k^2 \in B_k = [\pi^2(k-1)^2, \pi^2 k^2]$ and the spectrum is absolutely continuous on $\mathbb{R}_+$.

We next apply Theorem 2.3 to compute the spectrum of the periodic RKM when the vertex potential strength is a constant $q$, independent of $n$. We fix the length edge to be one.

**Theorem 2.6.** *For the Hamiltonian $H(q)$ with constant vertex potential $q \in \mathbb{R}$ on a metric tree with constant length $\ell = 1$ the spectrum is given by*

$$\sigma(H(q)) = \Big\{ \lambda \in \mathbb{R} \,\Big|\, \big|\xi_b(\sqrt{\lambda}, q)\big| \leq \frac{2b^{1/2}}{b+1} \Big\} \cup \{\, \pi^2 k^2 \,|\, k \in \mathbb{N} \,\}, \tag{2.22}$$

*where*

$$\xi_b(\mu, q) = \cos\mu + \frac{q \sin\mu}{\mu(b+1)}, \qquad \xi_b(\mathrm{i}\mu, q) = \cosh\mu + \frac{q \sinh\mu}{\mu(b+1)}$$

*for $\mu > 0$ and $\xi_b(0, q) = 1 + q/(b+1)$. Furthermore,*

$$\sigma(H(q)) = \bigcup_{k=1}^{\infty} \big( B_k(q) \cup \{\pi^2 k^2\} \big) \tag{2.23}$$

*where $B_k(q)$ are closed intervals. In addition, the spectrum is purely absolutely continuous on $\bigcup_k B_k(q)$.*

*The bands satisfy $B_k(q) \subset [(k-1)^2\pi)^2, k^2\pi^2]$ for $k \geq 2$. In addition, $B_1(q) \subset [0, \pi^2]$ if and only if $q \geq -(b^{1/2} - 1)^2$, and $B_1(q) \subset (-\infty, 0)$ if and only if $q < -(b^{1/2} + 1)^2$.*

*If $b = 1$ and $q \neq 0$, then the intervals $B_k(q)$ $(k \geq 2)$ touch only one of the points $\pi^2 k^2$ or $\pi^2(k-1)^2$. If $b > 1$, the points $\pi^2 k^2$ never lie in the union of the bands $\bigcup_k B_k(q)$. In particular, if $b > 1$ or $b = 1$ and $q \neq 0$, all gaps are open.*

*Proof.* The spectral characterization is an application of Theorem 2.3 using (3.3b). The case $b = 1$ has been analyzed in [AGHKH88, Thm. 2.3.3]. □

## 3. Random quantum tree graphs and localization

3.1. **Random quantum tree graphs.** We consider now random perturbations of the length sequence $\{\ell_n\}$ or the vertex potential strength $\{q_n\}$. Let $(\Omega_1, \mathbb{P}_1)$ be a probability space and $(\Omega, \mathbb{P}) := (\Omega_1, \mathbb{P}_1)^{\mathbb{N}}$ the product probability space. In our applications, $\Omega_1$ will always be a compact interval. To exclude unnecessary complications (see e.g. (3.12)), we assume that $\operatorname{supp} \mathbb{P}_1 = \Omega_1$ where $\operatorname{supp} \mathbb{P}_1$ is the largest closed subset such that the complement is of $\mathbb{P}_1$-measure 0.

We can define the notion of *ergodicity* on such spaces: There is a canonical (right) shift function $(\tau_{n_0}\omega)(n) := \omega_{n_0+n}$ preserving the probability measure $\mathbb{P}$ on $\Omega$. Note that $\tau_n = \tau_1^{\circ n}$.

**Definition 3.1.** A measure preserving map $\tau_1 \colon \Omega \longrightarrow \Omega$ is called *ergodic* if any measurable set $A \in \mathcal{F}$ with $\tau_1(A) = A$ satisfies $\mathbb{P}(A) \in \{0, 1\}$.

From the Kolmogorov 0-1 law it follows that the (right) shift is an ergodic action on $\Omega$ (cf. e.g. [S79, p. 26]).

**Definition 3.2.** The **Random Length Model** (RLM) is a *random length* quantum tree graph defined by an iid sequence $\{\ell_n\}$ of random variables $\ell_n \colon \Omega_1 \longrightarrow (0, \infty)$ satisfying (2.2) $\mathbb{P}_1$-almost surely. We denote the corresponding family of quantum tree graphs and Laplacians by $\{T(\omega)\}$ and $\{\Delta_{T(\omega)}\}$.

The **Random Kirchhoff Model** (RKM) is a *random Hamiltonian* on a radial quantum tree graph $T$ given by an iid sequence $\{q_n\}$ of random variables $q_n \colon \Omega_1 \longrightarrow (0, \infty)$ satisfying (2.9) $\mathbb{P}_1$-almost surely. We denote the corresponding family of Hamiltonians on the (fixed) quantum tree graph $T$ by $\{H(\omega)\}$. For simplicity, we assume that $\ell_n = 1$ for all $n$.

To unify the notation, we denote both operators by $H(\omega)$ acting on $T(\omega)$. Since $H(\omega)$ is radial (for almost all $\omega$), we can apply the symmetry reduction Theorem 2.1 and obtain a family of random operators $H_n(\omega)$. As a consequence of the ergodicity, we obtain:

**Theorem 3.3.** *The spectral components of the spectrum of $H(\omega)$ are almost surely constant, i.e., there exist subsets $\Sigma_\bullet$ such that $\sigma_\bullet(H(\omega)) = \Sigma_\bullet$ for almost all $\omega \in \Omega$. In addition, the spectral sets $\Sigma_\bullet$ are determined by the corresponding almost sure spectrum of the Hamiltonian $H_1(\omega)$ on $\mathsf{L}_2(\mathbb{R}_+)$. Here, $\bullet$ labels either the pure point* (pp)*, the absolutely continuous* (ac) *or singularly continuous* (sc) *spectrum.*

*Proof.* The first statement is standard for random operators (see e.g. [PF92]). The last statement follows easily from Theorem 2.1 and the fact that $H_{n+1}(\omega) = H_1(\tau_n\omega)$ and $H_1(\omega)$ have the same almost sure spectral components for all $n$. □

**Theorem 3.4.** *The almost sure spectrum is given by*

$$\Sigma = \overline{\bigcup_{\omega_1 \in \Omega_1} \sigma(H_1(\omega_1 \mathbb{1}))},$$

*where $\omega_1 \mathbb{1} \in \Omega$ is the element with the same entry $\omega_1$ in each component and $H_1(\omega_1 \mathbb{1})$ is periodic. Assuming that $\Omega_1$ is a compact interval, we have in the RLM*

$$\Sigma = \bigcup_{k=1}^{\infty} \bigcup_{\ell \in \Omega_1} \frac{1}{\ell^2} \big(B_k \cup \{\pi^2 k^2\}\big)$$

$$= \bigcup_{k=1}^{\infty} \Big( \big[\frac{1}{\ell_+^2} \min B_k, \frac{1}{\ell_-^2} \max B_k\big] \cup \big[\frac{\pi^2 k^2}{\ell_+^2}, \frac{\pi^2 k^2}{\ell_-^2}\big] \Big), \tag{3.1}$$

*where $\Omega_1 = [\ell_-, \ell_+]$ and the intervals $B_k$ are defined in (2.21), and in the RKM, we have*

$$\Sigma = \bigcup_{k=1}^{\infty} \bigcup_{q \in \Omega_1} \big(B_k(q) \cup \{\pi^2 k^2\}\big)$$

$$= \bigcup_{k=1}^{\infty} \Big( \big[\min B_k(q_-), \max B_k(q_+)\big] \cup \{\pi^2 k^2\} \Big), \tag{3.2}$$

*where $\Omega_1 = [q_-, q_+]$ and $B_k(q)$ is defined in (2.23). If $b > 1$ or $b = 1$ and $0 \notin [q_-, q_+]$ then $\Sigma$ has infinitely many gaps close to $\pi^2 k^2$.*

*Proof.* The spectrum of the periodic operator was calculated in Thms. 2.5–2.6. Note that in both models, the band edges depend continuously and monotonically on the random parameter and the union is locally finite, so the union of compact intervals is still a closed set. $\square$

In order to prove that $H_1(\omega)$ has pure point spectrum almost surely, we need to control the growth of generalized eigenfunctions. We have already seen in the previous section, that it is enough to control the growth of nontrivial solutions of the random discrete map $\vec{F}_\lambda = \vec{F}_\lambda(\omega, \cdot) \colon \mathbb{N} \longrightarrow \mathbb{C}^2$ of (2.17). The random transfer matrix $T_\lambda(n) = T_\lambda(\omega_n)$ in the RLM has the form, for $\lambda > 0$,

$$T_\lambda(\omega_n) = D(b) R_\mu(\mu \ell(\omega_n)) = \begin{pmatrix} b^{1/2} \cos(\mu \ell(\omega_n)) & \dfrac{b^{1/2}}{\mu} \sin(\mu \ell(\omega_n)) \\ -b^{-1/2} \mu \sin(\mu \ell(\omega_n)) & b^{-1/2} \cos(\mu \ell(\omega_n)) \end{pmatrix} \tag{3.3a}$$

where $\ell \colon \Omega_1 \longrightarrow (0, \infty)$ is the single edge random length perturbation. For the RKM, we have

$$\begin{aligned} T_\lambda(\omega_n) &= D(b) S\big(q(\omega_n)\big) R_\mu(\mu) \\ &= \begin{pmatrix} b^{1/2} \cos \mu & \dfrac{b^{1/2}}{\mu} \sin \mu \\ b^{-1/2} \Big( -\mu \sin \mu + q(\omega_n) \cos \mu \Big) & b^{-1/2} \Big( \cos \mu + \dfrac{q(\omega_n)}{\mu} \sin \mu \Big) \end{pmatrix} \end{aligned} \tag{3.3b}$$

where the second equality holds for $\lambda > 0$. Here, $q \colon \Omega_1 \longrightarrow (0, \infty)$ is the single site random potential perturbation. In the case $\lambda < 0$, one has to replace $R_\mu(\mu)$ by $R^{\mathrm{h}}_\mu(\mu)$ with $\mu = \sqrt{|\lambda|}$. If $\lambda = 0$, then $R_0(0) = S(1)^{\mathrm{tr}}$.

3.2. **Lyapunov exponents.** As we have seen we can control the growth of generalized eigenfunctions via the growth of random matrices. We will provide therefore some general results on Lyapunov exponents and exponentially decaying solutions of recursion equations.

Assume that $T\colon \Omega_1 \times \Sigma_0 \longrightarrow \mathrm{SL}_2(\mathbb{R})$, $(\omega_1, \lambda) \mapsto T_\lambda(\omega_1)$ is measurable where $\Sigma_0 \subset \mathbb{R}$ is a measurable set. We assume that

$$\mathbb{E}_1(\ln \|T_\lambda^{-1}\|) < \infty. \tag{3.4}$$

Note that $\|A\| \geq 1$ for $A \in \mathrm{SL}_2(\mathbb{R})$. We set

$$U_\lambda(\omega, n) := T_\lambda(\omega_n) \cdot \ldots \cdot T_\lambda(\omega_1), \qquad U_\lambda(\omega, 0) := \mathbb{1}. \tag{3.5}$$

Clearly,

$$U_\lambda(\omega, n_1 + n_0) = U_\lambda(\tau_{n_0}\omega, n_1) U_\lambda(\omega, n_0), \tag{3.6}$$

i.e., $U_\lambda$ is a *multiplicative cocycle*, cf. [PF92, (11.23)].

We define the *Lyapunov exponent*

$$\gamma(\omega, \lambda) := \lim_{n\to\infty} \frac{1}{n} \ln \|U_\lambda(\omega, n)\| \tag{3.7}$$

where $\|\cdot\|$ is the operator norm of $2 \times 2$-matrices defined by $\|A\| := \sup_{v \in \mathbb{R}^2} |Av|/|v|$. The limit is nonrandom:

**Lemma 3.5.** *Suppose that the single transfer matrix $T_\lambda(\cdot)$ satisfies the integrability condition* (3.4)*. Then, there exists a measurable set $S_1 \subset \Omega \times \Sigma_0$ such that $S_1(\lambda) := \{\, \omega \mid (\omega, \lambda) \in S_1 \,\} \subset \Omega$ has full measure, and the limit* (3.7) *exists and is finite for all $(\omega, \lambda) \in S_1$. In addition, the limit is nonrandom, i.e.,*

$$\gamma(\lambda) := \mathbb{E}(\gamma(\cdot, \lambda)) = \gamma(\omega, \lambda)$$

*for all $\omega \in S_1(\lambda)$. Finally, $\gamma(\lambda) \geq 0$.*

*Proof.* We apply the subadditive ergodic theorem [PF92, Prop. 6.3] and have to verify that

$$\mathbb{E}(\ln \|U_\lambda(\cdot, n)\|) \geq C_\lambda n$$

for $n \geq 0$ and some constant $C_\lambda \in \mathbb{R}$. A simple norm estimate using $\|AB\| \geq (\|A^{-1}\| \|B^{-1}\|)^{-1}$ shows that $C_\lambda = -\mathbb{E}_1(\ln \|T_\lambda^{-1}\|)$ is enough. The measurability of $S_1$ follows from the measurability of $(\omega_1, \lambda) \to T_\lambda(\omega_1)$. $\square$

We parameterize the set of all directions in $\mathbb{R}^2$ (up to sign) by $\theta \in [0, \pi)$, or more abstractly by points in the real projective line $\mathrm{P}(\mathbb{R}^1)$ and sometimes write $\vec{F}_\lambda \sim \theta$ if the nonzero vector $\vec{F}_\lambda \in \mathbb{R}^2$ is in the direction $\theta$, i.e., a multiple of $(\sin\theta, \cos\theta)^{\mathrm{tr}}$, where *tr* denotes transpose.

We denote

$$\vec{F}_\lambda(\omega, \theta, n) := U_\lambda(\omega, n) \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \tag{3.8}$$

the propagation of the initial vector $\vec{F}(0) \sim \theta$. Clearly, $\vec{F}_\lambda(\omega, \theta, \cdot)$ solves the recursion equation

$$\vec{F}_\lambda(\omega, \theta, n+1) = T_\lambda(\omega_n) \vec{F}_\lambda(\omega, \theta, n), \quad \vec{F}(0) = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}. \tag{3.9}$$

We want to turn the positivity of the Lyapunov exponent into exponential bounds on the solution of the above recursion equation. To do so, we need the following deterministic version of the Oseledec theorem (cf. [CL90, Thm IV.2.4]):

**Theorem 3.6.** *Suppose that $U(n) \in \mathrm{SL}_2(\mathbb{R})$ for all $n \geq 1$ such that*

(i) $\lim_{n\to\infty} \frac{1}{n} \ln \|U(n)\| = \gamma$ *exists, $\gamma < \infty$ and*
(ii) $\lim_{n\to\infty} \frac{1}{n} \ln \|T(n)\| = 0$

*where $T(n) := U(n)U(n-1)^{-1}$ is the single transition matrix. Then there exists a nonzero vector $\vec{F}(0) \in \mathbb{R}^2$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \ln |U(n)\vec{F}(0)| = -\gamma \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{n} \ln |U(n)\vec{F}| = \gamma \tag{3.10}$$

*where $\vec{F}$ is linearly independent of $\vec{F}(0)$ in the latter case. In particular, the solution $\vec{F}(n) := U(n)\vec{F}(0)$ of the recursion equation $\vec{F}(n+1) = T(n)\vec{F}(n)$ with initial vector $\vec{F}(0)$ has almost exponential decay rate $-\gamma$, i.e.,*

$$\forall \varepsilon > 0 \quad \exists C(\varepsilon) > 0: \quad |\vec{F}(n)| \le C(\varepsilon)\mathrm{e}^{-(\gamma-\varepsilon)n}. \tag{3.11}$$

*Remark* 3.7. The previous theorem already indicates that we cannot expect to show exponential decay directly for the initial condition $\theta = 0$ (corresponding to a Dirichlet boundary condition at 0); moreover, we need the spectral averaging arguments of Appendix E. But the Dirichlet boundary condition is crucial in the symmetry reduction (see Theorem 2.1 or Theorem A.6), not for the first reduction step, but for the subsequent ones.

We will apply this theorem to $U(n) = U_\lambda(\omega, n)$ for fixed $\omega$ and $\lambda$ in Theorem 3.15. Clearly, in this case $\vec{F}(0)$ and $C(\varepsilon)$ also depend on $\lambda$ and $\omega$.

To ensure the positivity of the Lyapunov exponent we use the Furstenberg theorem [Fur63]:

**Theorem 3.8.** *Denote by $G_\lambda$ the smallest closed subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by all matrices $T_\lambda(\omega_1)$, $\omega_1 \in \Omega_1$. If $G$ is noncompact and no subgroup of finite index is reducible then $\gamma(\lambda) > 0$.*

A sufficient condition for $\gamma(\lambda) > 0$ is the following (cf. [Ish73, Thm. 4.1], [IM70]):

**Theorem 3.9.** *Suppose that $\{\, T_\lambda(\omega_1) \,|\, \omega_1 \in \Omega_1 \,\} \subset \mathrm{SL}_2(\mathbb{R})$ contains at least two elements with no common eigenvectors then $\gamma(\lambda) > 0$.*

The following lemma reduces the possibilities in our application, since we are only interested in transfer matrices associated to spectral parameters $\lambda$ in the almost sure spectrum:

**Lemma 3.10.** *Assume that the almost sure spectrum is the union of the periodic spectrum, i.e.,*

$$\Sigma = \bigcup_{\omega_1 \in \Omega_1} \sigma(H(\omega_1 \mathbb{1})). \tag{3.12}$$

*Suppose in addition, that the set*

$$N := \{\, (\omega_1, \lambda) \in \Omega_1 \times \Sigma \,|\, |\operatorname{tr} T_\lambda(\omega_1)| = 2 \,\} \tag{3.13}$$

*has $(\mathbb{P}_1 \otimes \boldsymbol{\lambda})$-measure 0, where $\boldsymbol{\lambda}$ denotes Lebesgue measure. Finally, suppose that there is a set $\Sigma_0 \subset \Sigma$ so that for all $\lambda \in \Sigma_0$, there exist at least two different* elliptic *matrices $T_1, T_2$ in $\{\, T_\lambda(\omega_1) \,|\, \omega_1 \in \Omega_1, \text{and } \lambda \in \Sigma_0 \,\} \subset \mathrm{SL}_2(\mathbb{R})$ having no common eigenvectors. Then $\gamma(\lambda) > 0$ for all $\lambda \in \Sigma_0$.*

*Proof.* Due to the second assumption, for almost all $\lambda \in \Sigma$, the set

$$N(\lambda) = \{\, \omega_1 \,|\, |\operatorname{tr} T_\lambda(\omega_1)| = 2 \,\}$$

has probability 0 so that the set of $\lambda$ such that $T_\lambda$ is elliptic or hyperbolic forms a support of $\mathbb{P}_1$. We have to show that there are at least two matrices in $\Omega_1 = \operatorname{supp} \mathbb{P}_1$ with no common eigenvectors. If both are elliptic, we are done due to our assumption. If one is elliptic and the other hyperbolic, they can never have a common eigenvector, since the eigenvectors of the first are nonreal, and the second are real. The case that both matrices are hyperbolic is not of interest, since $\lambda \in \Sigma$ implies that at least one of the matrices is not hyperbolic due to our first assumption. The result now follows from Theorem 3.9. $\square$

In cases when the transfer matrix is complicated, the following criteria is useful:

**Corollary 3.11.** *Suppose that* (3.12) *and* (3.13) *are true. Assume in addition, that for all* $\lambda \in \Sigma_0$ *there exist two* noncommuting *elliptic matrices in* $\{\, T_\lambda(\omega_1) \,|\, \omega_1 \in \Omega_1 \,\} \subset \mathrm{SL}_2(\mathbb{R})$*. Then* $\gamma(\lambda) > 0$ *for all* $\lambda \in \Sigma_0$*.*

*Proof.* If the matrices $T_1$ and $T_2$ do not commute, they differ in at least one eigenspace. Since $T_1$ and $T_2$ are elliptic and real, all eigenvectors are nonreal, and the second eigenspace is obtained from the first one by conjugation. In particular, $T_1$ and $T_2$ have no common eigenspace. $\square$

3.3. **Lyapunov exponents for the RLM and RKM.** In this subsection we show that under suitable assumptions on the single site random perturbation, the Lyapunov exponent of the transfer matrices (3.3) are positive. In addition we show that (3.4) and Assumption (ii) of Theorem 3.6 are fulfilled. We will need all these results in the next subsection in order to prove exponential localization.

**Lemma 3.12.** *Assume that* $\lambda > 0$ *lies in the almost sure spectrum of* $H_1(\omega)$ *in the RLM. Suppose furthermore that the branching number* $b > 1$ *and that there are at least two different values* $\ell_1, \ell_2 \in \Omega_1$ *such that* $\mu(\ell_1 - \ell_2) \notin \pi\mathbb{Z}$*. Then* $\gamma(\lambda) > 0$*. If* $b = 1$*, then* $\gamma(\lambda) = 0$ *for all* $\lambda > 0$*.*

*In particular, if* $b > 1$ *and* $\Omega_1$ *contains at least two different length* $\ell_1$ *and* $\ell_2$ *then* $\gamma(\lambda) > 0$ *for almost all* $\lambda > 0$*.*

*Proof.* We want to apply Lemma 3.10. The first two conditions are fulfilled and we only have to check that the eigenvectors of $T_i := T_\lambda(\omega_i)$, i.e., $\{\vec{e}_{1,+}; \vec{e}_{1,-}\}$ and $\{\vec{e}_{2,+}; \vec{e}_{2,-}\}$, never have an eigenspace in common in the elliptic case. A simple calculation shows that the eigenvectors are linear dependent iff $\sin \mu(\ell_1 - \ell_2)(b-1) = 0$, i.e., $\mu(\ell_1 - \ell_2) = k\pi$ or $b = 1$. In the latter case we can calculate $\gamma(\lambda) = 0$ explicitly. The last statement follows since $\{\, \mu^2 \,|\, \mu(\ell_1 - \ell_2) \in \pi\mathbb{Z} \,\}$ is a countable set iff $\ell_1 \neq \ell_2$. $\square$

**Lemma 3.13.** *Assume that there are* $q_1, q_2 \in \Omega_1$ *such that* $q_1 \neq q_2$ *and that* $\lambda \in \Sigma$*. If* $\mu = \sqrt{\lambda} \notin \pi\mathbb{N}$ *then* $\gamma(\lambda) > 0$*. If* $\mu \in \pi\mathbb{N}$ *then* $\gamma(\lambda) = \frac{1}{2}\ln b$*. In particular,* $\gamma(\lambda) > 0$ *for almost all* $\lambda > 0$*.*

*Proof.* Again, we apply Lemma 3.10. The first two assumptions are also satisfied in RKM. One can easily see that the eigenvectors of an *elliptic* transfer matrix associated to $q_1$ are linearly dependent on the ones associated to $q_2$ iff $\sin \mu = 0$ or $q_1 = q_2$. The Lyapunov exponent for $\lambda = \mu^2$ with $\mu \in \pi\mathbb{N}$ can easily be calculated since $T_\lambda(q) = \pm D(b)$ and the largest eigenvalue of $U_\lambda(\omega, n)$ is always $b^n$. $\square$

**Lemma 3.14.** *In both models, the integrability condition* (3.4) *and the condition* (ii) *in Theorem 3.6 are fulfilled.*

*Proof.* The norm of the transfer matrix can be estimated by

$$\|T_\lambda(\omega_n)\| \leq \|D(b)\| \|R_+(\mu\ell_n)\| \leq b^{1/2}$$

in the random length model (here, we only need to consider $\lambda > 0$ since $H = \Delta_{T(\omega)} \geq 0$). The same estimate holds for the inverse of $T_\lambda(\omega_n)$. In the random potential model, we have

$$\|T_\lambda(\omega_n)\| \leq \|D(b)\| \|S(-q_n)\| \|R_\pm(\mu)\| \leq b^{1/2}(1 + \max\{|q_-|, |q_+|\})\mathrm{e}^{\mu},$$

$\mu := \sqrt{|\lambda|}$, and similarly for the inverse. Therefore, the norms are independent of $n$. In particular, (3.4) and Assumption (ii) of Theorem 3.6 are fulfilled for both models. $\square$

3.4. **Exponential localization on the tree graph.** Here, we show that in both random models of Definition 3.2 localization holds. Denote by $H_1(\omega)$ the Hamiltonian on $\mathbb{R}_+$ with Dirichlet boundary condition $f(0) = 0$.

**Theorem 3.15** ([Kot86]). *Assume that $\gamma(\lambda) > 0$ for Lebesgue-almost all $\lambda \in \Sigma_0$ and $\Sigma_0 \subset \mathbb{R}$. Assume in addition, that the spectral averaging formula* (E.1) *holds. Then $\sigma(H_1(\omega)) \cap \Sigma_0$ is almost surely pure point, i.e., if $\Sigma_\bullet$ denote the almost sure spectrum (respectively, almost sure spectral components) of $H_1(\omega)$, then*

$$\Sigma \cap \Sigma_0 = \Sigma_{\mathrm{pp}} \cap \Sigma_0 \qquad \text{and} \qquad \Sigma_{\mathrm{c}} \cap \Sigma_0 = \emptyset.$$

*In addition, almost all eigenfunctions of $H_1(\omega)$ on the half-line $[0, \infty)$ decay with almost exponential decay rate $\gamma(\lambda)$ in the sense of* (3.17)*.*

*Proof.* Without loss of generality, we assume that $\gamma(\lambda) > 0$ for *all* $\lambda \in \Sigma_0$ (just exclude the exceptional set of measure 0 from $\Sigma_0$). We decompose $\Omega$ into its first and remaining component, i.e., $\omega = (\omega_1, \hat{\omega}) \in \Omega_1 \times \hat{\Omega} = \Omega$ and set

$$S := \{\, (\hat{\omega}, \lambda) \in \hat{\Omega} \times \Sigma_0 \mid \lim_n \frac{1}{n} \ln \|U_\lambda(\hat{\omega}, n)\| > 0 \,\}. \tag{3.14}$$

It follows from standard arguments that $S$ is measurable. In addition, $S(\hat{\omega}) = \{\, \lambda \in \Sigma_0 \,|\, (\hat{\omega}, \lambda) \in S \,\}$ is a tail event, i.e., $S(\hat{\omega})$ does not depend on a finite number of random variables. From Lemma 3.5 and the assumption $\gamma(\lambda) > 0$ we see that the set of energies $S_1$, defined in Lemma 3.5, has full $(\hat{\mathbb{P}} \otimes \boldsymbol{\lambda})$-measure. Since $S_1 \subset S$, the set $S$ has full $(\hat{\mathbb{P}} \otimes \boldsymbol{\lambda})$-measure. In particular, for $(\hat{\omega}, \lambda) \in S$, Assumption (i) of Theorem 3.6 is fulfilled. We have already seen that Assumption (ii) is always fulfilled. Therefore, there exists $\theta_0 = \theta_0(\hat{\omega}, \lambda)$ such that

$$\lim_{n \to \infty} \frac{1}{n} \ln |\vec{F}_\lambda(\hat{\omega}, n, \theta)| = \begin{cases} -\gamma(\lambda), & \theta = \theta_0 \\ \gamma(\lambda), & \theta \neq \theta_0 \end{cases} \tag{3.15}$$

where

$$\vec{F}_\lambda(\hat{\omega}, n, \theta) = U_\lambda(\hat{\omega}, n) \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}.$$

Let $f$ be the generalized eigenfunction on $\mathbb{R}_+$ associated to $\vec{F}_\lambda(\hat{\omega}, \cdot, \theta_0)$. Since $\vec{F}_\lambda(\hat{\omega}, n, \theta_0)$ decays exponentially in $n$, we see from (2.20), that then $f \in \mathsf{L}_2(\mathbb{R}_+)$. Now, the remaining point to show is, that $\theta_0 = 0$, i.e., that $f$ satisfies a Dirichlet boundary condition at 0.

Denote the measure associated to $H_1(\omega)$ in Corollary B.5 by $\rho_\omega$. Due to Lemma D.16, the Weyl-Titchmarsh function $m$ associated to $H_1(\omega)$ is the Borel transform of the measure $\rho_\omega$ and we can apply the results on spectral averaging of Appendix E. In particular, using Fubini and the spectral averaging formula (E.5), we obtain

$$\begin{aligned} \int_{\Omega_1} \int_{\hat{\Omega}} \rho_{(\omega_1, \hat{\omega})}(S(\hat{\omega})^{\mathrm{c}}) \,\mathrm{d}\hat{\mathbb{P}}(\hat{\omega}) \,\mathrm{d}\mathbb{P}_1(\omega_1) &= \int_{\hat{\Omega}} \int_{\Omega_1} \rho_{(\omega_1, \hat{\omega})}(S(\hat{\omega})^{\mathrm{c}}) \,\mathrm{d}\mathbb{P}_1(\omega_1) \,\mathrm{d}\hat{\mathbb{P}}(\hat{\omega}) \\ &\le C_5 \int_{\hat{\Omega}} \boldsymbol{\lambda}(S(\hat{\omega})^{\mathrm{c}}) \,\mathrm{d}\hat{\mathbb{P}}(\hat{\omega}) = C_5 (\hat{\mathbb{P}} \otimes \boldsymbol{\lambda})(S^{\mathrm{c}}) = 0 \end{aligned} \tag{3.16}$$

where $\boldsymbol{\lambda}$ denotes Lebesgue measure. This means that for $\mathbb{P}$-almost all $\omega = (\omega_1, \hat{\omega})$, we have $\rho_\omega(S(\hat{\omega})^{\mathrm{c}}) = 0$, i.e., $S(\hat{\omega})$ is a support for the spectral measure $\rho_\omega$. Fix now such an $\omega$.

We show in Theorem C.18 that the spectral measure is also supported on $\Sigma_\omega$, the set of eigenvalues having a polynomial bounded eigenfunction. The set of energies $S(\hat\omega)\cap\Sigma_\omega$ is a support for the spectral measure $\rho_\omega$. For any $\lambda\in S(\hat\omega)\cap\Sigma_\omega$, there is a generalized eigenfunction $\varphi$ of $H_1(\omega)$ with eigenvalue $\lambda$ and having polynomial growth. In addition, since $(\hat\omega,\lambda)\in S$, we have constructed an eigenfunction $f\in L^2$ from the coefficients $\vec F_\lambda(\hat\omega,\cdot,\theta_0)$ as in (3.15). From Lemma D.12 we see that the Wronskian $W(f,\varphi)(t_n+)$ of two generalized eigenfunctions is independent of $n$. Since $\varphi(t_n+)$ and $\varphi'(t_n+)$ are polynomially bounded in $n$ (cf. Theorem C.18) and since $f(t_n+)$ and $f'(t_n+)$ are almost exponentially decaying (cf. (3.15)) we see that

$$\lim_n W(f,\varphi)(t_n+)=\lim_n\bigl(f'(t_n+)\varphi(t_n+)-f(t_n+)\varphi'(t_n+)\bigr)=0.$$

In particular, $W(f,\varphi)(0)=0$ and $f$, $\varphi$ satisfy the same boundary condition at 0, namely $\theta_0=0$, i.e., $f(0)=0$.

Consequently, each $\lambda\in S(\hat\omega)\cap\Sigma_\omega$ is an $\mathsf{L}_2$-eigenfunction of $H_1(\omega)$, i.e., that $\rho_\omega(\{\lambda\})>0$ for all $\lambda$ in a support of the spectral measure. Since a spectral measure is a Borel measure and the Hilbert space is separable, the support must be countable. This implies that the measure is pure point since a continuous measure cannot be supported on a countable set. □

*Remark* 3.16. The spectral averaging used in (3.16) is basically Kotani's trick. We may weaken the spectral averaging formula (E.1) in the following way: We assume that (E.1) is fulfilled for all $\lambda\in\Sigma_k\subset[\lambda_-,\lambda_+]=:\Sigma_0$ with an $k$-dependent constant $C_5=C_5(k)$ and where $\Sigma_k$ is an increasing sequence such that $\bigcup_k\Sigma_k=:\Sigma_\infty$ equals $\Sigma_0$ Lebesgue-almost everywhere. In the RKM, we will see that $\Sigma_k$ is just $\Sigma_0$ with a "security" distance from the points $k^2\pi^2$ tending to 0 as $k\to\infty$.

We can still use Kotani's trick in this case: As in (3.16) it follows that for each $k\in\mathbb{N}$ there is a set of full measure $\Omega(k)$ such that $\rho_\omega(S(\hat\omega)^{\mathrm c}\cap\Sigma_k)=0$ for all $\omega\in\Omega(k)$. The intersection $\Omega(\infty)$ of all $\Omega(k)$ has still full measure, and for $\omega\in\Omega(\infty)$, we have

$$\rho_\omega\bigl(S(\hat\omega)^{\mathrm c}\cap\Sigma_\infty\bigr)=\rho_\omega\Bigl(\bigcup_k\bigl(S(\hat\omega)^{\mathrm c}\cap\Sigma_k\bigr)\Bigr)\le\sum_k\rho_\omega\bigl(S(\hat\omega)^{\mathrm c}\cap\Sigma_k\bigr)=0.$$

The rest of the argument in the proof of Theorem 3.15 remains the same, replacing $\Sigma_0$ by $\Sigma_\infty$.

On a tree graph, we need to precise the meaning of *exponential decay*:

**Definition 3.17.** We say that a sufficiently smooth function $f$ on the tree graph $T$ has *almost exponential (pointwise) decay rate* $\beta>0$ if for all $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$|f(x)|+|f'(x)|\le C_\varepsilon\mathrm{e}^{-(\beta-\varepsilon)d(o,x)}\tag{3.17}$$

for all $x\in T$ where $f'(x)$ is defined in (4.1) for $x\in V$.

*Remark* 3.18. (i) Due to the assumption (2.2) and since a generalized eigenfunction has the form (2.12) on the edge it suffices to ensure

$$|f(v)|+|f'(v)|\le C_\varepsilon\mathrm{e}^{-(\beta-\varepsilon)n}$$

for *vertices* $v\in V$ at generation $n$ only.

(ii) Note that if $f$ has almost exponential pointwise decay rate $\gamma>0$ on the half-line, then the associated radial function $\widetilde f$ on the tree graph with constant branching number $b\ge1$ has almost exponential pointwise decay rate $\gamma+(\ln b)/2$ due to the fact that in the symmetry reduction, we have the relation $f(d(o,x))=b^{(n-1)/2}\widetilde f(x)$ for $x$ in an edge at generation $n$.

Summarizing the results, we have shown:

**Theorem 3.19.** *Suppose that the random length quantum tree $T(\omega)$, respectively, the random Hamiltonian $H(\omega)$ on a radial quantum tree graph $T$ with branching number $b$, have a single site random perturbation with absolutely continuous and bounded distribution $\eta$ on $\Omega_1 = [\ell_-, \ell_+] \subset (0, \infty)$, respectively, $\Omega_1 = [q_-, q_+] \subset \mathbb{R}$. Suppose in addition, that $b > 1$ in the random length model (RLM) and that $b \geq 1$ in the random Kirchhoff model (RKM). Then the Kirchhoff Laplacian $\Delta_{T(\omega)}$, respectively, the Hamiltonian $H(\omega)$, has almost sure spectrum $\Sigma$ given in Theorem 3.4 and the spectrum is almost surely pure point. In addition, the eigenfunctions have almost exponential decay rate $\gamma(\lambda) + (\ln b)/2$ where $\gamma(\lambda)$ denotes the Lyapunov exponent.*

*Proof.* Clearly, the assertion is local in energy. Let $\Sigma_0 \subset (0, \infty)$ be a bounded interval. Due to Theorem 3.3 it suffices to consider $H_1(\omega)$ only. We have seen in Lemmas 3.12–3.13 that the Lyapunov exponent is positive almost everywhere on the almost sure spectrum $\Sigma$. Due to the assumptions on $\Omega_1$, (2.2) and (2.9) are fulfilled, so that the results on bounds on generalized eigenfunctions of Appendix C apply.

A proof of the spectral averaging assumption (E.1) is given in Corollaries E.7–E.8 for the RLM and RKM, respectively. The exceptional set $\Sigma_k$ in the RKM consists of the zeros of $\sin(\sqrt{\lambda})$, i.e., the Dirichlet spectrum of a single edge $e \cong (0, 1)$ with a security distance of order $1/k$. We finally apply Theorem 3.15 (taking Remark 3.16 into account) and the result follows. ☐

*Remark* 3.20.
(i) The case $b = 1$ in the RKM has been considered by Ishii [Ish73]. In this case, the almost sure spectrum is $[\inf \Sigma, \infty)$ where $\inf \Sigma \geq 0$ if $q \geq 0$ and $\inf \Sigma$ is given as the solution of $\operatorname{tr} T_{-\mu^2}(q_-) = 2$, i.e., $2\cosh(\mu) + q_- \sinh(\mu)/(\mu\sqrt{2}) = 2$ where $q_- = \inf \Omega_1$ and localization holds everywhere in $\Sigma$. Localization has been shown by Delyon, Simon and Souillard ([DSS85, Thm. 1.3. (i)]).
(ii) The case $b = 1$ in the RLM is of course uninteresting, since in this case, the tree Hamiltonian is the free Laplacian on $[0, \infty)$ with a Dirichlet boundary condition at 0 and has therefore purely absolutely continuous spectrum (see also Lemma 3.12).

## 4. General tree-like graphs

In this section, we show that our methods also apply to a more general class of metric graphs, namely to tree graphs, where an edge at generation $n$ is replaced by a decoration graph $G_n$. In this case, also the branching number $b = 1$ is of interest, since it includes line-like models like the necklace model considered in [KS04]. We only mention the necessary changes and begin with a general definition of radial tree-like graphs.

4.1. **Tree-like graphs.** We will construct a radial tree-like graph from a radial tree-graph $T = (V(T), E(T), \partial)$ by an edge decoration. We first need some notation for the decoration graph:

Let $G_* = (V(G_*), E(G_*), \partial)$ be a compact quantum graph. We fix two different vertices $o_0, o_1 \in V(G_*)$ sometimes called *boundary* or *connecting vertices* of the decoration graph $G_*$. In addition, we denote by $V_0(G_*) := V(G_*) \setminus \{o_0, o_1\}$ the the set of *inner* vertices of $G_*$.

Here, and in the sequel we use the abbreviation

$$f'(v) := f'_{G_*}(v) := \sum_{e \in E_v(G_*)} f'_e(v) \tag{4.1}$$

for the sum over the *inwards* derivative, i.e., $f'(v)$ is the *flux into the vertex* where $E_v(G_*)$ is defined in (2.1) and

$$f'_e(v) := \begin{cases} -f'_e(0), & \text{if } v = \partial_- e \\ f'_e(\ell_e), & \text{if } v = \partial_+ e \end{cases} \tag{4.2}$$

the *inward* derivative of $f_e$ at $v$. Note that $f'(v)$ depends on the graph; i.e., for a subgraph $S$ of $G_*$ or a graph $S$ containing $G_*$, we have in general $f'_S(v) \neq f'_{G_*}(v)$.

The Hilbert space $\mathsf{L}_2(G_*)$ associated to the decoration graph $G_*$ is given by $\mathsf{L}_2(G_*) := \bigoplus_{e \in E(G_*)} \mathsf{L}_2(e)$ with norm given as in (2.3). We define the Sobolev space of order 1 on $G_*$ as

$$\mathsf{H}^1(G_*) := \Big\{ f \in \bigoplus_{e \in E(G_*)} \mathsf{H}^1(e) \,\Big|\, f_{e_1}(v) = f_{e_2}(v), \quad \forall e_1, e_2 \in E(G_*), v \in V(G_*) \Big\} \tag{4.3}$$

with norm given by

$$\|f\|^2_{\mathsf{H}^1(G_*)} := \sum_{e \in E(G_*)} \big( \|f\|_e^2 + \|f'\|_e^2 \big). \tag{4.4}$$

The Sobolev space of order 2 on $G_*$ is then

$$\mathsf{H}^2(G_*) := \Big\{ f \in \bigoplus_{e \in E(G_*)} \mathsf{H}^2(e) \,\Big|\, f \in \mathsf{H}^1(G_*), \quad f'_{G_*}(v) = 0 \quad \forall v \in V_0(G_*) \Big\}. \tag{4.5}$$

with norm defined via

$$\|f\|^2_{\mathsf{H}^2(G_*)} := \sum_{e \in E(G_*)} \big( \|f\|_e^2 + \|f'\|_e^2 + \|f''\|_e^2 \big). \tag{4.6}$$

In particular, we pose the boundary conditions only at the *inner* vertices, *not* at the connecting vertices $o_0, o_1$. Hence, the differential operator $H_{G_*}$ acting on each edge as in (2.5) with domain $\mathsf{H}^2(G_*)$ is *not* self-adjoint.

We now define the edge decoration:

**Definition 4.1.** We say that a metric graph $G$ is obtained from a metric graph $T$ by an *edge decoration with a metric graph $G_*$ at the edge $t \in E(T)$* if we replace $t$ in $T$ by the graph $G_*$ where $\partial_\pm t \in V(T)$ is identified with two distinct vertices $o_0$, $o_1 \in V(G_*)$ ($o_0 \neq o_1$), i.e., $\partial_- t \cong o_0$ and $\partial_+ t \cong o_1$ (see Figure 2).

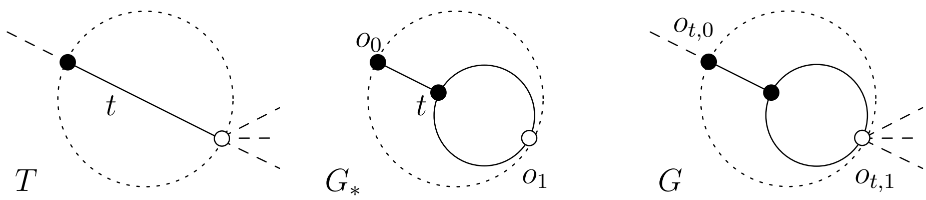


FIGURE 2. Decorating a graph $T$ with a graph $G_*$: The graph $T$ (solid and dashed) is decorated by replacing the edge $t \in E(T)$ with a graph $G_*$, and we call the new vertices $o_{t,j}$, $j = 0, 1$.

We embed $V(T) \hookrightarrow V(G)$ and $V(G_*) \hookrightarrow V(G)$. If e.g. $G_*$ consists of a single edge $e$ only, the edge decoration with $G_*$ does not change the original graph $T$.

**Definition 4.2.** A *tree-like* metric graph associated to a tree graph $T$ is a graph $G$ obtained from a (generally infinite) tree graph $T$ by edge decoration with $G_*(t)$ at each tree edge $t \in E(T)$. A *radial tree-like* metric graph is a tree-like graph $G$ where the decoration graph $G_*(t)$ depends only on the generation of $t$, i.e., there exists a sequence of compact metric graphs $\{G_n\}_n$ such that $G_n = G_*(t)$ for all $t \in E(T)$ with $\operatorname{gen} t = n$.

We label $o_{t,0} = \partial_- t$ and $o_{t,1} = \partial_+ t$ the start/end vertex of $t \in E(T)$ considered as vertices in the decoration graph $G_*(t)$. Obviously, a radial tree-like metric graph is determined by the sequence of decoration graphs $\{G_n\}_n$, including the edge lengths, and the sequence of branching numbers $\{b_n\}_n$.

The notion extends to *quantum* graphs, i.e., metric graphs with a Hamiltonian. Another notation for the right/left "derivative" at the connecting vertices $o_0$ and $o_1$ of $G_*$ will be useful, namely

$$f^\dagger(v) := -\sum_{e \in E_v(G_*)} f'_e(v) = -f'(v) \qquad \text{at } v = o_0 \tag{4.7a}$$

$$f^\dagger(v) := \sum_{e \in E_v(G_*)} f'_e(v) + q(v)f(v) = f'(v) + q(v)f(v) \qquad \text{at } v = o_1 \tag{4.7b}$$

with the notation $f'$ introduced in (4.1)–(4.2). Here $q(v)$ denotes the vertex potential strength at the vertex $o_1$. For simplicity, we assume that the vertex potential has support only at the vertex $o_1$, i.e., $q$ is determined by the single number $q(o_1) \in \mathbb{R}$. The different signs for the vertex $o_0$ and $o_1$ are due to our convention in (4.2) considering always the *inward* derivative at a vertex. This notation allows us to express the boundary condition for the Hamiltonian of a *radial tree-like quantum graph* in a simple way (see also Remark 4.4):

**Definition 4.3.** A *radial tree-like quantum graph* is a radial tree-like metric graph $G = (V(G), E(G), \partial, \ell)$ together with a vertex potential strength $q \colon V(T) \longrightarrow \mathbb{R}$ such that there exists a sequence $\{q_n\}_n$ with $q(v) = q_n$ for all vertices $v \in V(T)$ in generation $n$ of the underlying tree.[2] The corresponding Hamiltonian $H = H_G$ is given by

$$(Hf)_e = -f''_e \tag{4.8}$$

on each edge, for functions $f \in \operatorname{dom} H_G$, where $\operatorname{dom} H_G$ is the set of those functions $f$ such that $f, f'' \in \mathsf{L}_2(G) = \bigoplus_{t \in E(T)} \mathsf{L}_2(G_*(t))$ such that $f = \{f_t\}_t$ with $f_t := f\restriction_{G_*(t)}$ satisfies

$$f(o) = 0 \qquad \text{and} \qquad f_t \in \mathsf{H}^2(G_*(t)), \quad t \in E(T) \tag{4.9}$$

(in particular, $f_t$ satisfies the inner boundary conditions as in (4.3) and (4.5)), and

$$f_{t_1}(v) = f_{t_2}(v) \qquad \text{and} \qquad f^\dagger_{t_1}(v) = f^\dagger_{t_2}(v) \tag{4.10}$$

for all $t_1, t_2 \in E(T)$ meeting in a common *tree* vertex $v \in E(T) \subset E(G)$.

*Remark* 4.4. The previous characterization of the boundary condition explains why we introduced the notion (4.7). Note that the vertex potential strength is hidden in the notation. In the case when each decoration graph $G_*$ is a single edge $(0,1)$ without vertex potentials, $f^\dagger(v)$ for $v = o_0$ and $v = o_1$ is just the usual right and left derivative of $f$, respectively.

Clearly, a radial tree-like quantum graph is determined by the sequence of quantum graphs $\{G_n\}$, the sequence of vertex potential strength $\{q_n\}_n$ and the sequence of branching numbers $\{b_n\}$. We mention some examples falling into the class of radial tree-like metric graphs:

**Example 4.5.** (i) *Simple tree graphs:* The simplest example of a tree-like graph is of course a tree graph itself. A radial tree graph is completely determined by the sequences of edge lengths $\{\ell_n\}$, vertex potential strengths $\{q_n\}$ and branching numbers $\{b_n\}_n$ where $b_n \geq 1$.

[2] For simplicity, we assume that there is only one vertex potential on each decoration graph $G_*(t)$, located at the ending point $o_{1,t}$.

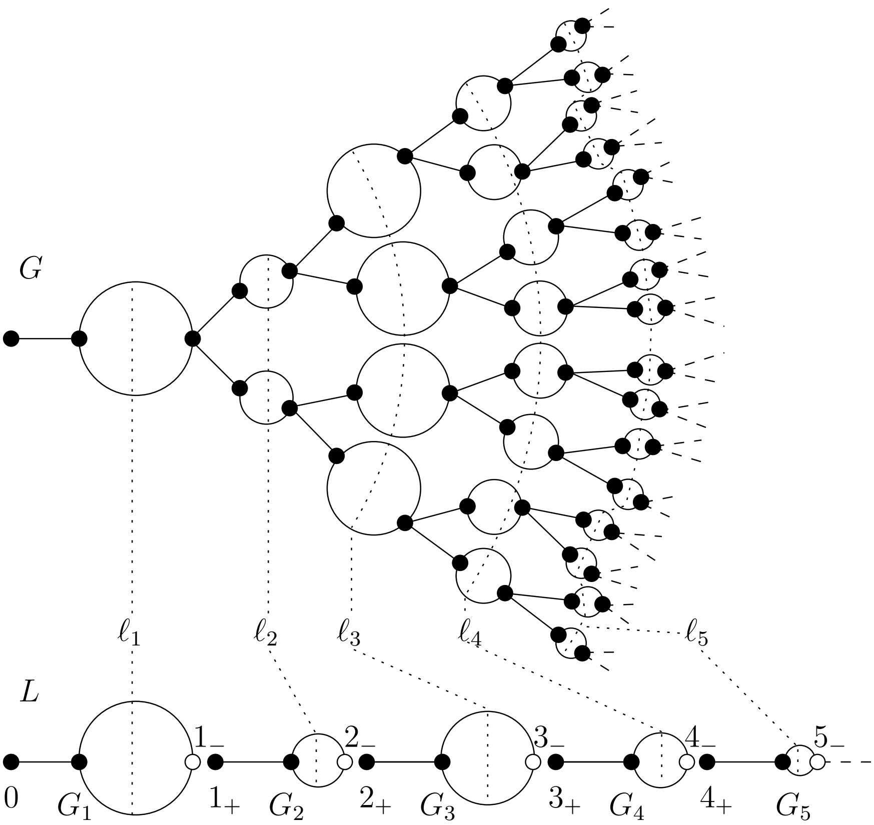


FIGURE 3. A tree-like graph $G$ with branching number $b = 2$ and a necklace decoration with $p = 2$ as in Example 4.5 (iii). The random variable $\ell_n$ in each generation $n$ is the length of the edges of the necklace decoration.

(ii) *Graph decoration at the ending point:* (a) Let $\hat{G}_*$ be a finite graph. If we attach an edge $e$ of length $\ell_e \geq 0$ to $\hat{G}_*$ we obtain a decoration graph $G_* = \hat{G}_* \cup \{e\}$ with starting point $o_0$ being the free end of the attached edge and with ending point being any vertex of the decoration graph (even the other vertex of the attached edge).

(b) For example, if $\hat{G}_*$ consists of a loop of length 1, we obtain a decoration of the radial tree graph with base edge of length $\ell_n$ and a decoration loop of length 1 at each generation $n$.

We refer to this model as the *loop decoration model.*

(iii) *Necklace or onion decoration:* If $G_*$ consists of an edge $e_0$ of length 1 starting at $o_0$ together with $p$ edges of length $\ell$ joining the ending point of $e_0$ with the ending vertex $o_1$, we obtain a *(branched, half-line) onion* or *necklace decoration model* ($p = 2$). Clearly, the decorated tree graph is determined by the sequence of lengths $\{\ell_n\}_n$, the (constant) edge number $p$ and the (constant) branching number $b$ (see Figure 3). We will allow that the length of the loop is 0, i.e., $\ell_n \in [0, \ell_+]$, in the sense that the loop degenerates to a single vertex.

(iv) *Line graphs:* If the branching numbers $b_n$ all equal to 1, we obtain a line-like graph. For example, the previous (half-line) necklace decoration model is similar to the model already considered in [KS04] (see also Section 5.3).

(v) *Kirchhoff models:* We can add a vertex potential at the ending vertex $o_1$ of a fixed decoration graph $G_*$. The corresponding decorated tree graph has the same decoration graph at all steps, but a sequence of vertex potential strength $\{q_n\}_n$ at a vertex of generation $n$.

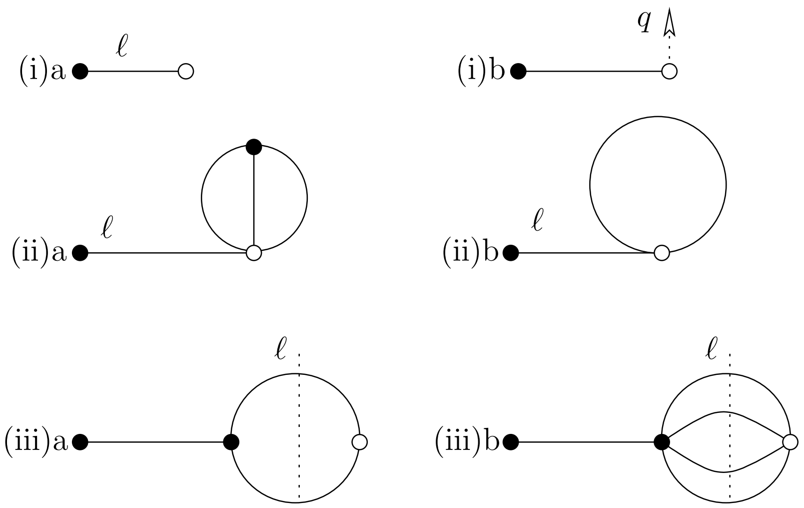


FIGURE 4. The decoration graphs of Example 4.5 together with the (random) parameter: (i) the simple RLM resp. RKM model; (ii)a decoration with a onion; (ii)b decoration with a loop at the ending point; (iii)a necklace/onion decoration.

We will give some natural conditions on the parameters in order that our examples satisfy the needed assumptions on the decoration graphs $\{G_n\}$ given in Assumption 4.8.

4.2. **Line-like graphs and symmetry reduction.** As on a simple quantum tree graph, we can profit from the symmetry reduction (see Section 4.2). To do so, we need the notion of a *line-like* graph associated to the sequence $\{G_n\}_n$ of decoration graphs and the sequence of branching numbers $\{b_n\}_n$.

On each decoration graph $G_n$, we specify two different vertices $(n-1)_+ = o_{n,0}$, $n_- = o_{n,1} \in V(G_n)$. We sometimes simply write $n-1 = (n-1)_+$ or $n = n_-$ if it is clear that they belong to $V(G_n)$ (e.g., $0 = 0_+$).

**Definition 4.6.** A *line-like metric graph* $L_n = L_{n,\infty}$ starting at $n$ is obtained from the union of $G_k$, $n < k$ by identifying $k_+ \in V(G_k)$ with $k_- \in V(G_{k+1})$ for $n < k$. Similarly, we denote by $L_{n_0,n_1}$ the line-like graph obtained as concatenation of $G_k$, $n_0 < k \le n_1$, and set $L := L_0$ for the entire line-like graph.

The norm on $\mathsf{L}_2(L_n)$ is defined by

$$\|f\|^2_{L_n} := \sum_{k>n} \|f\|^2_{G_k}.$$

Clearly, $L_n$ is determined by the sequence of graphs $\{G_k\}_{k>n}$. Similarly, the notion of a line-like quantum graph can be defined:

**Definition 4.7.** A *line-like quantum graph* is given by a line-like metric graph $L_n = (V(L_n), E(L_n), \partial, \ell)$ together with a sequence $\{b_k\}_{k>n}$ of positive numbers and a sequence of vertex potential strength $\{q_k\}_{k>n} \subset \mathbb{R}$. The corresponding Hamiltonian $H_{L_n}$ acts on each edge as in (4.8) for functions $f \in \operatorname{dom} H_{L_n}$ where $\operatorname{dom} H_{L_n}$ is the set of those functions $f$ such that $f, f'' \in \mathsf{L}_2(L_n) = \bigoplus_{k>n} \mathsf{L}_2(G_k)$ such that $f = \{f_k\}_k$ with $f_k := f\restriction_{G_k}$ satisfies

$$f_1(n) = 0 \qquad \text{and} \qquad f_k \in \mathsf{H}^2(G_k), \quad k > n \tag{4.11}$$

(in particular, $f_k$ satisfies the inner boundary conditions as in (4.3) and (4.5)), and

$$f_k(k_-) = b_k^{-1/2} f_{k+1}(k_+) \qquad \text{and} \qquad f_k^\dagger(k_-) = b_k^{1/2} f_{k+1}^\dagger(k_+) \tag{4.12}$$

for all $k > n$. Again, the vertex potential strength $q_k$ is hidden in (4.12) in the symbol $f_k^\dagger(k_-)$ (cf. (4.7)).

We can now associate a line-like metric graph $L = L_0$ to a radial tree-like metric graph: Let $\{t_k\} \subset E(T)$ be an infinite path in the tree graph $T$ such that $\operatorname{gen} t_n = n$. In particular, the path starts at $\partial_- t_1 = o$. We denote $L_0$ the quantum subgraph of $G$ corresponding to the path $\{t_k\}$ in $T$. Similarly, let $L_n$ be the quantum subgraph of $L_0$ starting at generation $n$. For example if $G$ is a simple tree graph, $L_n$ is isometric to the half line $[0, \infty)$. In general, $L_n$ is isometric to the concatenation of the decoration graphs $G_k := G_*(t_k)$ where $k_- = o_{t_k,1} \cong o_{t_{k+1},0} = k_+$ are identified ($k > n$). Clearly, $L_n$ is a line-like graph. Similarly, a tree-like quantum graph, i.e., a tree-like metric graph $G$ with Hamiltonian $H_G$ determines uniquely a sequence of line-like quantum graphs $\{L_n\}_n$ with Hamiltonians $\{H_{L_n}\}_n$.

We will see that the converse is also true: The family $\{H_{L_n}\}_n$ of Hamiltonians on the line-like graphs $L_n$ determines uniquely the (spectral) behavior of the quantum graph $G$ with Laplacian $H_G$. Namely, due to the symmetry reduction in Theorem A.6, we can reduce the spectral analysis of $H_G$ on $G$ to the analysis of the family $\{H_{L_n}\}_n$ on the line-like graphs $L_n$. Note that as in the simple tree graph case, the functions on the tree-like graph and on the line-like graph differ by a weight factor (although we denoted both by $f$), see Remark 2.2.

We will need some assumptions on the decoration graphs $\{G_n\}$ and the vertex potential strength — like the assumptions (2.2) and (2.9) for a simple tree graph — for example to ensure the self-adjointness of $H_G$ and $H_L$, and the bounds on generalized eigenfunctions.

**Assumption 4.8.** We say that the sequence of quantum decoration graphs $\{G_n\}_n$ is *uniform* if there exist finite constants $\ell_\pm > 0$, $0 < \kappa \le 1$ and $q_\pm > 0$ such that each member $G_* = (V, E, \partial, \ell, q(o_1)) \in \{G_n\}_n$ satisfies the following conditions:

$$d_{G_*}\big(o_0, o_1\big) \ge \ell_-, \tag{4.13a}$$

$$\ell_e \ge \kappa \ell_-, \qquad e \in E_{o_0} \tag{4.13b}$$

$$\deg o_0 = 1 \tag{4.13c}$$

$$\ell(G_*) := \sum_{e \in E(G_*)} \ell_e \le \ell_+ \tag{4.13d}$$

$$q_- \le q(o_1) \le q_+. \tag{4.13e}$$

The first three assumptions assure that each decoration graph $G_*$ is "long" enough and does not branch at the starting vertex $o_0$ (this will be needed in order to calculate the Green's function, cf. Lemma D.15). The fourth condition is a global upper bound on the decoration graph (cf. (2.2)). Assumption (4.13e) is a global bound on the strength of the vertex potential (cf. (2.9)).

Note that all our assumptions are fulfilled on a simple tree graph, i.e., when $G_n$ consists of a single edge with vertex potential at the ending vertex (Example 4.5 (i)). The same is true for Example 4.5 (iii). In addition, in Example 4.5 (ii), the assumptions are fulfilled once there is a lower bound on the base edge length $\ell_n \ge \ell_- > 0$ or the end vertex $n_-$ of $G_n$ does not lie on the base edge. In Example 4.5 (v) we only need to assure that in the (constant) decoration graph $G_*$ the vertex $o_0$ has degree 1 and a bounded vertex potential (cf. (4.13e)).

We summarize the various results needed later which are proven in the appendix (cf. Theorem A.6, Lemma C.10, Thms. C.17–C.18).

**Theorem 4.9.** *Assume that the sequence of decoration graphs $\{G_n\}$ is uniform and of polynomial length growth (i.e, it satisfies Assumptions* (4.13)*). Assume in addition that $G$ is the radial tree-like quantum graph with decoration graphs $\{G_n\}$ and branching number sequence $\{b_n\}$. Denote by $L_n$ the associated line-like graph $L_n$ starting at vertex $n$ and by $H_{L_n}$ its Hamiltonian. Then $H_G$ defined in Definition 4.3 is self-adjoint on* $\operatorname{dom} H_G$*. Furthermore,*

$$H_G \cong H_1 \oplus \bigoplus_{n=2}^{\infty} (\oplus b_0 \cdot \ldots \cdot b_{n-2}(b_{n-1}-1))H_n$$

*where $(\oplus m)H_n$ means the $m$-fold copy of $H_n$. Each operator $H_n = H_{L_{n-1}}$ is self-adjoint on* $\operatorname{dom} H_{L_n}$ *as defined in Definition 4.7.*

*In addition, the spectrum of $H_n$ is supported by polynomially bounded generalized eigenfunctions $\varphi$. More precisely, $\varphi(k_+)$ and $\varphi^\dagger(k_+)$ are bounded by $k$ times a constant depending only on the constants of* (4.13) *and the eigenvalue.*

4.3. **Transfer matrices.** As in the tree graph case, we need control over the growth of generalized eigenfunctions of the Hamiltonian $H = H_L$ of a line-like graph. A generalized eigenfunction here is a function satisfying

$$Hf = -f'' = \lambda f \tag{4.14}$$

on each edge such that $f$ satisfies all inner boundary conditions (i.e., $f\restriction_{G_n} \in \mathsf{H}^2(G_n)$) and all connecting boundary conditions (4.12) except at 0 (and there is no integrability condition at $\infty$).

We can calculate the solutions explicitly, since on each edge, the solution still has the form (2.12) with coefficients determined by the boundary conditions. Namely, we can define the *transfer* or *monodromy matrix* $T_\lambda(n)$ for the decoration graph $G_n$ as follows: For a given $\vec{F}(n-1) = (F_{n-1}, F'_{n-1})^{\mathrm{tr}} \in \mathbb{C}^2$ let $f$ be a solution of (4.14) such that $f \in \mathsf{H}^2(G_n)$, i.e., $f$ satisfies all inner boundary conditions (cf. (4.5)) and $f((n-1)_+) = F_{n-1}$ and $f^\dagger((n-1)_+) = F'_{n-1}$. The transfer matrix is then defined as in (2.17) via

$$\vec{F}(n) = T_\lambda(n)\vec{F}(n-1), \qquad \text{and} \qquad \vec{F}(0) \in \mathbb{C}\begin{pmatrix} 0 \\ f^\dagger(0+) \end{pmatrix}. \tag{4.15}$$

where

$$\vec{F}(n) := \vec{F}(n_+) := \begin{pmatrix} f(n_+) \\ f^\dagger(n_+) \end{pmatrix}. \tag{4.16}$$

We sometimes write

$$T_\lambda(x, G_n)\vec{F}(n-1) = f(x) \tag{4.17}$$

for the solution $f$ of the eigenvalue equation on $G_n$ with initial data $\vec{F}(n-1)$.

Note that in contrast to the simple tree graph case where $G_n$ is a single edge, the transfer matrix might not be defined for all energies $\lambda \in \mathbb{C}$. We specify an exceptional set $E(G_n)$ in (D.11) for which the transfer matrix might not be defined. The set $E(G_n)$ roughly consists of the spectrum of the Dirichlet operator on $G_n$, i.e., the self-adjoint operator $H^{\mathrm{D}}_{G_n}$ with boundary condition $f((n-1)_+) = 0$ and $f(n_-) = 0$. In addition, there might be more exceptional energies expressed via the Dirichlet-to-Neumann map on $G_n$. We call the values in $E(G_n)$ the *exceptional energies* of $G_n$. The exceptional set $E(L)$ of the line-like graph $L$ consisting of the concatenation of all $G_n$'s is the union of all exceptional sets. In particular, if $\lambda \notin E(L)$, then the transfer matrix $T_\lambda(n)$ is uniquely defined as below and has determinant 1 (Lemma D.6).

In some concrete examples, it is easier to directly determine the set of values $\lambda$ for which the transfer matrix is not defined. The direct calculation has the advantage, that the set of values for which $T_\lambda(n)$ is not defined may be smaller than the set $E(L)$ defined abstractly in Definition D.9. This phenomena occurs for the simple tree graph: The abstract setting would yield the Dirichlet spectrum of a single edge, namely $E(L) = \{\, \pi^2 k^2/\ell_n^2 \mid k \in \mathbb{N}, n \in \mathbb{N}\,\}$, but the direct calculation of Section 2.4 shows, that the transfer is defined for all values of $\lambda$ (cf. also Lemma D.6).

We will give the transfer matrices and Dirichlet-to-Neumann maps for the examples cited below:

**Example 4.10.** (ii) *Graph decoration at the ending point:* The transfer matrix of the decoration graph associated to the energy $\lambda$ is given by

$$T_\lambda(n) = D(b) T_\lambda(\hat{G}_*) R_\mu(\mu \ell_n) \tag{4.18}$$

where $T_\lambda(\hat{G}_*)$ is the transfer matrix with respect to the decoration graph $G_*$ and where $\mu = \sqrt{\lambda}$.

If $G_*$ is a graph attached to the end point of the edge (i.e., the connecting points $(n-1)_+$ and $n_-$ lie on the base edge), then $T_\lambda(\hat{G}_*) = S(r_\lambda)$ where $r_\lambda = r_\lambda(\hat{G}_*)$ is the Dirichlet-to-Neumann map associated to the graph $\hat{G}_*$ with $o_1$ as single boundary point, i.e., $r_\lambda = \varphi^\dagger(o_1)$ where $\varphi$ is the unique solution of $H_{G_*}\varphi = \lambda\varphi$ with $\varphi(o_1) = 1$. The Dirichlet-to-Neumann map is defined for all $\lambda \notin E(G_*)$ where $E(G_*)$ is the spectrum of the Dirichlet Hamiltonian $H^{\mathrm{D}}_{G_*}$ (with Dirichlet boundary condition at $o_1 \in V(\hat{G}_*)$). The transfer matrix is similar to the one of the RKM, i.e.,

$$\begin{aligned} T_\lambda(n) &= D(b) S(r_\lambda) R_\mu(\mu\ell_n) \\ &= \begin{pmatrix} b^{1/2}\cos(\mu\ell_n) & \dfrac{b^{1/2}}{\mu}\sin(\mu\ell_n) \\ \dfrac{-\mu\sin(\mu\ell_n) + r_\lambda\cos(\mu\ell_n)}{b^{1/2}} & \dfrac{\cos(\mu\ell_n) + \dfrac{r_\lambda}{\mu}\sin(\mu\ell_n)}{b^{1/2}} \end{pmatrix}, \end{aligned} \tag{4.19}$$

but now with an energy depending vertex potential strength and the random parameter being a length perturbation. The Dirichlet-to-Neumann map is

$$\Lambda(\ell, \lambda) = \mu \begin{pmatrix} -\cot\mu\ell & \dfrac{1}{\sin\mu\ell} \\ -\dfrac{1}{\sin\mu\ell} & \cot\mu\ell + \dfrac{r_\lambda}{\mu} \end{pmatrix}. \tag{4.20}$$

Concretely, in the loop decoration model (with a loop of length 1), we have

$$T_\lambda(n) = D(b) S(r_\lambda) R_\mu(\mu\ell_n) \quad \text{with} \quad r_\lambda = -2\mu\tan(\mu/2) \tag{4.21}$$

with exceptional set $E(\ell) = \{\, \pi^2 k^2 \mid k \in \mathbb{N}\,\}$ independent of $\ell$.

(iii) *Necklace or onion decoration:* Here, the transfer matrix is

$$\begin{aligned} T_\lambda(n) &= D(b) R_{p\mu}(\ell_n\mu) R_\mu(\mu) \\ &= \begin{pmatrix} b^{1/2}\cos_{\frac{1}{p}}(\mu, \ell_n) & \dfrac{b^{1/2}\sin_p(\mu,\ell_n)}{p\mu} \\ -b^{-1/2} p\mu \sin_{\frac{1}{p}}(\mu,\ell_n) & b^{-1/2}\cos_p(\mu,\ell) \end{pmatrix} \end{aligned} \tag{4.22}$$

defined for all $\lambda > 0$ where

$$\sin_p(\mu,\ell) := \sin(\mu\ell)\cos\mu + p\cos(\mu\ell)\sin\mu \tag{4.23a}$$

$$\cos_p(\mu,\ell) := \cos(\mu\ell)\cos\mu - p\sin(\mu\ell)\sin\mu. \tag{4.23b}$$

The Dirichlet-to-Neumann map is

$$\Lambda(\ell, \lambda) := \frac{p\mu}{\sin_p(\mu, \ell)} \begin{pmatrix} -\cos_{\frac{1}{p}}(\mu, \ell) & 1 \\ -1 & \cos_p(\mu, \ell) \end{pmatrix} \tag{4.24}$$

with exceptional set $E(\ell)$ consisting of the Dirichlet spectrum of the decoration graph (cf. Figure 5), i.e., of those $\lambda = \mu^2$ such that $\sin_p(\mu, \ell) = 0$ or $\sin(\mu\ell) = 0$.[3]

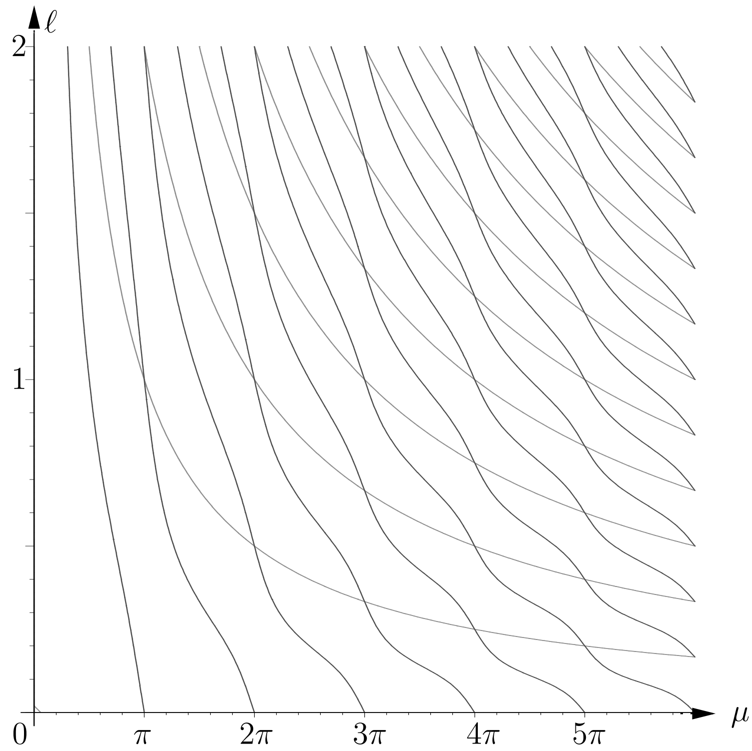


FIGURE 5. The Dirichlet spectrum of the necklace decoration ($p = 2$). The zeros of $\sin_p(\mu, \ell)$ are plotted in dark grey, the zeros of $\sin(\mu\ell)$ are plotted in light grey.

(iv) *Line-like graphs:* Here, we have

$$T_\lambda(n) = T_\lambda(G_n) \tag{4.25}$$

where $T_\lambda(G_n)$ is the transfer matrix of the (random) decoration graph $G_n$.

(v) *Kirchhoff models:* The transfer matrix is just

$$T_\lambda(n) = D(b)S(q_n)\hat{T}_\lambda(G_*) \tag{4.26}$$

where $q_n$ denotes the strength of the vertex potential at $n_-$ and where $\hat{T}_\lambda(G_*)$ is the transfer matrix of the fixed decoration graph $G_*$.

We end this section with a typical example for the *periodic* spectrum. Note that the onion decoration was also considered in [AEL94] as a line-like graph ($b = 1$) considering the band-gap ratio of the periodic operator.

[3] Note that the Dirichlet spectrum of the necklace decoration graph consists of the squares $\lambda = \mu^2$ of the zeros of $\sin_p(\mu, \ell) = 0$ *and* of $\sin(\mu\ell) = 0$ (cf. Figure 5). The latter zeros correspond to eigenfunctions living only on the loop. These zeros do not appear as poles in the Dirichlet-to-Neumann map, since in its definition, the end vertices of the loop edges are identified as one vertex $o_1$ (cf. also Lemma D.3).

**Theorem 4.11.** *Suppose that all length are the same $\ell_n = \ell$ in the necklace/onion decoration model Example 4.5 (iii) with branching number $b \geq 1$ and edge decoration number $p \geq 2$. Then the spectrum of the corresponding Laplacian on the decoration graph $G = G(\ell)$ is given by*

$$\sigma(\Delta_{G(\ell)}) = \{\, \mu^2 \mid |(b^{1/2} + b^{-1/2}) \cos_\kappa(\mu, \ell)| \leq 2 \,\} \cup \{\, \mu^2 \mid \sin_p(\mu, \ell) = 0 \,\}$$
$$= \bigcup_{k=1}^{\infty} \big(B_k(\ell) \cup \{\lambda_k(\ell)\}\big), \qquad \kappa := \frac{b + p^2}{p(b+1)}, \quad (4.27)$$

*where $B_k(\ell)$ are compact intervals and $\lambda_k(\ell)$ is the $k$th Dirichlet eigenvalue of a single decoration graph with length $\ell$ (cf.* (4.23) *for the notation* $\cos_p$ *etc.).*

*Proof.* The spectral characterization is a simple consequence of Theorem 2.3 and (4.22). Note that $\operatorname{tr} T_\lambda$ is nonconstant. ☐

*Remark* 4.12. (i) The square roots of the band edges as functions of $\ell$ (i.e., the solutions $\mu = \mu_k^\pm(\ell)$ of the equation $(b^{1/2} + b^{-1/2}) \cos_\kappa(\mu, \ell) = \pm 2$) satisfy

$$\mu_k'(\ell) = -\mu\Big(\ell + \kappa \frac{\sin_{\frac{1}{\kappa}}(\mu, \ell)}{\sin_\kappa(\mu, \ell)}\Big)^{-1}.$$

Numerical examples show that $\mu_k'(\ell) < 0$, i.e, that the band edges are monotonically decreasing in $\ell$. Furthermore, if $b$ or $p$ are very large, the bands are very narrow. In addition, for small $b$ and large $p$ (i.e, if $\kappa \gg 1$), the bands are almost constant if $\ell$ is not an integer.

(ii) In the case $b = p$, i.e., if the branching number equals the number of decoration edges in the loop, we have an interesting phenomena: First, $\kappa = 1$ and $\mu_k'(\ell) = -\mu(\ell + 1)^{-1} < 0$. Furthermore, $\cos_\kappa(\mu, \ell) = \cos(\mu(\ell + 1))$ and $\sin_\kappa(\mu, \ell) = \sin(\mu(\ell + 1))$, i.e., the absolutely continuous spectrum is exactly the same as for the RLM with length $\ell + 1$ (cf. Theorem 2.5). In this sense, the transport properties of the branched necklace model and the simple RLM are the same, i.e., for the transport properties, it is irrelevant, whether there are loops or the loops are opened at the end point (in order to obtain a RLM with length $\ell + 1$).

## 5. Localization for random tree-like quantum graphs

### 5.1. **General random models.**

Here, we assume that the symmetric, radial tree-like quantum graph $G = (V, E, \partial, \ell, q)$ which is completely determined by the sequence of decoration graphs $\{G_n\}_n$ together with the sequence of branching numbers $\{b_n\}$ is random in the following sense:

**Definition 5.1.** Let $\mathcal{G}$ be a family of compact quantum decoration graphs. We say that the radial tree-like quantum graph $G$ is constructed *randomly* from the set $\mathcal{G}$, if there is an iid sequence of random variables $\{G_n, b_n\}_n$ with values in $\mathcal{G}$ and $\{b_-, \dots, b_+\}$, respectively, such that $G(\omega)$ has the decoration graph $G_n(\omega)$ and the branching number $b_n(\omega)$ at generation $n$. Similarly, the sequence of iid random variables $\{G_n, b_n\}$ determines a *random line-like quantum graph.*

We fix a probability measure $\mathbb{P}_1$ on $\Omega_1 := \mathcal{G} \times \{b_-, \dots, b_+\}$. Clearly, we can consider a random radial tree-like or line-like quantum graph $G(\omega)$ or $L(\omega)$ as a random variable on the product measure space $(\Omega, \mathbb{P}) := (\Omega_1, \mathbb{P}_1)^{\mathbb{N}}$.

We are mostly interested in *minimal* random models, since one expects localization at least for high disorder. In all our application, the class of decoration quantum graphs $\mathcal{G}$ will depend only on one real parameter. For example in the RLM, $\Omega_1$ consists of quantum graphs $G = G(\ell)$ of a single edge and fixed branching number $b$. The random parameter is the length, so we can set $\ell \in [\ell_-, \ell_+] =: \Omega_1$. In the RKM, we have a similar model, now $\Omega_1 := [q_-, q_+]$.

In order to copy the proof of localization of Kotani as in Theorem 3.15, we need some further adaptations, mainly due to the fact, that several constants tend to $\infty$ if we approach the exceptional set. Here, and in the sequel, $\Lambda_{ij}(\omega_1, \lambda)$ are the components of the Dirichlet-to-Neumann map of the decoration graph $G(\omega_1)$ defined in Definition D.2.

We need more assumptions for the random model. Let $\Sigma_0 \subset \mathbb{R}$ be a bounded interval.

**Assumption 5.2.** We say that the random radial tree-like graph $G = G(\omega)$ with decoration graphs in $\mathcal{G}$ is *good in the compact spectral interval* $\Sigma_0$ if the following conditions are fulfilled:

(i) The family of decoration graphs $\mathcal{G}$ is *uniform*, i.e., each decoration graph $G(\omega_1) \in \mathcal{G}$ satisfies (4.13) $\mathbb{P}_1$-almost surely with *uniform* constants.

(ii) The single site probability space $\Omega_1$ is the union of finitely many compact intervals with its Borel $\sigma$-algebra, and the eigenvalues of the Dirichlet Laplacian $\Delta^{\mathrm{D}}_{G(\omega_1)}$ depend piecewise analytically on $\omega_1$. In addition, the Dirichlet-to-Neumann map $\Lambda(\omega_1, \lambda)$ depend analytically on $\omega_1$ whenever $\lambda \notin \sigma(\Delta^{\mathrm{D}}_{G(\omega_1)})$. Both maps are assumed to be continuous up to the border of $\Omega_1$.

(iii) We assume that the exceptional set $E(G(\omega_1)) \subset \mathbb{R}$ of each decoration graph (cf. (D.11)) is discrete.

(iv) There is a constant $C = C(\lambda)$ such that[4]
$$\mathbb{E}_1(\ln \|T_\lambda(\cdot)\|) \le C_\lambda$$

(v) There exists an increasing sequence of real numbers $\{\lambda_k\}$ and for each $k$ a sequence $\{\delta_{k,n}\}_n$, $\delta_{k,n} \to 0$ as $n \to \infty$ such that the spectral averaging formula (E.1) holds in the compact energy interval $[\lambda_k + \delta_{k,n}, \lambda_{k+1} - \delta_{k+1,n}] \cap \Sigma_0$.

*Remark* 5.3. (i) We believe that Assumption 5.2 (iii) is generally true (under some mild conditions), although we are not aware of a proof. Since this condition is always satisfied in our examples, we state it as an assumption (see also Remark D.7 (i)).

(ii) Assumption 5.2 (v) is usually fulfilled only for models if the single site random distribution is absolutely continuous, i.e., if there is a nonnegative function $\eta \in \mathsf{L}_\infty(\Omega_1)$ such that $\mathrm{d}\mathbb{P}_1(\omega_1) = \eta(\omega_1)\,\mathrm{d}\omega_1$.

Typically, the sequence $\{\lambda_k\}_k$ consists of the Dirichlet spectrum of the decoration graph (with length $\ell \in \partial\Omega_1$ in random lengths models). In some random lengths models, the exceptional set is not needed, e.g. in Example 4.5 (ii) or the RLM of Section 3.

We set
$$E_0 := \{\, (\omega_1, \lambda) \mid \lambda \in E(G_*(\omega_1)) \,\}. \tag{5.1}$$
The next lemma assures that $E_0$ is still a "small" set:

**Lemma 5.4.** *There exists $a > 0$ such that*
$$E_k := \Big\{\, (\omega_1, \lambda) \in \Omega_1 \times \Sigma_0 \;\Big|\; \operatorname{dist}(\lambda, \sigma(H^{\mathrm{D}}_{G(\omega_1)})) < \eta_k \text{ or } |\Lambda_{01}(\omega_1, \lambda)| < \eta_k \,\Big\}$$
*with $\eta_k = k^{-2a}$ fulfills $\sum_k (\mathbb{P}_1 \otimes \boldsymbol{\lambda})(E_k) < \infty$. Furthermore, $E_0 := \bigcap_k E_k$ and $(\mathbb{P}_1 \otimes \boldsymbol{\lambda})(E_0) = 0$.*

[4]Note that $\|A\| = \|A^{-1}\|$ for $A \in \mathrm{SL}_2(\mathbb{R})$ (see also (3.4)).

Note that $E(G_*(\omega_1))$ is a *closed* set (and therefore measurable) and that it consists of $\sigma(H^{\mathrm{D}}_{G(\omega_1)})$ and those $\lambda \in \Sigma_0$ such that $\Lambda_{01}(\omega_1, \lambda) = 0$.

*Proof.* By assumption, the Dirichlet eigenvalues $\lambda_k(\omega_1)$ depend piecewise analytically on $\omega_1$ and that $\Lambda_{01}(\omega_1, \lambda)$ is analytic (by assumption it is analytic in $\omega_1$ and by the series representation (D.9) it is also analytic in $\lambda$). Therefore, the thickened exceptional set $E_k$ lies in a strip of order $k^{-2}$ around $E_0$ if we choose $\eta_k = k^{-2a}$ for some $a > 0$. Since $\Omega_1 \times \Sigma_0$ is compact, the sum over the measures is finite. The second assertion is an easy consequence of the first Borel-Cantelli lemma, see for example [S79, Thm. 3.1]. $\square$

Next, we need several lemmas ensuring that we have a *global* $\mathsf{L}_2$-estimate as in (2.20) on the eigenfunction $T_\lambda(\cdot, G(\omega_1))\vec{F}_0$ defined in (4.17) on a sufficiently large subset of $\Omega \times \Sigma_0$:

**Lemma 5.5.** *There exists a sequence $\{C'_k\}_k$ growing at most polynomially such that*

$$E'_k := \Big\{ (\omega_1, \lambda) \in \Omega_1 \times \Sigma_0 \setminus E_0 \,\Big|\, \exists \vec{F}_0 \neq \vec{0}\colon\ \|T_\lambda\big(\cdot, G(\omega_1)\big)\vec{F}_0\| > C'_k|\vec{F}_0| \Big\} \tag{5.2}$$

*satisfies $E'_k \subset E_k$. In particular, $E'_0 := \bigcap_k E'_k \subset E_0$ has $(\mathbb{P}_1 \otimes \boldsymbol{\lambda})$-measure 0.*

*Proof.* Let

$$\widetilde{E}_k := \bigcup_{i,j=0,1} \{\, (\omega_1, \lambda) \in \Omega_1 \times \Sigma_0 \setminus E_0 \,|\, |\Lambda_{ij}(\omega_1, \lambda)| > \widetilde{C}_k \,\} \tag{5.3}$$

where

$$\widetilde{C}_k := \sup\{\, |\Lambda_{ij}(\omega_1, \lambda))| \,|\, (\omega_1, \lambda) \in \Omega_1 \times \Sigma_0 \setminus E_k, \quad i, j = 0, 1 \,\}. \tag{5.4}$$

Note that the supremum $\widetilde{C}_k$ exists since $E_k$ is an open set and $\Omega_1 \times \Sigma_0$ is compact by Assumption 5.2 (ii). In addition, $\widetilde{C}_k$ is bounded by the supremum of the entries of the Dirichlet-to-Neumann map on the set $K_k$ of $(\omega_1, \lambda)$ with $\mathrm{dist}(\lambda, \sigma(H^{\mathrm{D}}_{G(\omega_1)}) \geq \eta_k$ only. But since $K_k$ is compact, and since the Dirichlet-to-Neumann map is meromorphic with simple poles (see (D.9)), we have $|\Lambda_{ij}(\omega, \lambda)| \leq \widetilde{C}/\eta_k$ for a constant $\widetilde{C} > 0$ independent of $k$. In particular, $\widetilde{C}_k \leq \widetilde{C}/\eta_k = O(k^{2a})$ as in Lemma 5.4.

By definition, we have $(E_k)^{\mathrm{c}} \subset (\widetilde{E}_k)^{\mathrm{c}}$. Furthermore, we can bound the norm of $T_\lambda(\cdot, G(\omega_1))$ estimated in (D.15) by

$$C'_k := \Big(1 + \frac{1 + \sup \Sigma_0}{\eta_k}\Big)\|E\|\Big(1 + \frac{\widetilde{C}_k + 1}{\eta_k}\Big)$$

for $(\omega_1, \lambda) \in (E_k)^{\mathrm{c}}$. Therefore, $(E_k)^{\mathrm{c}} \subset (E'_k)^{\mathrm{c}}$ and $C'_k = O(k^{6a})$ follows. $\square$

Similarly, we can show that the set where the norm of the transfer matrix is not bounded, is small:

**Lemma 5.6.** *There exists a sequence $\{C''_k\}_k$ growing at most polynomially such that*

$$E''_k := \{\, (\omega_1, \lambda) \in \Omega_1 \times \Sigma_0 \setminus E_0 \,|\, \|T_\lambda(\omega_1)\| > C''_k \,\} \tag{5.5}$$

*satisfies $E''_k \subset E_k$. In particular, $E''_0 := \bigcap_k E''_k \subset E_0$ has $(\mathbb{P}_1 \otimes \boldsymbol{\lambda})$-measure 0.*

*Proof.* The transfer matrix has been expressed in terms of the Dirichlet-to-Neumann map in (D.12). Using an appropriate matrix norm, we see that $\|T_\lambda(\omega_1)\|$ can be estimated by $C''_k = p(\widetilde{C}_k)/\eta_k$ for $(\omega_1, \lambda) \in (E_k)^{\mathrm{c}}$ where $p(C)$ is a universal polynomial of degree 2, monotone in $C$. As in the previous lemma, $C''_k = O(k^{6a})$ and again, $(E_k)^{\mathrm{c}} \subset (E''_k)^{\mathrm{c}}$. $\square$

Let $\pi_n \colon \Omega \times \Sigma_0 \longrightarrow \Omega_1 \times \Sigma_0$, $(\omega, \lambda) \mapsto (\omega_n, \lambda)$ be the projection onto the $n$th component. Furthermore, we set

$$S_0 := \bigcap_n \pi_n^-((E_0)^c) = \{\, (\omega, \lambda) \in \Omega \times \Sigma_0 \,|\, (\omega_n, \lambda) \notin E_0 \text{ for all } n \,\},$$
$$S_k := \pi_k^-((E_k)^c) = \{\, (\omega, \lambda) \in S_0 \,|\, (\omega_k, \lambda) \notin E_k \,\}.$$

**Lemma 5.7.** *The sets $S_0$ and $S_k$ are measurable. Furthermore, $S_0$ and $\underline{S}_\infty := \liminf S_k := \bigcup_{n\in\mathbb{N}} \bigcap_{k\ge n} S_k$ have full $(\mathbb{P} \otimes \boldsymbol{\lambda})$-measure. In particular, for $(\omega, \lambda) = (\omega_1, \omega_2, \dots, \lambda) \in S_0$, the transfer matrix $T_\lambda(\omega_n)$ is defined for all $n$ and for $(\omega, \lambda) \in \underline{S}_\infty$ there exists $n \in \mathbb{N}$ such that*

$$\|T_\lambda(\cdot, G(\omega_k))\| \le C'_k \qquad \text{and} \qquad \|T_\lambda(\omega_k)\| \le C''_k \tag{5.6}$$

*for all $k \ge n$, i.e., the norm of the solution operator and the transfer matrix is bounded by constants depending on $k$ and which are of polynomial growth.*

*Proof.* Clearly, $S_0$ and $S_k$ are measurable since $E_0$ and $E_k$ are (by Assumption 5.2 (ii)). Furthermore, $(\omega, \lambda) \in S_0$ iff $\lambda \notin E(G(\omega_n))$ for all $n$, i.e., for these $\omega$ and $\lambda$, the transfer matrix $T_\lambda(\omega_n)$ is defined for all $n$. In addition,

$$(\mathbb{P} \otimes \boldsymbol{\lambda})((S_0)^c) = \lim_n (\mathbb{P} \otimes \boldsymbol{\lambda})\Big(\bigcup_{k\le n} \pi_k^-(E_0)\Big) \le \sum_n \big((\mathbb{P}_1 \otimes \boldsymbol{\lambda})(E_0)\big) = 0$$

due to the continuity of the measure and Lemma 5.4.

Next, we have

$$\sum_k (\mathbb{P} \otimes \boldsymbol{\lambda})((S_k)^c) = \sum_k (\mathbb{P}_1 \otimes \boldsymbol{\lambda})(E_k) < \infty$$

due to the independence of the family $\{S_k\}_k$ and Lemma 5.4. It follows from the Borel-Cantelli lemma for the complement $(\underline{S}_\infty)^c$ that $\underline{S}_\infty$ has full measure in $\Omega \times \Sigma_0$. The norm estimates are simple consequences of the definitions of $E'_k$, respectively, $E''_k$, and the fact that $E'_k, E''_k \subset E_k$ (see Lemmas 5.5–5.6). □

Now, the results of Section 3.2 extends to the case when the transfer matrices are only defined on $(E_0)^c$ instead of $\Omega_0 \times \Sigma_0$. Similarly, the product transfer matrix $U_\lambda(\omega, n)$ is defined for $(\omega, \lambda) \in S_0$ instead of the full product $\Omega \times \Sigma_0$.

**Theorem 5.8.** *Let $H(\omega)$ be the random Hamiltonian on a random tree-like graph $G(\omega)$ with constant branching number $b \ge 1$ such that Assumption 5.2 is fulfilled. Assume in addition, that the Lyapunov exponent satisfies $\gamma(\lambda) > 0$ for almost all $\lambda \in \Sigma_0$. Then localization holds for all energies in the almost sure spectrum, i.e., $\sigma(H(\omega)) \cap \Sigma_0$ is almost surely pure point.*

*In addition, there exists a set $S_0 \subset \Omega \times \Sigma_0$ of full $(\mathbb{P} \otimes \boldsymbol{\lambda})$-measure such that all eigenfunctions associated to $\lambda$ and $H(\omega)$ on the tree-like graph with $(\omega, \lambda) \in S_0$ decay with almost exponential decay rate $\beta := \gamma(\lambda) + (\ln b)/2$ of an eigenfunction $f$ on the tree-like graph in the sense that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|f(x)| \le C_\varepsilon \mathrm{e}^{-(\beta-\varepsilon)d(o,x)} \tag{5.7}$$

*for all $x \in T$.*

*Remark* 5.9. We expect that also for the exceptional values $(S_0)^c$ we have exponential decaying or even compactly supported eigenfunctions; this can be seen in most examples directly. A general proof would need more analysis on the behavior in the exceptional set (see also Section D.1).

*Proof.* We argue as in the proof of Theorem 3.15 and stress only the needed changes here. We define the set $S$ as in (3.14), but intersected with $\underline{S}_\infty$ (in particular, all transfer matrices are defined). From Lemma 3.5 (note Assumption 5.2 (iv)), we see that there exists a set of full measure $S_1 \subset S_0$ such that $\gamma(\lambda)$ exists for $\omega \in S_1(\lambda)$ a.s. Now, together with the assumption $\gamma(\lambda) > 0$ it follows that $S_1 \subset S$ and in particular, $S$ has full measure in $\Omega \times \Sigma_0$ (or in $\hat{\Omega} \times \Sigma_0$, what is the same). For $(\hat{\omega}, \lambda) \in S$, Assumption (i) of Theorem 3.6 is fulfilled. Next, Assumption (ii) follows from Lemma 5.7: since $C_k''$ has polynomial growth, we have

$$\lim_k \frac{1}{k} \ln \|T_\lambda(\omega_k)\| \le \lim_k \frac{1}{k} \ln C_k'' = 0$$

provided $k$ is large enough. We therefore get a nontrivial solution $\vec{F}(\hat{\omega}, \cdot, \theta_0) \in \ell_2(\mathbb{N}, \mathbb{C}^2)$ of the discretized eigenvalue equation. To see that the associated eigenfunction $f$ on the line-like graph is in $\mathsf{L}_2(L)$, we note that

$$\|f\|_L^2 = \sum_{k=1}^{\infty} \|T_\lambda(\cdot, G(\omega_k)) \vec{F}(\hat{\omega}, k-1, \theta_0)\|^2_{G(\omega_k)}$$

and estimate the norms by $C_k' |\vec{F}(\hat{\omega}, k-1, \theta_0)|$ if $k \ge n$ for some $n \in \mathbb{N}$ large enough due to Lemma 5.7. Since the convergence only depends on the behavior of the tail of the sum and since $|\vec{F}(\hat{\omega}, k-1, \theta_0)|$ decays exponentially in $k$ (see (3.15)), we have $f \in \mathsf{L}_2(L)$.

If $\rho_\omega$ denotes the spectral measure of $H(\omega)$, note that due to Lemma D.16, the Weyl-Titchmarsh function $m$ associated to $H(\omega)$ is the Borel transform of a measure $\hat{\rho}_\omega$ and the spectral measure has the decomposition $\rho_\omega = \hat{\rho}_\omega + \rho_{\omega,\mathrm{pp}}$ into disjoint measures where $\rho_{\omega,\mathrm{pp}}$ is already pure point and has support in $E(L(\omega)) = (S_0(\omega))^{\mathrm{c}}$. The rest of the localization proof is similar to the proof of Theorem 3.15 replacing the measure $\rho_\omega$ by $\hat{\rho}_\omega$. Note Remark 3.16 for the weaker Assumption 5.2 (v) on the spectral averaging condition.

From the almost exponential decay of $\vec{F}(\hat{\omega}, n, 0)$ and (5.6) it follows that $\Phi_\varepsilon f \in \mathsf{L}_2(L)$ for $\varepsilon > 0$ where $\Phi_\varepsilon$ is the exponential weight $\Phi_\varepsilon(x) = \mathrm{e}^{(\gamma(\lambda)-\varepsilon)n}$ for $x \in G_n$ and $f$ is the associated eigenfunction on the line-like graph. Theorem C.18 implies the almost exponential pointwise decay of $f$ on $L$ in the sense that for $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|f(x)| \le C_\varepsilon \Phi_\varepsilon(x)^{-1}$ for $x \in G_n$, $n \in \mathbb{N}$. Finally, from (4.13a) and (4.13d) it follows that we can replace the discontinuous weight function $\Phi_\varepsilon$ by $\mathrm{e}^{-(\gamma(\lambda)-\varepsilon)d(0,x)}$ on the line-like graph. The additional exponential decay $(\ln b)/2$ for an eigenfunction on the *tree*-like graph comes from the symmetry reduction (see Remark 3.18 (ii)). ☐

5.2. **Examples.** We are now able to check the assumptions in our concrete examples:

**Theorem 5.10.** *Suppose that the decoration graph consists of a single edge of length $\ell$ with a fixed graph $\hat{G}_*$ attached at the ending point (Example 4.5 (ii)(b), Figure 4 (ii)(b)). Assume that the single site perturbation of the decoration at the ending point model with branching number $b \ge 1$ has an absolutely continuous distribution with bounded density $\mathrm{d}\mathbb{P}_1(\ell) = \eta(\ell)\,\mathrm{d}\ell$ and support in $\Omega_1 := [\ell_-, \ell_+]$ for $0 < \ell_- < \ell_+$. Then localization holds for all energies in the almost sure spectrum and the eigenfunctions decay almost exponentially with rate $\gamma(\lambda) + (\ln b)/2$ in the sense of* (5.7).

*Proof.* Fix a compact spectral interval $\Sigma_0$. We have to check that the decoration graphs are "good" in $\Sigma_0$ in the sense of Assumption 5.2. Clearly, the decoration graphs satisfy the uniformity assumptions (4.13). Furthermore, the dependence of the Dirichlet eigenvalues and the Dirichlet-to-Neumann map (cf. (4.20)) on the random parameter $\ell$ is (piecewise) analytic.

The exceptional set $E(\ell)$ consists only of the Dirichlet spectrum of a single decoration graph $\hat{G}_*$ and of the values $\lambda = \mu^2$ such that $\sin(\mu\ell) = 0$. In particular, $E(\ell)$ is discrete. The integrability condition is fulfilled since the norm of the transfer matrix $T_\lambda(n)$ (cf. (4.19)) can easily be estimated by a constant depending only on $\lambda \notin E(\ell)$. The spectral averaging is established in Corollary E.7.

The Lyapunov exponent is positive: It is easy to see (due to the analytic dependence on $\ell$) that the two assumptions (3.12) and (3.13) of Corollary 3.11 are fulfilled. The third condition is also satisfied since one can always find two noncommuting matrices $T_\lambda(\ell_i)$, $\ell_i \in [\ell_-, \ell_+]$. □

**Theorem 5.11.** *Assume that the single site perturbation of the necklace or onion model (with $p \geq 2$ loop edges) of Example 4.5 (iii) (see also Figure 4 (iii)) with branching number $b \geq 1$ has an absolutely continuous distribution with bounded density $\mathrm{d}\mathbb{P}_1(\ell) = \eta(\ell)\,\mathrm{d}\ell$ and support in $\Omega_1 := [0, \ell_+]$. Then localization holds for all energies in the almost sure spectrum and the eigenfunctions decay almost exponentially with rate $\gamma(\lambda) + (\ln b)/2$ in the sense of* (5.7).

*Proof.* Fix a compact spectral interval $\Sigma_0$. Again, we have to check that the decoration graphs are "good" in $\Sigma_0$ in the sense of Assumption 5.2. Clearly, the necklace, respectively, onion, decoration graphs satisfy the uniformity assumptions (4.13). Furthermore, the dependence of the Dirichlet eigenvalues and the Dirichlet-to-Neumann map (cf. (4.24)) on the random parameter $\ell$ is (piecewise) analytic; in addition, $\Lambda(0, \lambda)$ corresponds to the Dirichlet-to-Neumann map of a single edge (i.e., the case when the loop of length $\ell$ degenerates to a point), hence the dependence is continuous up to the border of $\Omega_1 = [0, \ell_+]$.

The exceptional set consists only of the Dirichlet spectrum of a single decoration graph $G_*(\ell)$ and is therefore a discrete subset of $\mathbb{R}$.

The integrability condition is fulfilled since the norm of the transfer matrix $T_\lambda(n)$ (cf. (4.22)) can easily be estimated by a constant depending only on $\Sigma_0$ and $p$.

For the spectral averaging, we use Lemma E.5: Note that the representation of the Möbius transformation of the inverse transfer matrix $\hat{T}_z(\ell)^{-1} = R_w(-w)R_{pw}(-\ell w)$ (cf. (E.8)) holds with

$$A_w := \frac{p}{p^2 \sin^2 w + \cos^2 w} \qquad \text{and} \qquad B_w := -\frac{(p^2-1) w \sin w \cos w}{p^2 \sin^2 w + \cos^2 w}$$

where $w^2 = z$. Now,

$$A_w = \frac{p}{p^2 \sin^2 \mu + \cos^2 \mu} - \mathrm{i}\varepsilon \frac{2p(p^2-1)\sin\mu\cos\mu}{(p^2 \sin^2 \mu + \cos^2 \mu)^2} + O(\varepsilon^2)$$

for $w = \mu + \mathrm{i}\varepsilon$ $(0 < \varepsilon \leq \varepsilon_0)$. Therefore, $\operatorname{Re} A_w = O(1)$, $\operatorname{Im} A_w = O(\varepsilon)$ and similarly, $B_w = O(1)$ with constants depending only on $\Sigma_0$ and $\varepsilon_0$. The winding number of the denominator of the Möbius transformation is bounded since $\ell \in [0, \ell_+]$ and the values of $\mu$ also lie in a compact interval. Here, the exceptional values $\{\lambda_k\}$ consists of the union of the Dirichlet spectrum $\Delta^{\mathrm{D}}_{G_*(\ell)}$ for the end points, i.e., $\ell = 0$ and $\ell = \ell_+$.

The Lyapunov exponent is positive: It is easy to see (due to the analytic dependence on $\ell$) that the two assumptions (3.12) and (3.13) of Corollary 3.11 are fulfilled. The third condition is also satisfied since one can always find two noncommuting matrices $T_\lambda(\ell_i)$, $\ell_i \in [0, \ell_+]$. □

**Theorem 5.12.** *Assume that we have a fixed decoration graph $G_*$ in each generation satisfying* (4.13c) *and that the set of zeros of the Dirichlet-to-Neumann matrix element $\Lambda_{01}(z)$ is a discrete subset of $\mathbb{R}$ (e.g. if $G_*$ is a necklace decoration). Assume in addition,* *that the single site perturbation is a vertex potential at the end point of each decoration graph with range* $q \in \Omega_1 := [q_-, q_+] \subset \mathbb{R}$. *Then localization holds for all energies in the almost sure spectrum with eigenfunctions having almost exponential pointwise decay rate* $\gamma(\lambda) + (\ln b)/2$ *on the tree-like graph in the sense of* (5.7).

*Proof.* We argue as in the previous proof. Assumptions (4.13) are clear. The Dirichlet-to-Neumann map in this case does not depend on the random parameter $q$. The condition on the exceptional set (here also independent of $q$) is fulfilled by assumption, as well as the integrability condition (since $q$ has its range in a compact interval). The spectral averaging holds due to Corollary E.8. To show that the Lyapunov exponent is positive we apply again Corollary 3.11. One can always find two noncommuting transfer matrices $T_\lambda(q_i) = D(b)S(q_i)T_\lambda(G_*)$ provided $T_\lambda(G_*)$ is not of the form $D(b)S(\kappa)$. Note that the latter can only happen for a countable set of $\lambda$'s since generally, a transfer matrix contains the rotation matrices $R_{p\mu}(\mu\ell_0)$ $(\lambda > 0)$. □

*Mixed examples.* We can also mix the examples, for example an edge decoration with $b = 1$, random length $\ell_1 \in [0, \ell_+]$ and a simple edge of random length $\ell_2 \in [\ell_-, \ell_+]$ and branching number $b = 2$. The probability space now consists of two components $\Omega_1 = \Omega_{1,1} \cup \Omega_{1,2}$.

5.3. **Full-line models.** So far, we only considered *rooted* radial quantum trees which lead to *half* line-like graphs. Our results on localization extend to *unrooted* trees leading to *full* line-like graphs. We do not present the details, but we illustrate the result in the case of a line-like graph, i.e., the branching number is $b = 1$, and the necklace decoration of Example 4.5 (iii) (see also Figure 4 (iii)). The random necklace model was originally treated by Kostrykin and Schrader in [KS04] where the authors showed discontinuity of the integrated density of states. We complete this study by proving Anderson localization for the random necklace model.

For $n \in \mathbb{Z}$, let $G_n(\omega) = G_*(\omega_n)$, be the necklace decoration of Example 4.5 (iii) (with $p = 2$ arcs of length $\ell_n = \omega_n$ forming the loop). Let $L(\omega)$, $\omega \in \Omega := \Omega_1^{\mathbb{Z}}$, be the line-like graph obtained by joining the decoration graph in a line unbounded in both directions. All the results of the random half-line models extend to full-line models. The spectrum of the *periodic* full model on $L = L(\ell)$ (with constant length $\ell = \ell_n$) is purely absolutely continuous and is given by

$$\sigma(\Delta_{L(\ell)}) = \left\{ \mu^2 \,\middle|\, |\cos(\mu\ell)\cos\mu - \tfrac{5}{4}\sin(\mu\ell)\sin\mu| \le 1 \right\} = \bigcup_{k=1}^{\infty} B_k(\ell) \tag{5.8}$$

(cf. (4.27)).

Our result on localization in this situation reads as follows:

**Theorem 5.13.** *Assume that the single site perturbation of the full-line necklace model (with* $p = 2$ *loop edges) of Example 4.5 (iii) has an absolutely continuous distribution with bounded density* $\mathrm{d}\mathbb{P}_1(\ell) = \eta(\ell)\,\mathrm{d}\ell$ *and support in* $\Omega_1 := [0, \ell_+]$. *Then localization holds for all energies in the almost sure spectrum* $\Sigma = \bigcup_{\ell \in \Omega_1} \sigma(\Delta_{L(\ell)})$ *with eigenfunctions having almost exponential decay rate* $\gamma_\pm(\lambda)$ *for* $x \to \pm\infty$ *in the sense of* (5.7).

*Proof.* The proof in the full-line model (cf. [KS87]) is similar to the proof of the half-line model, so we give only a sketch of the proof: We have already seen that the Lyapunov exponent for $n \to +\infty$ is positive; for $n \ge 0$ the transfer matrix from generation 0 to $-n$ is $U_\lambda(\omega, -n) = T_\lambda(\omega_{n-1})^{-1} \cdot \ldots \cdot T_\lambda(\omega_0)^{-1}$, and the same argument as for $n \to \infty$ shows that the Lyapunov exponent is positive also for $n \to -\infty$. As before, from the Oseledec theorem it follows that there exist generalized eigenfunctions $f_\pm$ on the positive, respective, negative, half-line model decaying exponentially where

$(f_\pm, f_\pm^\dagger)(0) \sim \theta_0^\pm = \theta_0^\pm(\omega,\lambda)$ for a set $S \subset \Omega \times \Sigma$ of full measure. Here, we have to show that $\theta_0^+ = \theta_0^-$ in order to assure that $f_+$ and $cf_-$ are the restrictions of an eigenfunction $f$ in the domain of $H(\omega)$ for a suitable constant $c \in \mathbb{C}$.

From the spectral averaging argument, we see that $S(\omega) \subset \Sigma$ is a support of the spectral measure (component) $\hat\rho_\omega$. Note that we need the estimate (E.1) for $T_z(\omega_1)$ (see the proof of Theorem 5.11) *and* $T_z(\omega_1)^{-1}$ (cf. Corollary E.7). The spectral measure is also supported on the set of eigenvalues $\lambda$ with polynomially bounded (generalized) eigenfunctions $\varphi$. The Wronskian of $\varphi$ and $f_+$, respectively, $f_-$, is constant, and 0 in the limit $n \to \pm\infty$, so that $f_\pm$ and $\varphi$ satisfy the same condition at 0, namely $\theta_0^+ = \theta_0^-$. ☐

## A. Symmetry reduction

For radial tree-like graphs $G = (V, E, \partial, \ell, q)$ associated to a tree graph $T = (V(T), E(T), \partial)$, we can profit from the symmetric structure of $G$ (for a definition of a radial tree-like graph we refer to Definition 4.2). The argument used here follows closely the symmetry reduction for the simple tree graph $T$ (cf. [NS00, SoS02, Sol04]).

On a rooted tree, we can define a partial order $\succeq$ on the set of vertices and edges as follows: If a vertex $v \in V(T)$ lies on the shortest path from $o$ to $v' \in V(T)$ we say that $v'$ *succeeds* $v$ ($v' \succeq v$). Similarly, an edge $t \in E(T)$ succeeds $v$ iff its start vertex succeeds $v$, i.e., $\partial_- t \succeq v$. The *vertex subtree* $T_{\succeq v}$ succeeding $v \in V$ is the graph of all edges and vertices succeeding $v$. The *edge subtree* $T_{\succeq t}$ is the subgraph of all edges and vertices succeeding $\partial_+ t$ together with the root $\partial_- t$ of the subtree. In particular, a vertex subtree $T_{\succeq v}$ is the union of all edge subtrees $T_{\succeq t}$ with $v = \partial_- t$.

Similarly, let $G_{\succeq v}$, respectively, $G_{\succeq t}$, be the *vertex*, respectively, *edge*, subgraph of the tree-like graph $G$ corresponding to the underlying tree subgraph $T_{\succeq v}$, respectively, $T_{\succeq t}$, i.e., $G_{\succeq v}$, respectively, $G_{\succeq t}$, consists of all decoration graphs $G_*(t')$ with $t' \in E(T_{\succeq v})$, respectively, $t' \in E(T_{\succeq t})$. We can associate a line-like graph $L_n$ to the radial tree-like graph $G_{\succeq v}$, $\operatorname{gen} v = n$, as in Section 4.2, consisting of the sequence $\{G_k\}_{k>n}$ of decoration graphs with branching number sequence $\{b_k\}_{k \ge n}$

Let $b = b_n$ be the branching number of $v \in V(T)$. The cyclic group $\mathbb{Z}_b$ acts on the vertex subgraph $G_{\succeq v}$ by shifting the $b$ succeeding edge subgraphs $G_{\succeq t}$, $\partial_- t = v$, in a cyclic way. The group action on $G_{\succeq v}$ lifts naturally to an unitary action on $\mathsf{L}_2(G_{\succeq v})$. We denote the action of $1 \in \mathbb{Z}_b$ by $Q_v$. Since 1 generates $\mathbb{Z}_b$, the operator $Q_b$ also generates the action on $\mathsf{L}_2(G_{\succeq v})$. Furthermore, $Q_v^b = \mathbb{1}$ and the eigenvalues of $Q_v$ are the $b$th unit roots $e_b^s$ of 1, $s = 0, \dots, b-1$. The corresponding eigenspaces are denoted by

$$\mathsf{L}_2^s(G_{\succeq v}) := \ker(Q_v^b - e_b^s \mathbb{1}).$$

**Definition A.1.** A function $f \in \mathsf{L}_2(G_{\succeq v})$ is called *$s$-radial at the tree vertex $v \in V(T)$* iff $f \in \mathsf{L}_2^s(G_{\succeq v})$ and if $f \in \mathsf{L}_2^0(G_{\succeq v'})$ for all succeeding tree vertices $v' \succ v$. The set of all $s$-radial functions is denoted by $\mathsf{L}_2^{s,\mathrm{rad}}(G_{\succeq v})$. A 0-radial function is simply called *radial.*

In other words, a function $f \in \mathsf{L}_2(G_{\succeq v})$ is $s$-radial iff $Q_v f = e_b^s f$ for $b = b_n$, $n = \operatorname{gen} v$, and if $f$ is invariant under the group action $Q_{v'}$ on the subsequent subgraph $G_{\succeq v'}$ for all $v' \succ v$. Clearly, such a function is completely determined by its value on the line-like graph $L_n$. We therefore define

$$J_v^s \colon \mathsf{L}_2^{s,\mathrm{rad}}(G_{\succeq v}) \longrightarrow \mathsf{L}_2(L_n) \qquad J_v^s f := \bigoplus_{k>n} (b_n \cdot \ldots \cdot b_{k-1})^{1/2} f\restriction_{G_k}. \tag{A.1}$$

**Lemma A.2.** *The operator $J_v^s$ is unitary.*

*Proof.* We have

$$\|J_v^s f\|_{L_n}^2 = \sum_{k>n} (b_n \cdot \ldots \cdot b_{k-1}) \|f\|_{G_k}^2 = \|f\|_{G_{\succeq v}}^2$$

since $b_n \cdot \ldots \cdot b_{k-1}$ is the total number of copies of $G_k$ contained in $G_{\succeq v}$ at generation $k$ and since for $f \in \mathsf{L}_2^{s,\mathrm{rad}}(G_{\succeq v})$ the value $\|f\|_{G_k}$ is independent of the choice of the decoration graphs $G_*$ at generation $k$. Finally, it can easily be seen that $J_v^s$ is surjective. □

**Lemma A.3.** *We have the decomposition*

$$\mathsf{L}_2(G) = \mathsf{L}_2^{0,\mathrm{rad}}(G_{\succeq o}) \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{\substack{v \in V(T)\\ \mathrm{gen}\, v = n}} \bigoplus_{s=1}^{b_n - 1} \mathsf{L}_2^{s,\mathrm{rad}}(G_{\succeq v}). \tag{A.2}$$

*Proof.* Since $b_0 = 1$ we can split off the first decoration graph $G_1$ and consider only $\mathsf{L}_2(G_{\succeq o_1})$ where $o_1$ denotes the vertex of generation 1. Note that $\mathsf{L}_2^{0,\mathrm{rad}}(G_{\succeq o}) = \mathsf{L}_2(G_1) \oplus \mathsf{L}_2^{0,\mathrm{rad}}(G_{\succeq o_1})$. From the eigenspace decomposition of $Q_{o_1}$ we obtain

$$\mathsf{L}_2(G_{\succeq o_1}) = \bigoplus_{s=0}^{b_1 - 1} \mathsf{L}_2^{s}(G_{\succeq o_1})$$

since $G = G_1 \cup G_{\succeq o_1}$ where $G_1$ is the decoration graph at generation 1. Next, we have

$$\mathsf{L}_2^{s}(G_{\succeq o_1}) = \mathsf{L}_2^{s,\mathrm{rad}}(G_{\succeq o_1}) \oplus \bigoplus_{\substack{v \in V(T)\\ \mathrm{gen}\, t = 2}} \Big( \mathsf{L}_2(G_{\succeq v}) \ominus \mathsf{L}_2^{0,\mathrm{rad}}(G_{\succeq v}) \Big) \tag{A.3}$$

since functions in $\mathsf{L}_2^{s}(G_{\succeq o_1}) \ominus \mathsf{L}_2^{s,\mathrm{rad}}(G_{\succeq o_1})$ vanish on the decoration graphs $G_*(t)$ of generation $\mathrm{gen}\, t = 2$. In addition, the radial component of functions on the subtrees $G_{\succeq v}$ is already contained in $\mathsf{L}_2^{s,\mathrm{rad}}(G_{\succeq o_1})$, therefore, $\mathsf{L}_2(G_{\succeq v}) \ominus \mathsf{L}_2^{0,\mathrm{rad}}(G_{\succeq v}) = \bigoplus_{s=1}^{b_2-1} \mathsf{L}_2^{s}(G_{\succeq v})$ so that

$$\mathsf{L}_2(G) = \mathsf{L}_2^{0,\mathrm{rad}}(G_{\succeq o}) \oplus \bigoplus_{s=1}^{b_1 - 1} \mathsf{L}_2^{s,\mathrm{rad}}(G_{\succeq o_1}) \oplus \bigoplus_{\substack{v \in V(T)\\ \mathrm{gen}\, v = 2}} \bigoplus_{s=1}^{b_2 - 1} \mathsf{L}_2^{s}(G_{\succeq v}).$$

Now we can decompose $\mathsf{L}_2^{s}(G_{\succeq v})$ as in (A.3) and obtain the desired formula (A.2) recursively. It remains to show that a function $f$ orthogonal to the direct sum in the right hand side of (A.2) vanishes. Clearly, such a function vanishes on the first decoration graph $G_1$. In addition, such a function must also vanish on the decoration graphs $G_*(2) := \bigcup_{t \in E(T), \mathrm{gen}\, t = 2} G_*(t)$ of generation 2 since $\mathsf{L}_2^{s,\mathrm{rad}}(G_{\succeq o_1}) \cap \mathsf{L}_2(G_*(2)) = \mathsf{L}_2^{s}(G_{\succeq o_1}) \cap \mathsf{L}_2(G_*(2))$. The same arguments holds for any subgraph $G_{\succeq v}$ so that $f$ vanishes on all decoration graphs $G_*(t)$, i.e., $f = 0$. □

We assume that the decoration graphs $G_*(t)$ satisfy the uniformity assumptions (4.13). We then define a quadratic form on the subgraph $G_{\succeq v}$ with domain

$$\operatorname{dom} \mathfrak{h}_{G_{\succeq v}} = \Big\{ f \in \bigoplus_{t \in E(T_{\succeq v})} \mathsf{H}^1(G_*(t)) \,\Big|\, f \in \mathsf{C}_\circ(G_{\succeq v}) \Big\} \tag{A.4}$$

where $\mathsf{C}_\circ(G_{\succeq v})$ denotes the space of continuous on $G_{\succeq v}$ vanishing at the root vertex $v$ of $G_{\succeq v}$. The quadratic form is defined as

$$\mathfrak{h}_{G_{\succeq v}}(f) = \sum_{t \in E(T_{\succeq v})} \big( \|f'\|_{G_*(t)}^2 + q(\partial_+ t) |f(t)|^2 \big). \tag{A.5}$$

As in Lemma C.8 it follows that $\mathfrak{h}_{G_{\succeq v}}$ is a closed quadratic form and relatively form-bounded w.r.t. the free form $\mathfrak{d}_{G_{\succeq v}}$ (where $q(v) = 0$) with relative bound 0. We denote the self-adjoint operator associated to $\mathfrak{h}_{G_{\succeq v}}$ by $H_{G_{\succeq v}}$. We now show that the orthogonal composition of the previous lemma also decomposes $H = H_G$:

**Lemma A.4.** *The components of the decomposition* (A.2) *are invariant subspaces of the Hamiltonian $H$ on $G$.*

*Proof.* We want to show that the domain of $\mathfrak{h} = \mathfrak{h}_G$ decomposes into

$$\operatorname{dom}\mathfrak{h} = \operatorname{dom}\mathfrak{h}_o^{0,\mathrm{rad}} \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{\substack{v\in V(T)\\ \operatorname{gen} v=n}} \bigoplus_{s=1}^{b_n-1} \operatorname{dom}\mathfrak{h}_v^{s,\mathrm{rad}}$$

where $\operatorname{dom}\mathfrak{h}_v^{s,\mathrm{rad}}$ is the space of all functions $f_v^s \in \mathsf{L}_2^{s,\mathrm{rad}}(G_{\succeq v}) \cap \mathsf{C}_\circ(G_{\succeq v})$ such that

$$\mathfrak{h}_{G_{\succeq v}}(f_v^s) < \infty. \tag{A.6}$$

In order to do so, we have to verify that if $f = \{f_v^s\}$ is the orthogonal decomposition of $f \in \operatorname{dom}\mathfrak{h}$ w.r.t. (A.2) then $f_v^s \in \operatorname{dom}\mathfrak{h}_v^{s,\mathrm{rad}}$: By definition, $f_v^s \in \mathsf{L}_2^{s,\mathrm{rad}}(G_{\succeq v})$ and $f_v^s \in \mathsf{C}_\circ(G_{\succeq v})$ follows from the continuity of $f \in \operatorname{dom}\mathfrak{h}$ and the fact that $f_v^s$ vanishes on $(G_{\succeq v})^{\mathrm{c}}$. Furthermore, if $f_v^s \in \operatorname{dom}\mathfrak{h}_{G_{\succeq v}}$ then $Q_v f_v^s \in \operatorname{dom}\mathfrak{h}_{G_{\succeq v}}$ and $\mathfrak{h}_{G_{\succeq v}}(Q_v f_v^s) = h_{G_{\succeq v}}(f_v^s)$, i.e., $Q_v$ leaves $\mathfrak{h}_{G_{\succeq v}}$ invariant; in particular, (A.6) is fulfilled. $\square$

Note that functions vanishing outside a subgraph $G_{\succeq v}$ must satisfy a Dirichlet condition at the root vertex $v$ since functions in $\operatorname{dom}\mathfrak{h}$ are continuous.

We now want to compare the radial quadratic form $\mathfrak{h}_v^{s,\mathrm{rad}}$ with a form on the line-like graph $L_n$, $n = \operatorname{gen} v$.

**Lemma A.5.** *The quadratic form $\mathfrak{h}_v^{s,\mathrm{rad}}$ is unitary equivalent to $\mathfrak{h}_{L_n}$ where*

$$\mathfrak{h}_{L_n}(g) := \sum_{k>n} \bigl(\|g'\|^2_{G_k} + q_k |f(k_-)|^2\bigr) \tag{A.7}$$

*and*

$$\operatorname{dom}\mathfrak{h}_{L_n} = \{\, f \in \bigoplus_{k>n} \mathsf{H}^1(G_k) \,|\, \forall k > n:\ f(k_-) = b_k^{-1/2} f(k_+), \quad f(0) = 0\,\}, \tag{A.8}$$

*i.e.,*

$$J_v^s(\operatorname{dom}\mathfrak{h}_v^{s,\mathrm{rad}}) = \operatorname{dom}\mathfrak{h}_{L_n} \qquad \textit{and} \qquad \mathfrak{h}_v^{s,\mathrm{rad}}(f) = \mathfrak{h}_{L_n}(J_v^s f).$$

*Proof.* The proof follows from a straightforward calculation. $\square$

Summarizing the results, we obtain (denoting $H_n = H_{L_{n-1}}$ the operator on $L_{n-1}$ associated to the quadratic form $\mathfrak{h}_{L_{n-1}}$):

**Theorem A.6.** *The Hamiltonian $H$ on a symmetric, radial tree-like quantum graph $G$ is unitary equivalent to*

$$H \cong H_1 \oplus \bigoplus_{n=2}^{\infty} (\oplus b_0 \cdot \ldots \cdot b_{n-2}(b_{n-1} - 1)) H_n$$

*where $(\oplus m)H_n$ means the $m$-fold copy of $H_n$.*

The domain of $H_n = H_{L_{n-1}}$ is given in Lemma C.10.

## B. Bounds on generalized eigenfunctions

In this section we provide $\mathsf{L}_2$-bounds on generalized eigenfunctions on quantum graphs. We start with a slightly more general setting. Let $X$ be a metric, $\sigma$-finite measure space with measure $\mu$. We usually denote the measure by $\mathrm{d}x = \mathrm{d}\mu(x)$. A slightly different setting using only Hilbert space arguments can be found in [PSW89, PS93].

**Assumption B.1.** Let $H$ be a semibounded, self-adjoint operator $H \geq \lambda_0$ in $\mathsf{L}_2(X) = \mathsf{L}_2(X, \mathrm{d}\mu)$. We assume that $H$ is *local*, i.e., $\operatorname{supp} Hf \subset \operatorname{supp} f$ for any $f \in \operatorname{dom} H$. In addition, we assume that the space of functions $f \in \operatorname{dom} H$ with compact support are dense in $\mathsf{L}_2(X)$. Denote $K := (H - \lambda_0 + 1)^{-m/2}$ the $\frac{m}{2}$-th power of the resolvent at $\lambda_0 \in \mathbb{R}$ ($m > 0$). Our main assumption in this section assures that $K$ is a Carleman operator, i.e.,

$$K \colon \mathsf{L}_2(X) \longrightarrow \mathsf{L}_\infty(X)$$

is bounded, or, equivalently,

$$\|f\|_{\mathsf{L}_\infty(X)} \leq C_1 \|(H - \lambda_0 + 1)^{m/2} f\|_{\mathsf{L}_2(X)}. \tag{B.1}$$

We will prove in the next section that on a quantum graph or a line graph, the above conditions are met with $m = 1$ for the graph Hamiltonian under suitable conditions on the model.

Carleman operators have a measurable kernel $k \colon X \times X \longrightarrow \mathbb{C}$ (cf. [S82, Cor. A.1.2]), i.e.,

$$(Kf)(x) = \int_X k(x, y) f(y)\, \mathrm{d}y$$

satisfying

$$\|K\|_{\mathsf{L}_2 \to \mathsf{L}_\infty} = \sup_{x \in X} \|k(x, \cdot)\|_{\mathsf{L}_2(X)} \leq C_1 < \infty.$$

We will show that $K$ and certain other functions of $H$ have an integral kernel also in a *weighted* $\mathsf{L}_2$-space.

**Assumption B.2.** Let $\Phi \in \mathsf{L}_2(X) \cap \mathsf{L}_\infty(X)$ be a bounded, square-integrable and *positive* function such that $\Phi$ is bounded away from 0 on any compact set.

To $\Phi$ we associate the Hilbert scaling (cf. [BS91])

$$\mathcal{H}_+ := \mathsf{L}_2(X, \mathrm{d}\mu_+) \hookrightarrow \mathcal{H} := \mathsf{L}_2(X, \mathrm{d}\mu) \hookrightarrow \mathcal{H}_- := \mathsf{L}_2(X, \mathrm{d}\mu_-)$$

where $\mathrm{d}\mu_\pm := \Phi^{\mp 2}\, \mathrm{d}\mu$ are weighted measures and $\mathcal{H}_\pm$ are normed by $\|f\|_\pm := \|\Phi^{\mp 1} f\|$. Then the inner product $\langle \cdot, \cdot \rangle \colon \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ extends to a dual (sesquilinear) pairing $(\cdot, \cdot) \colon \mathcal{H}_- \times \mathcal{H}_+ \longrightarrow \mathbb{C}$. In particular, $\mathcal{H}_-$ can be interpreted as the dual $(\mathcal{H}_+)^*$ with respect to this pairing. In addition, the multiplication with $\Phi$, respectively, $\Phi^{-1}$, becomes an isometry, i.e.,

$$\mathcal{H}_+ \underset{\Phi}{\overset{\Phi^{-1}}{\rightleftarrows}} \mathcal{H} \underset{\Phi}{\overset{\Phi^{-1}}{\rightleftarrows}} \mathcal{H}_-$$

and $(f, g) = \langle \Phi f, \Phi^{-1} g \rangle$ for $f \in \mathcal{H}_-$, $g \in \mathcal{H}_+$. Since $\Phi$ is bounded away from 0 on any compact set, the norms in $\mathcal{H}$ and $\mathcal{H}_\pm$ are equivalent for functions with support in fixed compact subset of $X$.

Our aim is to show that $T := \Phi K^2 \Phi$ or more generally, $T_\varphi := \Phi \varphi(H)^2 \Phi$ for fast enough decaying functions $\varphi$ (in particular, $\varphi = \mathbb{1}_I$, $I$ bounded) are of trace class as operators from $\mathcal{H}$ to $\mathcal{H}$ and have an integral kernel $t_\varphi$. Our Hilbert scaling allows us to consider $\widetilde{T}_\varphi := \varphi(H)^2$ as map $\mathcal{H}_+ \to \mathcal{H}_-$. It is still of trace class as product of the Hilbert Schmidt operators $\varphi(H) \colon \mathcal{H}_+ \longrightarrow \mathcal{H}$ and $\varphi(H) \colon \mathcal{H} \longrightarrow \mathcal{H}_-$ (cf. the lemma below) and has integral kernel $\widetilde{t}_\varphi(x,y) = \Phi(x)^{-1} t_\varphi(x,y)\Phi(y)^{-1}$ with respect to the pairing $(\cdot,\cdot)$, i.e.,

$$(\widetilde{T}_\varphi f, g) = \int_X \int_X \widetilde{t}_\varphi(x,y)\overline{f(x)} g(x)\,\mathrm{d}x\,\mathrm{d}y$$

for $f, g \in \mathcal{H}_+$. In a second step, apply the above considerations to $\varphi = \mathbb{1}_I$ and disintegrate the spectral resolution $\mathbb{1}_I(H)$ with respect to a spectral measure of $H$.

We start with showing that $\Phi K$ is a Hilbert-Schmidt operator (cf. [CL90, Prop. II.3.11]):

**Lemma B.3.** *Suppose that $\Phi \in \mathsf{L}_2(X) \cap \mathsf{L}_\infty(X)$ and that* (B.1) *is fulfilled. Then for any measurable function $\varphi \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$|\varphi(\lambda)| \le C_2(\lambda - \lambda_0 + 1)^{-m/2} \tag{B.2}$$

*for all $\lambda \ge \lambda_0$ the operator $\Phi\varphi(H)$ is Hilbert-Schmidt with Hilbert-Schmidt norm bounded by $C_3 := \|\Phi\|_{\mathsf{L}_2(X)} C_1 C_2$. In addition,*

$$\Phi\varphi(H)^2\Phi \colon \mathcal{H} \longrightarrow \mathcal{H}$$

*is trace class with trace bounded by $C_3^2$.*

*Proof.* The kernel of $\Phi K$ is $(x,y) \mapsto \Phi(x)k(x,y)$. By (B.1), its $\mathsf{L}_2(X \times X)$-norm is bounded by $\|\Phi\|_{\mathsf{L}_2(X)} C_1$ so that $\Phi K$ is Hilbert-Schmidt. In particular, $T = \Phi K(\Phi K)^*$ is trace class and

$$\|\Phi K^2 \Phi\|_{\mathcal{B}_1} = \operatorname{tr}(\Phi K^2 \Phi) \le \|\Phi K\|_{\mathcal{B}_2}^2 \le \|\Phi\|_{\mathsf{L}_2(X)}^2 C_1^2.$$

To pass to a general function $\varphi$, note that $\varphi(H)^2 \le C_2^2 K^2$ by the spectral theorem. The result follows from the monotonicity of the trace and $0 \le \Phi\varphi(H)^2\Phi \le C_2^2 \Phi K^2 \Phi$. ☐

In particular, the above lemma applies for the characteristic function $\mathbb{1}_I$ of a bounded, measurable set $I \subset \mathbb{R}$ with $C_2 := (\sup I - \lambda_0 + 1)^{m/2}$. Therefore, one can show that $E(I) := \Phi \mathbb{1}_I(H)\Phi$ defines a nonnegative, trace-class-operator-valued, strongly $\sigma$-additive measure, i.e, (i) $E(I) \ge 0$, (ii) $E(I)$ is trace class for all bounded and measurable $I \subset \mathbb{R}$ and (iii) $E(\biguplus I_n) = \operatorname{s-lim} \sum_n E(I_n)$.

We use the following lemma to disintegrate $E(I)$ (cf. [S82, Thm. C.5.1]:

**Lemma B.4.** *Suppose $E(\cdot)$ is a nonnegative, trace-class-operator-valued, strongly $\sigma$-additive measure. Then there exists a Borel measure $\rho$ (i.e., a measure on the Borel sets of $\mathbb{R}$, finite on all compact sets) and a measurable function $E \colon \mathbb{R} \longrightarrow \mathcal{B}_1(\mathcal{H})$ such that $E(\lambda) \ge 0$,*

$$E(I) = \mathrm{w}\text{-}\int_I E(\lambda)\,\mathrm{d}\lambda \qquad \textit{and} \qquad \operatorname{tr} E(\lambda) = 1 \quad \rho\textit{-a.e.}$$

*Proof.* Set $\rho(I) := \operatorname{tr} E(I)$ and $\rho_{ij}(I) := \langle \varphi_i, E(I)\varphi_j\rangle$ for an orthonormal basis $\{\varphi_i\}_i$ of $\mathcal{H}$. Clearly, $\rho$ is a Borel measure, as well as $\rho_{ij}$ are $\mathbb{C}$-valued Borel measures. Furthermore, $(\rho_{ij}(I))_{i,j\in J}$ is a nonnegative matrix for any finite subset $J \subset \mathbb{N}$. In addition, $\rho_{ij}$ is absolutely continuous w.r.t. $\rho$, so by the Radon-Nikodym theorem there exists a measurable function $e_{ij}$ such that $\mathrm{d}\rho_{ij}(\lambda) = e_{ij}(\lambda)\,\mathrm{d}\rho(\lambda)$ for all $i,j$. Using the fact that $\rho(I) = \sum_i \rho_{ii}(I)$ one sees that $\sum_i e_{ii}(\lambda) = 1$ a.e.

Define $E(\lambda)$ as the operator with associated matrix $(e_{ij}(\lambda))_{ij}$ in the basis $\{\varphi_i\}_i$. Clearly, $E(\lambda)$ has trace 1 and a limit argument shows that

$$\langle E(I)f, g\rangle = \sum_{ij} \overline{f_i}\, g_j\, \rho_{ij}(I) = \int \sum_{ij} \overline{f_i}\, g_j\, e_{ij}(\lambda)\,\mathrm{d}\rho(\lambda) = \int \sum_{ij} \langle E(\lambda)f, g\rangle\,\mathrm{d}\rho(\lambda)$$

where $f_i = \langle \varphi_i, f\rangle$ and $g_j = \langle \varphi_j, g\rangle$. ☐

As a corollary, we obtain

**Corollary B.5.** *The measure $\rho(I) := \operatorname{tr} E(I)$ associated to the trace-class-operator-valued, strongly $\sigma$-additive measure $E(I) := \Phi \mathbb{1}_I(H)\Phi$ is a spectral measure for $H$, i.e., $\rho(I) = 0$ iff $1_I(H) = 0$. If $\varphi$ satisfies* (B.2) *then $\int_{\mathbb{R}} |\varphi(\lambda)|^2 \,\mathrm{d}\rho(\lambda) < \infty$.*

*Furthermore, the disintegrated operator $E(\lambda)\colon \mathcal{H} \longrightarrow \mathcal{H}$ is also Hilbert-Schmidt and has a kernel $e_\lambda \in \mathsf{L}_2(X \times X)$ and $\|e_\lambda\|_{\mathsf{L}_2(X\times X)} \le 1$. In addition,*

$$\widetilde{E}(\lambda) := \Phi^{-1} E(\lambda) \Phi^{-1} \colon \mathcal{H}_+ \longrightarrow \mathcal{H}_-$$

*has the kernel*

$$\widetilde{e}_\lambda(x,y) := \Phi(x)^{-1} e_\lambda(x,y) \Phi(y)^{-1}$$

*and allows the disintegration formula*

$$(\mathbb{1}_I(H)f, g) = \int_I (\widetilde{E}(\lambda)f, g)\,\mathrm{d}\rho(\lambda) = \int_I \int_X \int_X \widetilde{e}_\lambda(x,y)\overline{f(x)}\, g(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho(\lambda)$$

*for all $f, g \in \mathcal{H}_+$ with $\widetilde{e}_\lambda \in \mathcal{H}_- \otimes \mathcal{H}_-$ and $\|\widetilde{e}_\lambda\|_{\mathcal{H}_-\otimes\mathcal{H}_-} \le 1$.*

*Proof.* Since $\ker \Phi = \{0\}$ and $\Phi^* = \Phi$ as multiplication operator, $\rho(I) = 0$ implies $\Phi \mathbb{1}_I(H) \Phi = 0$ and therefore $\mathbb{1}_I(H) = 0$. In particular, $\rho$ is a spectral measure. The fact that $\varphi \in \mathsf{L}_2(\mathbb{R}, \mathrm{d}\rho)$ follows from Lemma B.3, the definition of $\rho$ and the spectral calculus.

The remaining assertions are almost obvious: Trace class operators are also Hilbert-Schmidt and

$$\|e_\lambda\|_{\mathsf{L}_2(X\times X)} = \|E(\lambda)\|_{\mathcal{B}_2} \le \|E(\lambda)\|_{\mathcal{B}_1} = \operatorname{tr} E(\lambda) = 1$$

a.e. ☐

Under the same assumptions as before, we can now pass to more general functions of $H$ using a standard approximation argument.

**Lemma B.6.** *Let $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ be bounded and measurable. Then*

$$\begin{aligned}(\varphi(H)f, g) &= \int_{\mathbb{R}} \varphi(\lambda)(\widetilde{E}(\lambda)f, g)\,\mathrm{d}\rho(\lambda)\\ &= \int_{\mathbb{R}} \int_X \int_X \varphi(\lambda)\widetilde{e}_\lambda(x,y)\overline{f(x)}\, g(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\rho(\lambda)\end{aligned}$$

*for all $f, g \in \mathcal{H}_+$.*

We now want to show that $\widetilde{e}_\lambda(\cdot, y)$ solves the eigenvalue equation $(H - \lambda)u = 0$ in a generalized sense. To do so, set

$$\operatorname{dom} H_+ := \{\, f \in \mathcal{H}_+ \mid f \in \operatorname{dom} H, \quad Hf \in \mathcal{H}_+ \,\}, \qquad H_+ f := Hf.$$

The operator $H_+$ is a closed operator in $\mathcal{H}_+$. Since the space of functions $f \in \operatorname{dom} H$ with compact support is dense in $\mathcal{H}$ and since the norms on $\mathcal{H}$ and $\mathcal{H}_+$ are equivalent on a fixed compact set, $\operatorname{dom} H_+$ is dense in $\mathcal{H}_+$. Therefore, we can define the adjoint $H_- := (H_+)^*$ w.r.t. $(\cdot,\cdot)$, i.e., $f \in \operatorname{dom} \mathcal{H}_-$ iff $f \in \mathcal{H}_-$ and if there exist $H_- f \in \mathcal{H}_-$ such that

$$(H_- f, g) = (f, H_+ g)$$

for all $g \in \operatorname{dom} H_+$.

**Definition B.7.** A function $u \in \mathcal{H}_-$ is a *generalized eigenfunction* of $H$ with $\mathsf{L}_2$-growth rate $\Phi^{-2}$ for the eigenvalue $\lambda$ , if $H_- u = \lambda u$, i.e.,

$$(u, (H_+ - \lambda)g) = 0$$

for all $g \in \operatorname{dom} H_+$.

The next lemma assures that the integral kernel is a generalized eigenfunction (cf. [S82, Thm. C.5.2]):

**Lemma B.8.** *We have* $\widetilde{e}_\lambda(\cdot, y) \in \operatorname{dom} H_-$ *and*

$$h(y) := (\widetilde{e}_\lambda(\cdot, y), (H_+ - \lambda)g) = 0$$

*for all* $g \in \operatorname{dom} H_+$*,* $\mu$*-almost all* $y \in X$ *and* $\rho$*-almost all* $\lambda \in \mathbb{R}$*.*

*Proof.* Since $\widetilde{e}_\lambda \in \mathcal{H}_- \otimes \mathcal{H}_-$, we have $h \in \mathcal{H}_-$ and the above equation is equivalent to

$$\int_X \int_X \widetilde{e}_\lambda(x,y)\overline{f(x)}\,(H_+ - \lambda)g(y)\,\mathrm{d}x\,\mathrm{d}y = (\widetilde{E}(\lambda)f, (H_+ - \lambda)g) = 0 \tag{B.3}$$

for all $f \in \mathcal{H}_+$ and $g \in \operatorname{dom} H_+$ using the kernel representation of $\widetilde{E}(\lambda)$ in Corollary B.5 and the fact that $H_+g,\ g \in \mathcal{H}_+$. Now, we define a signed measure $\widetilde{\rho}(I) = \int_I (\widetilde{E}(\lambda')f, (H_+ - \lambda)g)\,\mathrm{d}\rho(\lambda')$ and obtain

$$(\widetilde{E}(\lambda)f, (H_+ - \lambda)g) = \lim_{I \searrow \{\lambda\}} \frac{\widetilde{\rho}(I)}{\rho(I)} = \lim_{I \searrow \{\lambda\}} \frac{1}{\rho(I)}\big(\mathbb{1}_I(H)f, (H_+ - \lambda)g\big)$$

using the Radon-Nikodym derivative and Corollary B.5. The left hand side equals

$$\lim_{I \searrow \{\lambda\}} \frac{1}{\rho(I)}\big\langle \mathbb{1}_I(H)f, (H - \lambda)g\big\rangle = \lim_{I \searrow \{\lambda\}} \frac{1}{\rho(I)}\big\langle \varphi_\lambda(H)f, g\big\rangle$$

since $(\cdot,\cdot) = \langle\cdot,\cdot\rangle$ on $\mathcal{H}$ and by the functional calculus with $\varphi_\lambda(\lambda') = (\lambda' - \lambda)\mathbb{1}_I(\lambda')$. Finally, the latter term equals

$$\lim_{I \searrow \{\lambda\}} \frac{1}{\rho(I)} \int_{\mathbb{R}} \varphi_\lambda(\lambda')\big(\widetilde{E}(\lambda')f, g\big)\,\mathrm{d}\rho(\lambda') = 0$$

by Lemma B.6 and the Radon-Nikodym derivative. ☐

We are now able to state our main result on the growth of generalized eigenfunction (cf. [S82, Thm. C.5.4]):

**Theorem B.9.** *Suppose* $H$ *is an operator with spectral measure* $\rho$ *satisfying Assumption B.1 and* $\Phi$ *is a weight function satisfying Assumption B.2. Then there exist a measurable disjoint decomposition*

$$\sigma(H) = \biguplus_{n \in \mathbb{N} \cup \{\infty\}} \Sigma_n$$

*(up to sets of* $\rho$*-measure* $0$*), and for each* $n \in \mathbb{N} \cup \{\infty\}$ *and* $j = 1, \dots, n$ *there exists a measurable function* $\varphi_j \colon \Sigma_n \times X \longrightarrow \mathbb{C}$ *such that* $\{\varphi_{\lambda,j} := \varphi_j(\lambda, \cdot)\}_{1 \le j \le n} \subset \operatorname{dom} H_-$ *are linearly independent,*

$$\sum_{j=1}^n \|\varphi_{\lambda,j}\|_-^2 = \sum_{j=1}^n \|\Phi\varphi_{\lambda,j}\|^2 = 1$$

*and* $(H_- - \lambda)\varphi_{\lambda,j} = 0$ *for* $\rho$*-almost all* $\lambda \in \Sigma_n$*.*

*In addition,*

$$\Sigma := \{\, \lambda \in \mathbb{R} \mid \exists \varphi \in \operatorname{dom} H_- \setminus \{0\} : \quad H_-\varphi = \lambda\varphi \,\} \tag{B.4}$$

*is a support for the measure* $\rho$*, i.e,* $\rho(\Sigma^{\mathrm{c}}) = 0$*.*

*Proof.* Let $\Sigma_n := \{\, \lambda \in \sigma(H) \mid \dim \operatorname{ran} E(\lambda) = n \,\}$. By Corollary B.5, there exists a orthonormal system of eigenfunctions $(\widetilde{\psi}_{\lambda,j})_j$ of $E(\lambda)$ with nonnegative eigenvalues such that

$$E(\lambda)\widetilde{\psi}_{\lambda,j} = \varepsilon_{\lambda,j}\widetilde{\psi}_{\lambda,j}.$$

We set $\psi_{\lambda,j} := \sqrt{\varepsilon}_{\lambda,j}\widetilde{\psi}_{\lambda,j}$. Then we have $e_\lambda(x,y) = \sum_j \overline{\psi_{\lambda,j}(x)}\psi_{\lambda,j}(y)$ for the kernel in the weak sense, i.e.,

$$\langle \overline{f} \otimes g, e_\lambda \rangle = \sum_j \langle f, \psi_{\lambda,j}\rangle \langle \psi_{\lambda,j}, g\rangle \tag{B.5}$$

for $f, g \in \mathcal{H}$. Now set $\varphi_{\lambda,j} := \Phi^{-1}\psi_{\lambda,j} \in \mathcal{H}_-$. Then

$$\sum_j \|\varphi_{\lambda,j}\|_-^2 = \sum_j \|\psi_{\lambda,j}\|^2 = \sum_j \varepsilon_{\lambda,j} = \|E(\lambda)\|_{\mathcal{B}_1} = 1.$$

Finally, from (B.3) we get

$$\begin{aligned} 0 &= \int_X \int_X \widetilde{e}_\lambda(x,y)\overline{f(x)}\,(H_+ - \lambda)g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_X \int_X e_\lambda(x,y)\Phi(x)^{-1}\overline{f(x)}\,\Phi(y)^{-1}(H_+ - \lambda)g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\overset{(B.5)}{=} \sum_j \langle \Phi^{-1}f, \psi_{\lambda,j}\rangle \langle \psi_{\lambda,j}, \Phi^{-1}(H_+ - \lambda)g\rangle \end{aligned}$$

for $f \in \mathcal{H}_+$ and $g \in \operatorname{dom} H_+$. Setting $f = \varphi_{\lambda,j_0}$ we obtain

$$0 = \big\langle \psi_{\lambda,j_0}, \Phi^{-1}(H_+ - \lambda)g\big\rangle = \big(\varphi_{\lambda,j_0}, (H_+ - \lambda)g\big)$$

for all $g \in \operatorname{dom} H_+$ and $\lambda \in \sigma(H) \setminus N_g$ where $\rho(N_g) = 0$.

We have to show that we can choose a set of measure 0 *independent* of $g$ (cf. [PS93, Proof of Thm. 2.2 (b)]): This can be done since $H_+$ is closed in $\mathcal{H}_+$, i.e., $\operatorname{dom} H_+$ with its graph norm is a Hilbert space, and $\mathcal{H}_+$ is separable ($\mathsf{L}_2(X, \mathrm{d}\mu_+)$ is $\sigma$-finite!). It follows that $\operatorname{dom} H_+$ with its graph norm is separable (this is true for a self-adjoint operator; for a general operator, note that $H_+$ and $|H_+|$ define the same graph norm).

Therefore, we can choose the union $N = \bigcup N_g$ for countable many $g$ and $N$ still has measure 0. Therefore, $\varphi_{\lambda,j} \in \mathcal{H}_-$ and $(H_- - \lambda)\varphi_{\lambda,j} = 0$ for $\rho$-almost all $\lambda \in \mathbb{R}$. We have therefore shown that $\sigma(H)$ is included in $\Sigma$ up to a set of $\rho$-measure 0, and therefore, $\Sigma$ is a support for $\rho$. ☐

Dealing with one-dimensional problems, we easily get more information on the eigenfunction expansion as a by-product of the previous theorem:

**Lemma B.10.** *Suppose that the vector space of generalized eigenfunctions in the sense of Definition B.7 is generated by compactly supported functions and a finite number of functions with infinite support. Then the weak eigenfunction expansion* (B.5) *holds pointwise almost everywhere, i.e.,*

$$\widetilde{e}_\lambda(x,y) = \sum_j \overline{\varphi_{\lambda,j}(x)}\varphi_{\lambda,j}(y) \tag{B.6}$$

*for $\mu$-almost all $x, y \in X$ and $\rho$-almost all $\lambda \in \mathbb{R}$.*

*Proof.* We can choose the orthogonal basis $\widetilde{\psi}_{\lambda,j}$ to have compact support except than a finite number of vectors. Then the weak sum (B.5) is indeed a locally finite sum, and therefore exists also pointwise. ☐

## C. Line-like graphs and bounds on generalized eigenfunctions

In this section we specify the analysis done in the previous section to line-like graphs. We will show that the assumptions made in the previous sections are fulfilled. In particular, we get integral bounds on generalized eigenfunctions. In the concrete situation here, we can also prove pointwise estimates on generalized eigenfunctions (Section C.3) and a spectral resolution of the spectral projector (Section C.2).

C.1. **Quadratic forms and operators on line-like graphs.** In this section, we determine the operator domain of the reduced Hamiltonian on a line-like graph $L$ and show that the operator is essentially self-adjoined on compactly supported functions. This has been shown for graphs with a global lower bound on *all* length, i.e., $\ell_e \ge \ell_- > 0$ for all $e \in E(L)$, and global bounds on the boundary conditions, i.e., $1 \le b_n \le b_+$ and $q_- \le q_n \le q_+$ for example in [Ku04] (see also [Car97] for the case of tree graphs). Although we assume that the lengths of the edges connecting the vertices $(n+1)_+$ and $n_-$ have a global lower bound (and some other conditions, see (4.13)), we want to allow edges of arbitrary small size inside the decoration graph.

For simplicity, we only consider the line-like graph $L = L_0$. Clearly, all statements hold similarly for $L_n$. We begin with some Sobolev-type estimates which follow from our assumptions on the decoration graphs in (4.13):

**Lemma C.1.** *Suppose that the decoration graph $G_*$ satisfies* (4.13)*. Then there exists $C_1' > 0$ such that*

$$|f(x)|^2 \le \varepsilon \|f'\|^2_{G_*} + \frac{4}{\varepsilon}\|f\|^2_{G_*}, \qquad 0 < \varepsilon \le \ell_-,\ x \in G_*,\ f \in \mathsf{H}^1(G_*) \tag{C.1}$$

$$|f^\dagger(o_0)|^2 \le C_1'\big(\|f''\|^2_{G_*} + \|f'\|^2_{G_*}\big), \qquad f \in \mathsf{H}^2(G_*). \tag{C.2}$$

*Proof.* Let $\varepsilon > 0$. For the first estimate, note that due to (4.13a), every point $x \in G_*$ has a path $\gamma$ of length larger than $\ell_-/2$ either to $o_0$ or to $o_1$, in particular, there is a nonclosed path $\gamma$ of length $|\gamma| = \varepsilon/2$ starting at $x$. Denote the sequence of segments between the vertices on $\gamma$ by $e_i$, $i = 1, \dots, n$ joining the vertices $x_i$, $1 \le i \le n-1$ and $x_0 = x$ and $x_n$. In addition, let $\chi$ be the affine linear function on $\gamma$ with $\chi(x_0) = 1$ and $\chi(x_n) = 0$ where $x_1$ is the endpoint of $\gamma$. In particular, $|\chi'(x)| = 1/|\gamma|$ along $\gamma$. Due to the continuity of $f$ at the vertices, we have

$$f(x) = (\chi f)(x) = \sum_{i=1}^n \big((\chi f_{e_i})(\partial_+ e_i) - (\chi f_{e_i})(\partial_- e_i)\big) = \sum_{i=1}^n \int_{e_i} (\chi f_{e_i})'(s)\,\mathrm{d}s.$$

A simple estimate using Cauchy-Schwartz yields

$$|f(x)|^2 \le 2|\gamma|\|f'\|^2_\gamma + \frac{2}{|\gamma|}\|f\|^2_\gamma.$$

The second estimate follows from

$$|f_e'(o_0)|^2 \le 2\kappa\ell_-\|f''\|^2_e + \frac{2}{\kappa\ell_-}\|f'\|^2_e$$

using the previous estimate for a path $\gamma$ lying completely in the *single* edge $e_0$ emanating $o_0$ (cf. (4.13c)). Assumption (4.13b) assures, that $\gamma$ can be chosen to have length $\kappa\ell_-$. Therefore,

$$\big|f^\dagger(o_0)\big|^2 = \big|f_{e_0}'(o_0)\big|^2 \le 2\kappa\ell_-\|f''\|^2_{e_0} + \frac{2}{\kappa\ell_-}\|f'\|^2_{e_0} \tag{C.3}$$

and in particular, the estimate follows with $C_1' := 2\max\{\kappa\ell_-, 1/(\kappa\ell_-)\}$. ☐

In order to compare several Sobolev spaces and operator domains, we need to define a cut-off function $\chi$ on a quantum graph leaving the vertex boundary conditions invariant:

**Lemma C.2.** *There exists a nonnegative function $\chi \in \mathsf{H}^2(G_*)$, smooth on each edge, constant near each vertex such that $\chi(o_0) = 1$, $\chi(O_1) = 0$ and*

$$\|\chi^{(m)}\|_\infty \le \Big(\frac{2}{\ell_-}\Big)^m$$

*for* $m = 0, \dots, 3$.

We call such functions $\chi$ *smooth cut-off functions.* Note that $f \in \mathsf{H}^2(G_*)$ iff $\chi f \in \mathsf{H}^2(G_*)$ since $\chi$ is constant near each vertex.

*Proof.* Choose a function $\hat\chi$ affine linear on each vertex with $\hat\chi(o_0) = 0$ and $\hat\chi(o_1) = 1$ and $0 \le \hat\chi(v) \le 1$ in such a way, that no slope exceeds the minimal needed slope $1/\ell_-$ on the shortest path from $o_0$ to $o_1$ (due to (4.13a)). Let $\chi$ be a slight modification of $\hat\chi$ such that $\chi$ is constant near each vertex, smooth on each edge and such that the $m$th derivative is bounded by $(2/\ell_-)^m$. $\square$

Associated to the branching number sequence $\{b_n\}_n$ and the vertex potential strength $\{q_n\}_n$ we define several Sobolev spaces on a line-like graph $L$:

$$\mathsf{H}^1(L) := \Big\{ f \in \bigoplus_{n \ge 1} \mathsf{H}^1(G_n) \,\Big|\, \forall n \ge 1\colon \quad f_n(n_-) = b_n^{-1/2} f_{n+1}(n_+) \Big\} \tag{C.4}$$

$$\mathsf{H}^2(L) := \Big\{ f \in \bigoplus_{n \ge 1} \mathsf{H}^2(G_n) \,\Big|\, \forall n \ge 1\colon \quad \begin{array}{l} f_n(n_-) = b_n^{-1/2} f_{n+1}(n_+), \\ f_n^\dagger(n_-) = b_n^{1/2} f_{n+1}^\dagger(n_+) \end{array} \Big\} \tag{C.5}$$

and

$$\operatorname{dom} H^{\max} := \Big\{ f \in \mathsf{L}_2(L) \,\Big|\, f'' = \{f_e''\}_e \in \mathsf{L}_2(L), \quad \forall n \ge 1\colon f_n'' \in \mathsf{H}^2(G_n) \\ \begin{array}{l} f_n(n_-) = b_n^{-1/2} f_{n+1}(n_+), \\ f_n^\dagger(n_-) = b_n^{1/2} f_{n+1}^\dagger(n_+) \end{array} \Big\} \tag{C.6}$$

where $f_n := f \restriction_{G_n}$. The corresponding norms are given by

$$\|f\|^2_{\mathsf{H}^1(L)} := \|f\|^2 + \|f'\|^2, \qquad \|f\|^2_{\mathsf{H}^2(L)} := \|f\|^2 + \|f'\|^2 + \|f''\|^2 \\ \text{and} \qquad \|f\|^2_{H^{\max}} := \|f\|^2 + \|f''\|^2. \tag{C.7}$$

We denote by $H^{\max} = H_L^{\max}$ the *maximal* Hamiltonian with domain $\operatorname{dom} H^{\max}$ acting as $(H^{\max} f)_e = -f_e''$ on each edge.

Our aim is to show that $\operatorname{dom} H^{\max} = \mathsf{H}^2(L)$ and that their norms are equivalent. The following lemma is a useful tool to get rid of the first derivative:

**Lemma C.3.** *Suppose that $\chi$ is a function smooth on each edge and constant near each vertex. Suppose in addition that $H$ is a self-adjoint operator in $\mathsf{L}_2(L)$ such that $\chi f \in \operatorname{dom} H$ if $f \in \operatorname{dom} H^{\max}$, $H \ge \lambda_0$ and such that $Hf = -f''$ for functions with support away from the vertices. Then*

$$\|(\chi f)'\| \le \|dR^{1/2}\| \Big( \|\chi\|_\infty \|f'' - \lambda_0 f\| \\ + \big( \|\chi''\|_\infty + 2\|dR^{1/2}\| \|\chi'\|_\infty + \|\chi\|_\infty \big) \|f\| \Big) \tag{C.8}$$

*for all $f \in \operatorname{dom} H^{\max}$ where $R := (H - \lambda_0 + 1)^{-1}$ and $df := f'$.*

*Proof.* The assumptions on $H$ imply that $\operatorname{dom} H \subset \operatorname{dom} H^{\max}$ and that $H(\chi f) = -(\chi f)'' = -\chi f'' - 2(\chi' f)' + \chi'' f$. Then we can write

$$(\chi f)' = dR(H - \lambda_0 + 1)(\chi f) = dR\big(\chi(-f'' + (-\lambda_0 + 1) f) - 2d(\chi' f) + \chi'' f\big).$$

Since $\chi'$ has support away from the vertices, $(\chi' f)' = -d^*(\chi' f)$ where $d^*$ is the adjoint of $d$. Using $\|dRd^*\| = \|dR^{1/2}\|^2$ we obtain the desired estimate. $\square$

**Lemma C.4.** *The space* $\operatorname{dom} H^{\max}_{\mathrm{c}}$ *of compactly supported functions (not necessarily disjoint from the root vertex) in* $\operatorname{dom} H^{\max}$ *is dense. In addition, there is a constant* $C_1'' > 0$ *such that*

$$\|f'\|^2_L \le C_1''\big(\|f''\|^2_L + \|f\|^2_L\big) \tag{C.9}$$

*holds for all* $f \in \operatorname{dom} H^{\max}$.

*Proof.* For a function $f \in \operatorname{dom} H^{\max}$ let $f_n := \chi_n f$ where $\chi_n$ is the smooth cut-off function with $\chi_n(n-1) = 0$ and $\chi_n(n) = 1$ on $G_n$ as constructed in Lemma C.2 extended on $G_k$ by 0 for $k < n$, respectively, by 1, for $k > n$. Now

$$\|f - f_n\| \le \|f\|^2_{L_n} \to 0 \tag{C.10}$$

as $n \to \infty$ since $f \in \mathsf{L}_2(L)$. Furthermore,

$$\begin{aligned}\|(f-f_n)''\| &\le \|((1-\chi_n)f''\| + \|\chi_n''f\| + 2\|(\chi_n'f)'\| \\ &\le \|f''\|_{L_n} + \frac{4}{\ell_-^2}\|f\|_{L_n} + 2\|(\chi_n'f)'\|. \end{aligned}\tag{C.11}$$

Now, the latter term can be estimated by

$$\|(\chi_n'f)'\|_{L_n} \le \frac{2}{\ell_-}\|f''\|_{L_n} + \Big(\frac{8}{\ell_-^3} + \frac{8}{\ell_-^2} + \frac{2}{\ell_-}\Big)\|f\|_{L_n}$$

applying the previous lemma with $H := \Delta^{\mathrm{D}}_{L_n}$, the Dirichlet Laplacian on $L_n$ defined via the quadratic form $\mathfrak{d}(f) = \|f'\|^2$ with domain $\mathsf{H}^1_\circ(L_n)$. Note that the estimate $\|dR^{1/2}\| \le 1$ is equivalent to $\|df\| \le \mathfrak{d}(f) + \|f\|^2$ which is obviously true. Since $f$ and $f'' = \{f_e''\}_e \in \mathsf{L}_2(L)$, the left hand side of (C.11) tends to 0 as $n \to \infty$. We have therefore shown that compactly supported functions are dense in $\operatorname{dom} H^{\max}$.

To show (C.9), we can restrict ourselves to compactly supported functions. Partial integration taking the inner boundary conditions into account yields

$$\|f'\|^2_{L_n} \le \|f''\|^2_{L_n} + \|f\|^2_{L_n} + |f^\dagger(0)f(0)|.$$

The last term can be estimated using (C.1)–(C.2) and $ab \le \eta a^2/2 + b^2/(2\eta)$ as

$$\begin{aligned}|f^\dagger(0)f(0)| &\le \sqrt{C_1'}\big(\|f'\| + \|f''\|\big)\big(\sqrt{\varepsilon}\|f'\| + \frac{2}{\sqrt{\varepsilon}}\|f\|\big) \\ &\le \sqrt{C_1'\varepsilon}\|f'\|^2 + \frac14\|f'\|^2 + \Big(\frac{8C_1'}{\varepsilon} + \frac{1}{\sqrt{\varepsilon}}\Big)\|f\|^2 + \Big(4\varepsilon + \frac{1}{\sqrt{\varepsilon}}\Big)\|f''\|^2\end{aligned}$$

provided $0 < \varepsilon \le \ell_-$ where all the norms are $\mathsf{L}_2$-norms on $G_1$. Choosing $\varepsilon = \min\{1/(16C_1'), \ell_-\}$ we obtain

$$|f^\dagger(0)f(0)| \le \frac12\|f'\|^2 + \Big(\frac{8C_1'}{\varepsilon} + \frac{1}{\sqrt{\varepsilon}}\Big)\|f\|^2 + \Big(4\varepsilon + \frac{1}{\sqrt{\varepsilon}}\Big)\|f''\|^2.$$

Subtracting the contribution of $\|f'\|^2$ on the right hand side we finally see that there is a constant $C_1''$ depending only on $C_1'$ and $\ell_-$ such that (C.9) holds. ☐

*Remark* C.5. Note that for a fixed decoration graph, we can prove the estimate

$$\|f'\|^2_{G_*} \le \widetilde{C}_1''\big(\|f''\|^2_{G_*} + \|f\|^2_{G_*}\big) \tag{C.12}$$

for all $f \in \mathsf{H}^2(G_*)$ similar as in the above proof. But to do so, we need an estimate on $f^\dagger(o_1)$ as in (C.2) assuming that there is no vertex potential. Therefore, the constant $\widetilde{C}_1''$ depend on the minimal length of all edges adjacent to $o_1$ and the vertex degree of $o_1$ similar to (C.3), which does *not* admit a global lower bound in our family of decoration graphs.

The next lemma deals with the Sobolev spaces $\mathsf{H}^m(L)$ and their relation with $\operatorname{dom} H^{\max}$:

**Lemma C.6.** *The spaces* $\mathsf{H}^1(L)$ *and* $\mathsf{H}^2(L)$ *with their natural norms given in* (C.7) *are Hilbert spaces. The spaces* $\mathsf{H}^2(L)$ *and* $\operatorname{dom} H^{\max}$ *are equal and have equivalent norms. In addition, the subspaces* $\mathsf{H}^1_{\mathrm{c}}(L)$*, respectively,* $\mathsf{H}^2_{\mathrm{c}}(L)$*, of functions in* $\mathsf{H}^1(L)$ *resp.* $\mathsf{H}^2(L)$ *with compact support (not necessarily away from the root vertex* 0*) are dense.*

*Proof.* The completeness of $\mathsf{H}^1(L)$ and $\mathsf{H}^2(L)$ follows from the fact that $\mathsf{H}^1(L)$, respectively, $\mathsf{H}^2(L)$, are closed subspaces in the Hilbert space $\bigoplus_n \mathsf{H}^m(G_n)$: Note that (C.1) and (C.2) imply the continuity of $f \mapsto f(n_\pm)$, respectively, $f \mapsto f^\dagger(n_\pm)$.

From (C.9) we see that $\operatorname{dom} H^{\max} \subset \mathsf{H}^2(L)$ and that the inclusion is continuous. The opposite inclusion is trivial. The density of the space of compactly functions in $\mathsf{H}^2(L)$ now follows from Lemma C.4. The similar assertion for $\mathsf{H}^1(L)$ follows in the same way. ☐

To summarize, we can characterize the domain of the maximal Hamiltonian as follows:

**Lemma C.7.** *The maximal Hamiltonian* $H^{\max} = H^{\max}_L$ *on* $L$ *with domain* $\operatorname{dom} H^{\max} = \mathsf{H}^2(L)$ *is a closed operator. In addition,* $f \in \operatorname{dom} H^{\max} = \mathsf{H}^2(L)$ *iff*

(i) $f, f'' = \{f''_e\}_e \in \mathsf{L}_2(L)$,
(ii) $f$ *satisfies all vertex boundary conditions at inner vertices* $V_0(G_n)$*, i.e.,* $f_{e_1}(v) = f_{e_2}(v)$ *for all* $e_1, e_2 \in E_v(G_n)$*,* $v \in V(G_n)$*, and* $f'_{G_n}(v) = 0$ *for all* $v \in V_0(G_n)$ *and* $n \geq 1$ *(cf.* (4.1) *for the notation);*
(iii) $f$ *satisfies all vertex boundary conditions* (4.12) *at the connecting vertices* $n_\pm$*,* $n \geq 1$.

*Proof.* The domain $\operatorname{dom} H^{\max}$ with its graph norm is a complete space by the previous lemma. The characterization of $\operatorname{dom} H^{\max}$ is just a reformulation of (C.6). ☐

We will see in Lemma C.10 that the maximal operator $H^{\max}$ is maximal in the sense that only a boundary condition at the root vertex 0 is missing in order to have a self-adjoint operator. We will impose a *Dirichlet boundary condition.*

Let $\mathfrak{h}_{G_n}$ be the quadratic form on $\mathsf{H}^1(G_n)$ defined by

$$\mathfrak{h}_{G_n}(f) = \sum_{e \in E(G_n)} \|f'\|_e^2 + q_n |f(n_-)|^2$$

where $\{q_n\}$ is the strength of the vertex potential satisfying (4.13e). We define the quadratic form on the line-like graph $L = L_0$ as

$$\mathfrak{h}_L(f) := \sum_{n \geq 1} \mathfrak{h}_{G_n}(f_n) \tag{C.13}$$

with domain

$$\mathsf{H}^1_\circ(L) := \operatorname{dom} \mathfrak{h}_L := \big\{\, f \in \mathsf{H}^1(L) \,\big|\, f(0) = 0 \,\big\} \tag{C.14}$$

where $\mathsf{H}^1(L)$ has been defined in (C.4).

Let $\mathfrak{d}_L$ be the "free" quadratic form, i.e., the form without vertex potential, namely

$$\mathfrak{d}_L(f) := \sum_{e \in E(L)} \|f'\|_e^2$$

for $f \in \mathsf{H}^1_\circ(L)$. Remember that the vertex potential in $\mathfrak{h}$ has support only at the ending vertices $n_-$ of $G_n$.

**Lemma C.8.** *The quadratic form $\mathfrak{d} = \mathfrak{d}_L$ is closed. Furthermore, $\mathfrak{h} = \mathfrak{h}_L$ is relatively form-bounded with respect to the form $\mathfrak{d}$ with relative bound 0. In particular, $\mathfrak{h}$ is a closed form on $\mathsf{H}^1_\circ(L)$.*

*Proof.* Since $\mathsf{H}^1_\circ(L)$ is a closed subspace of the Hilbert space $\mathsf{H}^1(L)$ with norm given by $\|f\|^2 + \mathfrak{d}(f)$, $\mathfrak{d}$ is a closed form on $\mathsf{H}^1(L)$. Furthermore, we have

$$\begin{aligned}|\mathfrak{h}(f) - \mathfrak{d}(f)| &\le q_0 \sum_{n\in\mathbb{N}} |f(n_-)|^2 \\ &\le q_0 \sum_{n\in\mathbb{N}} \Big(\varepsilon \|f'\|^2_{G_n} + \frac{4}{\varepsilon}\|f\|^2_{G_n}\Big) \le q_0\Big(\varepsilon\|f'\|^2_L + \frac{4}{\varepsilon}\|f\|^2_L\Big)\end{aligned}$$

for any $0 < \varepsilon < \ell_-$ using (C.1) and (4.13e) where $q_0 := \max\{|q_-|, |q_+|\}$, with shows the assertion. ☐

**Corollary C.9.** *We have $0 \le \mathfrak{d} \le 2(\mathfrak{h} - \lambda_0)$ where $\lambda_0 := -\max\{4q_0/\ell_-, 8q_0^2\} \le 0$. We might choose $\lambda_0 := 0$ if the strength of the vertex potential is nonnegative, i.e., $q \ge 0$.*

*Proof.* A simple application of the last estimate shows that

$$(1 - q_0\varepsilon)\mathfrak{d}(f) \le \mathfrak{h}(f) + \frac{4q_0}{\varepsilon}\|f\|^2.$$

Choose $\varepsilon = \min\{\ell_-, 1/(2q_0)\}$. If $q \ge 0$ then clearly $\mathfrak{d} \le \mathfrak{h}$. ☐

Denote the self-adjoint operator associated to the quadratic form $\mathfrak{h} = \mathfrak{h}_L$ by $H = H_L$, and similarly $\Delta^{\mathrm{D}}_L$ the operator associated to $\mathfrak{d}$.

**Lemma C.10.** *A function $f$ is in the domain of $H_L$ iff (i)–(iii) of Lemma C.7 are fulfilled and if*

(iv) $f(0) = 0$.

*Furthermore, $H_L$ is essentially self-adjoint on all functions $f \in \operatorname{dom} H_L$ with compact support.*

*Proof.* A function $f \in \operatorname{dom}\mathfrak{h}$ in the domain of the operator associated to $\mathfrak{h}$ satisfies $\mathfrak{h}(f, g) = \langle Hf, g\rangle$ where $Hf$ denotes an element in $\mathsf{L}_2(L)$. Choosing only functions $g$ with support inside an edge $e$, partial integration shows that $(Hf)_e = -f''_e$ in a distributional sense; and therefore $f_e \in \mathsf{H}^2(e)$. Taking general $g \in \operatorname{dom}\mathfrak{h}$, it is an easy exercise to see that the boundary terms from partial integration vanish iff $f$ satisfies the conditions of Lemma C.7 (ii) and (4.12) for all inner, respectively, connecting, vertices and $f(0) = 0$. Therefore all conditions (i)–(iv) are fulfilled.

If a function $f$ satisfies the condition (i), we know from (C.9) that also $f' \in \mathsf{L}_2(L)$. Together with (ii)–(iv) we have $f \in \operatorname{dom}\mathfrak{h}$. Furthermore, the same argument as before shows that for each $f$ there is $Hf := \{-f''_e\}_e$ such that $\mathfrak{h}(f, g) = \langle Hf, g\rangle$, i.e., $f$ is in the domain of the associated operator.

The essential self adjointness follows from Lemma C.6. ☐

Finally, we also need the following estimate in Section D.2:

**Lemma C.11.** *If $f \in \mathsf{H}^2(L)$ then $|(ff^\dagger)(n_+)| \to 0$ as $n \to \infty$.*

*Proof.* Suppose first that $f$ has compact support. Partial integration on the line-like graph $L_n$ and the boundary condition (4.12) yield

$$|(ff^\dagger)(n_+)| = |(ff^\dagger)(n_-)| \le |\langle Hf, f\rangle_{L_n}| + \|f'\|^2_{L_n} \le \|f\|^2_{\mathsf{H}^2(L_n)}.$$

This inequality extends to all functions $f \in \mathsf{H}^2(L)$ due to Lemma C.6. Now, if $f \in \mathsf{H}^2(L)$, then $\|f\|^2_{\mathsf{H}^2(L_n)} = \|f\|^2_{L_n} + \|f'\|^2_{L_n} + \|f''\|^2_{L_n} \to 0$ and the result follows. ☐

C.2. **Generalized eigenfunctions and integral kernels.** In this section we first provide the necessary estimate (B.1) in order to show that the results of Appendix B apply.

**Lemma C.12.** *Suppose that $H = H_L$ is the self-adjoint operator on the line-like graph $L = L_0$ with Dirichlet boundary condition at $0$ constructed as below. Then*

$$\|f\|^2_{\mathsf{L}_\infty(L)} \le 4\Big(\ell_- + \frac{1}{\ell_-}\Big)\|(H - \lambda_0 + 1)^{1/2} f\|^2_{\mathsf{L}_2(L)}$$

*for all $f \in \operatorname{dom} \mathfrak{h} = \operatorname{dom}(H - \lambda_0)^{1/2}$ where $\lambda_0$ is given in Corollary C.9 and $\ell_- > 0$ is defined in* (4.13a)*. In particular, Assumption B.1 is fulfilled with $m = 1$.*

*Proof.* We have

$$\begin{aligned}|f(x)|^2 \le \ell_- \|f'\|^2_{G_n} + \frac{4}{\ell_-}\|f\|^2_{G_n} \le 2\ell_-\big(\mathfrak{h}(f) - \lambda_0\|f\|^2_L\big) + \frac{4}{\ell_-}\|f\|^2_L \\ \le 4\Big(\ell_- + \frac{1}{\ell_-}\Big)\|(H - \lambda_0 + 1)^{1/2} f\|^2_{\mathsf{L}_2(L)}\end{aligned}$$

using (C.1) if $x \in G_n$ and $f \in \operatorname{dom} \mathfrak{h}$. In addition, $H$ is a local operator, so that Assumption B.1 is fulfilled. □

On a quantum graph, we can define the notion of a generalized eigenfunction as follows:

**Definition C.13.** We say that $\varphi$ is a *generalized eigenfunction* of the graph Hamiltonian $H$ on the line-like graph $L$ with eigenvalue $\lambda \in \mathbb{R}$ if $\varphi\restriction_e \in \mathsf{H}^2(e)$ and $-\varphi'' = \lambda\varphi$ on each edge $e$ and if $\varphi$ satisfies the vertex boundary conditions of Lemma C.7 (ii) at each inner vertex $v \in V_0(G_n)$ and the boundary conditions (4.12) at the connecting vertices $n_\pm$ and $f(0) = 0$.

Note that automatically, a generalized eigenfunction is smooth on each edge since it must have the form (2.12) on each edge.

The next lemma assures that the notion of generalized eigenfunction of Definition C.13 and Definition B.7 agree up to an integrability condition:

**Lemma C.14.** *Suppose that $\varphi \in \operatorname{dom} H_-$ and $H_-\varphi = \lambda\varphi$ then $\varphi$ is a generalized eigenfunction in the sense of Definition C.13. On the other hand, suppose that $\varphi$ is a generalized eigenfunction in the sense of Definition C.13 and that $\Phi\varphi \in \mathsf{L}_2(L)$, then $\varphi \in \operatorname{dom} H_- \subset \mathcal{H}_-$ and $H_-\varphi = \lambda\varphi$.*

*Proof.* For the first assertion, $H_-\varphi = \lambda\varphi$ is equivalent to

$$(\varphi, (H_+ - \lambda) f) = 0 \tag{C.15}$$

for all $f \in \operatorname{dom} H_+$. Using $f \in \mathsf{C}^\infty_{\mathrm{c}}(e)$ one sees that $-\varphi'' = \lambda\varphi$ in the distributional sense, and from regularity theory, we obtain $\varphi \in \mathsf{C}^\infty(e)$. It follows that for the boundary terms we have

$$\sum_{v \in V}\sum_{e \in E_v}\big(\overline{\varphi_e(v)}\, f'_e(v) - \overline{\varphi'_e(v)}\, f(v)\big) = 0$$

for all $f \in \operatorname{dom} H_+$. Using the argument of [KS99, Lem. 2.2] we see that $\varphi$ has to satisfy the same boundary conditions as $f$ at each vertex $v$.

For the converse we have $\varphi \in \mathcal{H}_-$ and one easily sees that (C.15) is fulfilled for all $f \in \operatorname{dom} H_+$ using partial integration and the boundary conditions for $f$ and $\varphi$. □

We prove next a representation of the integral kernel of the resolution of unity:

**Lemma C.15.** *Let $\lambda \in \mathbb{R}$. Then the integral kernel of $\widetilde{E}(\lambda)$ associated to the operator $H$ has the representation*

$$\widetilde{e}_\lambda(x,y) = \sum_j \overline{\varphi_{\lambda,j}(x)}\varphi_{\lambda,j}(y) \tag{C.16}$$

*where $\{\varphi_{\lambda,j}\}_j$ forms a basis of generalized eigenfunctions. Even if the family is infinite, the sum is locally finite and defined everywhere. In particular, $\widetilde{e}_\lambda(x,y)$ is continuous on each edge and satisfies the boundary condition at each vertex in $x$ and $y$.*

*In particular, if $\lambda$ is not an exceptional energy (cf. Definition D.9), i.e., $\lambda \notin E(L)$, then the sum reduces to*

$$\widetilde{e}_\lambda(x,y) = c_\lambda \overline{\varphi_\lambda(x)}\varphi_\lambda(y) \tag{C.17}$$

*where $c_\lambda = 1/\|\Phi\varphi_\lambda\|_L^2 \in \mathbb{R}$ and $\varphi_\lambda$ is the generalized eigenfunction with $\varphi_\lambda(0) = 0$ and $\varphi_\lambda'(0) = 1$. All statements hold for almost all $\lambda$ w.r.t. a spectral measure of $H$.*

*Proof.* A generalized eigenfunction is $\mathsf{C}^\infty$ on each edge and satisfies the boundary conditions due to Lemma C.14. Since the space of generalized eigenfunctions (without conditions at 0 and $\infty$) is generated by compactly supported functions and at most two noncompactly supported functions (cf. Lemma D.11), we can apply Lemma B.10 and obtain (C.16).

For the second assertion, note that for nonexceptional energies, the space of generalized eigenfunctions without conditions at 0 and $\infty$ is two-dimensional (cf. Lemma D.11). In addition, there is only one function satisfying the boundary condition $\varphi_\lambda(0) = 0$ and $\varphi_\lambda^\dagger(0) = 1$. The value of the normalization constant $c_\lambda$ follows from $1 = \|E(\lambda)\|_{\mathcal{B}_1} = \int_L \Phi(x)^2 \widetilde{e}_\lambda(x,x)\,\mathrm{d}x = c_\lambda\|\Phi\varphi_\lambda\|_L^2$. In addition, we have $c_\lambda \in (0,\infty)$ for almost all $\lambda$. □

We finally need an integral representation of the Green's functions:

**Corollary C.16.** *The Green's function (i.e., the kernel of $(H-z)^{-1}$) can be written as*

$$G_z(x,y) = \int_{\mathbb{R}} \frac{1}{\lambda - z} \widetilde{e}_\lambda(x,y)\,\mathrm{d}\rho(\lambda) \tag{C.18}$$

*for* all $x, y \in L$. *In particular, $G_z$ is continuous (even $\mathsf{C}^\infty$) outside the vertices and satisfies the boundary conditions of $H$ at each vertex in each variable.*

*Proof.* We obtain the kernel representation from Lemma B.6 a priori only for *almost* all $x, y \in L$, but the representation (C.16) assures that $G_z(x,y)$ is smooth outside the edges and satisfies the boundary conditions (since $\widetilde{e}_\lambda$ does). □

C.3. **Polynomial bounds on generalized eigenfunctions.** In this section we show weighted $\mathsf{L}_2$-bounds on generalized eigenfunctions on line-like graphs. In addition, we prove pointwise bounds on the eigenfunctions. To do so, we fix the weight function $\Phi$ needed in order to apply the results of Appendix B. The metric measure space $(X,\mu)$ will be the metric graph $L$ with its natural measure. For example, if $\Phi(x) = 1/n$ for $x \in G_n$ then we have

$$\|\Phi\|_L^2 = \sum_n \frac{1}{n^2}\ell(G_n) < \infty$$

by (4.13d). Therefore, Assumption B.2 is also fulfilled. From Theorem B.9 we obtain that a generalized eigenfunction $\varphi_\lambda$ satisfies $\|\Phi\varphi_\lambda\|_L < \infty$ for almost all $\lambda$ with respect to a spectral measure of $H$:

**Theorem C.17.** *Suppose that the assumptions* (4.13) *on the decoration graphs $\{G_n\}_n$ are fulfilled. Then the spectral measure is supported by those $\lambda$ for which there is a generalized eigenfunction $\varphi$ of polynomial growth rate (in the sense that $\|\Phi\varphi\|_L < \infty$).*

Our second main result is:

**Theorem C.18.** *Suppose the assumptions* (4.13) *are fulfilled. Let* $\varphi = \varphi_\lambda$ *be a generalized eigenfunction of* $H$ *in the sense of Definition B.7 and* $\Phi(x) > 0$ *for all* $x \in L$. *Then there exist* $C_3, C_3' > 0$ *such that*

$$|\Phi(x)\varphi(x)| \le C_3 w_n \|\Phi\varphi\|_{\mathsf{L}_2(G)}$$
$$|\Phi(n_+)\varphi^\dagger(n_+)| \le C_3' w_n \|\Phi\varphi\|_{\mathsf{L}_2(G)}$$

*for* $x \in G_n \subset L$ *and* $n \in \mathbb{N}$ *for almost all* $\lambda \in \mathbb{R}$ *with respect to a spectral measure of* $H$ *where*

$$w_n := \frac{\Phi_+(n)}{\Phi_-(n)} := \frac{\sup\{\,\Phi(x) \,|\, x \in G_n\,\}}{\inf\{\,\Phi(x) \,|\, x \in G_n^+\,\}}$$

*and* $G_n^+ := L_{n-2,n+1}$, *the concatenation of* $G_k$, $k = n-1, \dots, n+1$. *In particular, if* $\Phi(x) = 1/n$ *for* $x \in G_n$ *then* $w_n \le 2$ *and* $|\varphi(x)|$ *(*$x \in G_n$*) and* $|\varphi^\dagger(n_+)|$ *are polynomially bounded in* $n$.

*Remark* C.19. Note that we state the result only with respect to a spectral measure! Generally this is of course false, take for example $b_n = 2$, $H = \Delta_L$ and a function constant on each decoration graph $G_n$ satisfying the boundary condition $\varphi_n(n_-) = 2^{-1/2}\varphi_{n+1}(n_+)$. Then $H_+\varphi = 0$, but $\varphi$ has exponential growth since $\varphi_n(x) = 2^{(n-1)/2}\varphi(0)$ for $x \in G_n$. The important point here is that $0 \notin \sigma(H)$.

*Proof.* Let $x \in G_n$ and $\chi$ a function such that $\chi = 1$ on $G_n$, $\chi = 0$ on $G_k$, $|n-k| \ge 1$ and $\chi(n-1) = 0$, $\chi(n) = 1$, $\chi(n+1) = 1$ and $\chi(n+2) =$ on $G_{n-1}$ and $G_{n+1}$ as constructed in Lemma C.2. Note that $\chi$ has support in $G_n^+$. Now,

$$|\Phi(x)\varphi(x)|^2 \le \Phi_+(n)^2\big(\ell_-\|\varphi'\|^2_{G_n} + \frac{4}{\ell_-}\|\varphi\|^2_{G_n}\big)$$

due to (C.1). Furthermore, $\|df\|^2 \le 2\|(H-\lambda_0+1)^{1/2}f\|^2$ due to Corollary C.9 and therefore $\|dR^{1/2}\|^2 \le 2$. From Lemma C.3 we conclude that

$$\begin{aligned}\|\varphi'\|_{G_n} &\le \|(\chi\varphi)'\|_{G_n^+} \\ &\le \sqrt{2}\Big((\lambda-\lambda_0) + \frac{4}{\ell_-^2} + \frac{4\sqrt{2}}{\ell_-} + 1\Big)\|\varphi\|_{G_n^+} =: C_2\|\varphi\|_{G_n^+}. \qquad \text{(C.19)}\end{aligned}$$

Finally,

$$\begin{aligned}|\Phi(x)\varphi(x)|^2 &\le \Phi_+(n)^2\Big(\frac{4}{\ell_-} + \ell_- C_2^2\Big)\|\varphi\|^2_{G_n^+} \\ &\le \Big(\frac{\Phi_+(n)}{\Phi_-(n)}\Big)^2\Big(\frac{4}{\ell_-} + \ell_- C_2^2\Big)\|\Phi\varphi\|^2_L \le w_n^2\Big(\frac{4}{\ell_-} + \ell_- C_2^2\Big)\|\Phi\varphi\|^2_L =: w_n^2 C_3^2\|\Phi\varphi\|^2_L.\end{aligned}$$

The second assertion follows similarly:

$$\begin{aligned}|\Phi(n_+)\varphi^\dagger((n-1)_+)|^2 &\le \Phi_+(n)^2 C_1'\big(\lambda^2\|\varphi\|^2_{G_n} + \|\varphi'\|^2_{G_n}\big) \\ &\le w_n^2 C_1'(\lambda^2 + C_2^2)\|\varphi\|^2_{G_n^+} \le w_n^2 C_1'(\lambda^2 + C_2^2)\|\Phi\varphi\|^2_L =: C_3'^2\|\Phi\varphi\|^2_L.\end{aligned}$$

$\square$

## D. Transfer matrices and Weyl-Titchmarsh functions

D.1. **Transfer matrix for generalized eigenfunctions.** We want to prove in this section that the transfer matrix is defined up to an exceptional set. For a fixed sequence of graphs $\{G_n\}$, the exceptional set is countable. The main ingredient is the *Dirichlet-to-Neumann map* (see for example [Ong05] or [FOP04]). Let $G_*$ be one of the decoration graphs replacing an edge in the tree graph with two boundary vertices $o_0$ and $o_1$.

Denote by $\Delta^{\mathrm{D}}_{G_*}$ the Dirichlet operator on $G_*$, i.e., the self-adjoint operator on functions $u \in \mathsf{H}^2(G_*)$ satisfying $u(o_0) = 0$ and $u(o_1) = 0$. Denote its spectrum repeated according to multiplicity by $\{\lambda_k\}_k$ and the corresponding orthonormal basis of *real* eigenfunctions by $\{\varphi_k\}_k$. We first state some results on the solution map

$$H_z \colon \mathbb{C}^2 \longrightarrow \mathsf{H}^2(G_*) \tag{D.1}$$

of the Dirichlet problem, i.e., $f = H_z(F_0, F_1)$ solves the equation $(\Delta^{\max}_{G_*} - z)f = 0$ with initial data $f(o_0) = F_0$ and $f(1) = F_1$ for $z \notin \sigma(\Delta^{\mathrm{D}}_{G_*})$. We need the following extension map

$$E \colon \mathbb{C}^2 \longrightarrow \mathsf{H}^2(G_*), \tag{D.2}$$

i.e., $\widetilde{f} = E(F_0, F_1) \in \mathsf{H}^2(G_*)$ such that $f(o_0) = F_0$, $f(o_1) = F_1$ and $\widetilde{f}^\dagger(o_i) = 0$ for $i = 0, 1$. For example, $\widetilde{f} := E(F_0, F_1) := F_0\chi + F_1(1-\chi)$ is a possible choice where $\chi$ is the smooth cut-off function constructed in Lemma C.2. In particular, the derivatives of $\chi$ up to order 2 enter in $\|E\|$ so that $\|E\|$ can be bounded by a universal polynomial of degree 2 in $1/\ell_-$.

In addition, denote by $\Delta^{\max}_{G_*}$ the differential operator $\Delta_{G_*}$ with *maximal* domain, i.e., $f \in \operatorname{dom} \Delta^{\max}_{G_*}$ iff $f, \Delta_{G_*} f \in \mathsf{L}_2(G_*)$.

We can now give expressions for the Dirichlet solution map:

**Lemma D.1.** *For $z \notin \sigma(\Delta^{\mathrm{D}}_{G_*})$ the solution map $H_z$ in* (D.1) *is given by*

$$H_z = \bigl(\mathbb{1} - (\Delta^{\mathrm{D}}_{G_*} - z)^{-1}(\Delta^{\max}_{G_*} - z)\bigr)E \tag{D.3}$$

*and is bounded as map $H_z \colon \mathbb{C}^2 \longrightarrow \mathsf{L}_2(G_*)$ with norm estimated by*

$$\|H_z\| \le \Bigl(1 + \frac{1 + |z|}{d(z, \sigma(\Delta^{\mathrm{D}}_{G_*}))}\Bigr)\|E\|. \tag{D.4}$$

*In addition, $H_z$ is also bounded as map $H_z \colon \mathbb{C}^2 \longrightarrow \mathsf{H}^2(G_*)$. Furthermore, $z \mapsto H_z$ is norm-analytic with the series representation*

$$H_z(F_0, F_1) = -\sum_k \frac{1}{\lambda_k - z}\bigl(\varphi_k^\dagger(o_1)F_1 - \varphi_k^\dagger(o_0)F_0\bigr)\varphi_k \tag{D.5a}$$

$$= H_0(F_0, F_1) - \sum_k \frac{z}{\lambda_k(\lambda_k - z)}\bigl(\varphi_k^\dagger(o_1)F_1 - \varphi_k^\dagger(o_0)F_0\bigr)\varphi_k \tag{D.5b}$$

*with $\widetilde{f} = E(F_0, F_1)$ where the first series converges in $\mathsf{L}_2(G_*)$ and the second in $\mathsf{H}^2(G_*)$.*

*Proof.* First, $H_z$ is well defined as map from $\mathbb{C}^2$ into $\mathsf{L}_2(G_*)$. Next, it follows from $\operatorname{dom} \Delta^{\mathrm{D}}_{G_*} \subset \operatorname{dom} \Delta^{\max}_{G_*}$ that $f = H_z(F_0, F_1)$ solves $(\Delta^{\max}_{G_*} - z)f = 0$. In addition, $f(o_0) = \widetilde{f}(o_0) = F_0$ and similarly in $o_1$, since functions in the range of the resolvent vanish at the boundary. Furthermore, $H_z$ is bounded as map into $\mathsf{L}_2(G_*)$ or as map into $\operatorname{dom} \Delta^{\max}_{G_*}$ with the graph norm given by $\|f\|^2_{\Delta^{\max}_{G_*}} := \|\Delta^{\max}_{G_*} f\|^2 + \|f\|^2$. Now, due to (C.12), we have

$$\|f\|^2_{\mathsf{H}^2(G_*)} \le (\widetilde{C}_1'' + 1)\|f\|^2_{\Delta^{\max}_{G_*}}$$

but the constant $\widetilde{C}_1''$ is not uniform in the sense of Assumption 4.8. Nevertheless, $H_z$ is continuous as map into $\mathsf{H}^2(G_*)$.

Expanding the resolvent into a series of eigenvector we obtain

$$H_z(F_0, F_1) = \widetilde{f} - \sum_k \frac{1}{\lambda_k - z} \langle \varphi_k, (\Delta_{G_*}^{\max} - z)\widetilde{f} \rangle \varphi_k.$$

Since $\widetilde{f} \in \operatorname{dom} \Delta_{G_*}^{\max}$ the coefficients

$$a_k := \frac{1}{\lambda_k - z} \langle \varphi_k, (\Delta_{G_*}^{\max} - z)\widetilde{f} \rangle \qquad \text{and} \qquad \lambda_k a_k$$

form sequences in $\ell_2(\mathbb{N})$. It follows that the series converge in $\mathsf{L}_2(G_*)$ and in $\operatorname{dom} \Delta_{G_*}^{\max}$ with the graph norm, and therefore the series also converges in $\mathsf{H}^2(G_*)$. The first series representation follows from partial integration. Note that

$$b_k = \langle \varphi_k, \widetilde{f} \rangle - \frac{1}{\lambda_k - z} \langle \varphi_k, (\Delta_{G_*}^{\max} - z)\widetilde{f} \rangle = -\frac{1}{\lambda_k - z} \big( \varphi_k^\dagger(o_1) F_1 - \varphi_k^\dagger(o_0) F_0 \big)$$

is in $\ell_2(\mathbb{N})$ since $\widetilde{f}$ and $\Delta_{G_*}^{\max} \widetilde{f}$ are both in $\mathsf{L}_2(G_*)$. □

For $z \in \mathbb{C} \setminus \sigma(\Delta_{G_*}^{\mathrm{D}})$, we can define the Dirichlet-to-Neumann map:

**Definition D.2.** The *Dirichlet-to-Neumann map* $\Lambda(G_*, z) = \Lambda(z)$ is the $2 \times 2$-matrix defined by[5]

$$\Lambda_{ij}(z) = H_z(\vec{e}_j)^\dagger(o_i) \tag{D.6}$$

for $i, j = 0, 1$, where $\vec{e}_0 = (1, 0)$ and $\vec{e}_1 = (0, 1)$. Let

$$\Psi_i(\lambda) = \{\, \varphi_k^\dagger(o_i) \,|\, k \text{ with } \lambda = \lambda_k \,\} \tag{D.7}$$

be the vector of boundary derivatives with dimension equal to the multiplicity of the eigenvalue $\lambda$. We denote

$$\sigma_{\mathrm{red}}(\Delta_{G_*}^{\mathrm{D}}) := \big\{\, \lambda \in \sigma(\Delta_{G_*}^{\mathrm{D}}) \,\big|\, \Psi_0(\lambda) \neq 0 \text{ or } \Psi_1(\lambda) \neq 0 \,\big\} \tag{D.8}$$

the *reduced Dirichlet spectrum of* $G_*$.

The next lemma explains the reason for introducing the reduced spectrum:

**Lemma D.3.** *The Dirichlet-to-Neumann map $\Lambda(z)$ is meromorphic in $z$ with poles of order 1 in the reduced spectrum of $\Delta_{G_*}^{\mathrm{D}}$ and has the absolutely convergent series*

$$\Lambda_{ij}(z) = \Lambda_{ij}(0) + (-1)^j \sum_{\lambda \in \sigma_{\mathrm{red}}(\Delta_{G_*}^{\mathrm{D}})} \frac{z \Psi_{ij}(\lambda)}{\lambda(\lambda - z)} \tag{D.9}$$

*where $\Psi_{ij}(\lambda) := \Psi_i(\lambda) \cdot \Psi_j(\lambda) = \sum_{k, \lambda_k = \lambda} \varphi_k^\dagger(o_i) \varphi_k^\dagger(o_j)$. In addition, we have $\Lambda_{10}(z) = -\Lambda_{01}(z)$.*

*Proof.* The series representation (D.9) and the absolutely convergence follows from (D.5b) and the fact that $d_0 = (\cdot)^\dagger(o_0)$ is continuous on $\mathsf{H}^2(G_*)$ by (C.2) (a similar *nonuniform* estimate holds for $d_1$). Note that if $\lambda \in \sigma(\Delta_{G_*}^{\mathrm{D}}) \setminus \sigma_{\mathrm{red}}(\Delta_{G_*}^{\mathrm{D}})$, then $\Psi_0(\lambda) = 0$ and $\Psi_1(\lambda) = 0$, i.e., the pole $\lambda$ does not appear in the series. The last statement follows from (D.9) once we have $\Lambda_{01}(0) = -\Lambda_{10}(0)$: To see the last equality, note that if $f = H_0(F_0, F_1)$ is a harmonic function with boundary values $F_0$ and $F_1$, then

$$0 = \langle \Delta_{G_*}^{\max} f, f \rangle - \langle f, \Delta_{G_*}^{\max} f \rangle = \langle \Lambda(0) J\vec{F}, \vec{F} \rangle_{\mathbb{C}^2} - \langle \vec{F}, \Lambda(0) J\vec{F} \rangle_{\mathbb{C}^2}$$

[5] In this section, we assume that there is no vertex potential, i.e., $q(o_1) = 0$ (cf. (4.7)).

where $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. In particular, $\Lambda(0)^* = J\Lambda(0)J$ and since $\Lambda(0)$ is a real matrix, we obtain the claim. $\square$

*Remark* D.4. (i) Note that $\Lambda(z) \neq \Lambda(z)^{\mathrm{tr}}$ due to our definition of $f^\dagger$, with has a different orientation at $o_0$ and $o_1$.

(ii) We have seen the phenomena that the set of poles of $\Lambda(z)$ is smaller than the Dirichlet spectrum already in the necklace decoration model (see Footnote 3). For the $p-1$ linearly independent eigenfunctions $\varphi_i$ with eigenvalue $\lambda = (\pi k/\ell)^2$ living on the $p$ loop edges we have $\varphi_i = 0$ near $o_0$ and $\varphi_i^\dagger(o_1) = 0$ since $(\cdot)^\dagger(o_1)$ is the sum over all edges meeting in $o_1$. In particular, the boundary derivative vectors $\Psi_i(\lambda)$ vanish for $i = 0, 1$.

In order to define the transfer matrix, we need the following notion:

**Definition D.5.** We say that $\lambda \in \mathbb{C}$ is a *separating energy for* $G_*$ if there exists a nontrivial solution $\varphi$ of the equation $(\Delta_{G_*}^{\max} - \lambda)\varphi = 0$ such that $\vec{\Phi}_0 = \vec{0}$ or $\vec{\Phi}_1 = \vec{0}$ where $\vec{\Phi}_i = (\varphi, \varphi^\dagger)(o_i)$. Denote

$$\widetilde{E}(G_*) := \big\{\, \lambda \in \mathbb{C} \,\big|\, \lambda \text{ is a separating energy for } G_* \,\big\} \tag{D.10}$$

the set of separating energies for $G_*$.

We call $\lambda$ an *exceptional energy* $G_*$ iff $\lambda$ is an element of

$$E(G_*) := \big\{\, \lambda \in \mathbb{C} \setminus \sigma(\Delta_{G_*}^{\mathrm{D}}) \,\big|\, \Lambda_{01}(\lambda) = 0 \,\big\} \cup \sigma(\Delta_{G_*}^{\mathrm{D}}) \tag{D.11}$$

the set of exceptional energies.

We will see in the next two lemmas, that for nonexceptional energies, we can uniquely define the transfer matrix.

**Lemma D.6.** (i) $E(G_*) \setminus \sigma(\Delta_{G_*}^{\mathrm{D}}) \subset \widetilde{E}(G_*) \subset \mathbb{R}$.

(ii) *Let $z \notin E(G_*)$, i.e., $z$ is not in the Dirichlet spectrum and $\Lambda_{01}(z) \neq 0$. Then for each $\vec{F}_0 \in \mathbb{C}$ there exists a unique solution of the equation $\Delta_{G_*}^{\max} f = zf$ with $f \in \mathsf{H}^2(G_*)$ and $\vec{F}_0 = (f(o_0), f^\dagger(o_0))$. We set $\vec{F}_1 = (f(o_1), f^\dagger(o_1)) \in \mathbb{C}^2$ and denote the solution by $T_z(x)\vec{F}_0 := T_z(x, G_*)\vec{F}_0 = f(x)$ for $x \in G_*$.*

*The* transfer *or* monodromy *matrix $T_z = T_z(G_*)$ is uniquely defined by $\vec{F}_1 = T(z)\vec{F}_0$. The transfer matrix is unimodular, i.e.,* $\det T_z(G_*) = 1$, *and satisfies*

$$T_z = T_z(G_*) = \frac{1}{\Lambda_{01}(z)} \begin{pmatrix} -\Lambda_{00}(z) & 1 \\ -\det \Lambda(z) & \Lambda_{11}(z) \end{pmatrix}. \tag{D.12}$$

*The transfer matrix is still uniquely defined for $\lambda \in \sigma(\Delta_{G_*}^{\mathrm{D}}) \setminus \sigma_{\mathrm{red}}(\Delta_{G_*}^{\mathrm{D}})$ and $\Lambda_{01}(\lambda) \neq 0$, although the solution "map" $T_\lambda(\cdot)$ is no longer uniquely defined.*

(iii) *Suppose that $\lambda_k \in \sigma_{\mathrm{red}}(\Delta_{G_*}^{\mathrm{D}})$ is a simple eigenvalue such that $\varphi_k^\dagger(o_0) \neq 0$ and $\varphi_k^\dagger(o_1) \neq 0$. Then the transfer matrix $T_z$ has an analytic continuation into $\lambda_k$ given by*

$$T(\lambda_k) = \begin{pmatrix} \dfrac{\varphi_k^\dagger(o_0)}{\varphi_k^\dagger(o_1)} & 0 \\ T_{21}(\lambda_k) & \dfrac{\varphi_k^\dagger(o_1)}{\varphi_k^\dagger(o_0)} \end{pmatrix}. \tag{D.13}$$

*with*

$$T_{21}(\lambda_k) = \frac{1}{\varphi_k^\dagger(o_0)\varphi_k^\dagger(o_1)}\Big(-z\sum_{n\neq k}\frac{\big(\varphi_k^\dagger(o_0)\varphi_n^\dagger(o_1)-\varphi_k^\dagger(o_1)\varphi_n^\dagger(o_0)\big)^2}{\lambda_k(\lambda_n-\lambda_k)}+$$
$$\varphi_k^\dagger(o_0)^2\Lambda_{11}(0)-\varphi_k^\dagger(o_1)^2\Lambda_{00}(0)-2\varphi_k^\dagger(o_0)\varphi_k^\dagger(o_1)\Lambda_{01}(0)\Big). \quad \text{(D.14)}$$

*Proof.* (i) A separating energy is an eigenvalue for a self-adjoint operator, i.e., $\widetilde{E}(G_*)\subset\mathbb{R}$. Note that the boundary condition depends on the fixed solution $\varphi$. Let $\lambda\in E(G_*)$ and $\lambda\notin\sigma(\Delta^{\mathrm{D}}_{G_*})$ $\lambda\in\sigma_{\mathrm{red}}(\Delta^{\mathrm{D}}_{G_*})$ then $\Lambda_{01}(\lambda)=0$ and therefore also $\Lambda_{10}(\lambda)=0$, so that $\Lambda(\lambda)$ is a diagonal matrix. In this case, there exist two linearly independent solutions $\varphi^{(0)}$, $\varphi^{(1)}$ of the eigenvalue equation such that

$$\vec{\Phi}_0^{(0)}=\begin{pmatrix}1\\ \Lambda_{00}(\lambda)\end{pmatrix},\quad \vec{\Phi}_1^{(0)}=\begin{pmatrix}0\\0\end{pmatrix},\quad \vec{\Phi}_0^{(1)}=\begin{pmatrix}0\\0\end{pmatrix},\quad \vec{\Phi}_1^{(1)}=\begin{pmatrix}1\\ \Lambda_{11}(\lambda)\end{pmatrix}$$

where $\vec{\Phi}_i^{(j)}=(\varphi^{(j)},\varphi^{(j)\dagger})(o_i)$.

(ii) If $\Lambda_{01}(z)\neq 0$ then a simple calculation shows that the transfer matrix $T_z(G_*)$ is given by (D.12). Furthermore, $\det T_z(G_*)=1$ since $\Lambda_{10}(z)=-\Lambda_{01}(z)$. Note that if $\lambda\in\sigma(\Delta^{\mathrm{D}}_{G_*})\setminus\sigma_{\mathrm{red}}(\Delta^{\mathrm{D}}_{G_*})$ then the derivatives of all Dirichlet solutions with eigenvalue $\lambda$ vanish at both boundary points $o_0$ and $o_1$ and can therefore be added to a solution $f$ without infecting the boundary vectors $\vec{F}_0$ and $\vec{F}_1$ and in particular, the transfer matrix $T_\lambda$.

(iii) The last assertion follows by a straightforward calculation. □

*Remark* D.7. (i) We do not show in general that $E(G_*)$ is discrete, but this is always fulfilled in our examples: The only point to check for the discreteness is that $\Lambda_{01}(z)$ is not constant.

(ii) The name *separating energy* comes from the fact that if e.g. $\lambda\in E(G_*)\setminus\sigma(\Delta^{\mathrm{D}}_{G_*})$ then $\Lambda_{01}(\lambda)=0$; we have seen in the proof that there exist two independent separating solutions. In particular, the recursion equation is "interrupted" or "separated" at such a decoration graph.

(iii) We do not give the possible extension of the solution map $T_z(\cdot)$ into (parts) of the Dirichlet spectrum, although the norm estimate (D.15) in the next lemma is quite rough. But in our applications, it does not matter if our exceptional set is larger then necessary.

Next, we give an expression for the solution of the eigenvalue equation on $G_*$ in terms of $\vec{F}(0)$. Its proof follows immediately from Lemma D.1 and a simple calculation.

**Lemma D.8.** *Let $z\notin E(G_*)$ then the solution map $T_z(\cdot)\colon\mathbb{C}^2\longrightarrow\mathsf{H}^2(G_*)$ defined in Lemma D.6 (ii) is given by*

$$T_z(\cdot)\begin{pmatrix}F_0\\F_0'\end{pmatrix}=H_z\begin{pmatrix}F_0\\F_1\end{pmatrix}\quad\text{with}\quad\begin{pmatrix}F_0\\F_1\end{pmatrix}:=\begin{pmatrix}1 & 0\\ -\dfrac{\Lambda_{00}(z)}{\Lambda_{01}(z)} & \dfrac{1}{\Lambda_{01}(z)}\end{pmatrix}\begin{pmatrix}F_0\\F_0'\end{pmatrix}$$

*and defines a continuous map from $\mathbb{C}^2$ into $\mathsf{L}_2(G_*)$, respectively, $\mathsf{H}^2(G_*)$. Its norm as map into $\mathsf{L}_2(G_*)$ is bounded by*

$$\|T_z(\cdot)\|\le\Big(1+\frac{1+|z|}{d(z,\sigma(\Delta^{\mathrm{D}}_{G_*}))}\Big)\|E\|\Big(1+\frac{|\Lambda_{00}(z)|+1}{|\Lambda_{01}(z)|}\Big) \quad \text{(D.15)}$$

*where the norm $\|E\|$ of the extension operator* (D.2) *is bounded by a universal polynomial of degree* 2 *in* $1/\ell_-$.

We now consider a sequence of decoration graphs $G_n$, attached to a line-like graph $L$:

**Definition D.9.** For a line-like graph $L$ consisting of the concatenations of $\{G_n\}_n$, we say that $\lambda \in \mathbb{R}$ is an *exceptional energy for* $L$ if $\lambda$ is an exceptional energy for at least one decoration graph $G_n$. Denote

$$E(L) := \bigcup_n E(G_n) \tag{D.16}$$

the set of all exceptional energies for $L$.

**Definition D.10.** Let $f$ be a generalized eigenfunction associated to the eigenvalue $z$ in the sense of Definition C.13, but without condition at the vertex 0 and no decay condition at $\infty$. We set $\vec{F}(v) := (f(v), f^\dagger(v))$ and $\vec{F}(n) := \vec{F}(n_+)$, where $n_+$ is the starting vertex of $G_{n+1}$.

A generalized eigenfunction is called *essentially noncompactly supported* if there is $n_0 \in \mathbb{N}$ such that $\{\, n \in \mathbb{N} \mid \vec{F}(n) \neq \vec{0}\,\} = \{\, n \in \mathbb{N} \mid n \geq n_0 \,\}$ and any linear combination of $f$ does not contain a compactly supported eigenfunction.

The next lemma makes an assertion about the dimension of the space of generalized eigenfunctions:

**Lemma D.11.** *If $z \notin E(L)$, i.e., if $z$ is nonexceptional for the line-like graph $L$, then the the space of generalized eigenfunctions $f$ (without condition at* 0 *and* $\infty$*) with eigenvalue $z$ is completely determined by the solution space of the recursion equation*

$$\vec{F}(n) = D(b_n)T_z(G_n)\vec{F}(n).$$

*In addition, the solution space is two-dimensional. Finally, if $\lambda \in E(L)$, then the space of essentially not-compactly supported generalized eigenfunctions has dimension at most* 2.

*Proof.* The first assertion is a simple consequence of Lemma D.6. The second assertion follows from the fact that an essentially noncompactly supported eigenfunction $f$ is completely determined by its start vector (the first nonvanishing vector $\vec{F}(n_0)$. ☐

The next lemma is a simple consequence of the definition of the transfer matrix:

**Lemma D.12.** *Suppose that $f$ and $g$ are two generalized eigenfunctions in the sense of Definition C.13 associated to the same eigenvalue $\lambda$, but without condition at the vertex* 0*. Then the so-called* Wronskian

$$W(f,g)(n) := f^\dagger(n_+)g(n_+) - f(n_+)g^\dagger(n_+) \tag{D.17}$$

*is independent of $n$. In addition, if $\lambda \notin E(L)$, $W(f,g) = 0$, $f \neq 0$ and $f(0) = 0$ then also $g(0) = 0$.*

*Proof.* The generalized eigenfunctions can be constructed via

$$\vec{F}_\lambda(n) = U_\lambda(n)\vec{F}(0) \qquad \text{and} \qquad \vec{G}_\lambda(n) = U_\lambda(n)\vec{G}(0).$$

In particular, the Wronskian is given by

$$\begin{aligned} W(f,g)(n) = \det(\vec{F}_\lambda(n), \vec{G}_\lambda(n)) &= \det U_\lambda(n)(\vec{F}_\lambda(0), \vec{G}_\lambda(0)) \\ &= \det U_\lambda(n) W(f,g)(0) \end{aligned}$$

and the result follows from the fact that the transfer matrices $U_\lambda(n) = T_\lambda(G_n) \cdot \ldots \cdot T_\lambda(G_1)$ are unimodular (cf. (D.12)).

The last assertion follows from the fact, that a generalized eigenfunction $f$ is uniquely determined by $\vec{F} = (f(0), f^\dagger(0))$ if $\lambda \notin E(L)$ (see Lemma D.11), so $f \neq 0$ implies

$\vec{F}(n_+) \neq 0$ for some $n$ (Lemma D.6) and therefore $\vec{F}(0) \neq 0$. Finally, $f(0) = 0$ implies $f^\dagger(0) \neq 0$ and therefore also $g(0) = 0$ since $W(f, g) = 0$. $\square$

D.2. **Weyl-Titchmarsh functions.** In this section we define the Weyl-Titchmarsh function associated to an operator on a line-like graph $L = L_0$. It will encode the corresponding spectral measure. This function will be used in Appendix E in order to show that the corresponding *averaged* measure is absolutely continuous with respect to the Lebesgue measure.

The Weyl-Titchmarsh function is defined as follows: Let $\psi = \psi_z$ be a generalized eigenfunction solving

$$H\psi = z\psi \tag{D.18}$$

in the sense of Definition C.13, but without condition at the vertex 0. Here, $z \in \mathbb{C}_+ := \{\, z \in \mathbb{C} \,|\, \operatorname{Im} z > 0 \,\}$ is in the upper half-plane. The aim of the next lemma is to show that there is exactly one such function $\psi$ satisfying $\psi \in \mathsf{L}_2(L)$ and $\psi(0) = 1$.

To give (D.18) a proper meaning, we let $H$ be the maximal operator $H^{\max}$ as defined in (C.6) with domain $\operatorname{dom} H^{\max} = \mathsf{H}^2(L)$. We derived an equivalent characterization of $\mathsf{H}^2(L)$ in Lemma C.7.

**Lemma D.13.** *For each $z \in \mathbb{C}_+$ there exists exactly one function $\psi = \psi_z \in \operatorname{dom} H^{\max}$ such that $H^{\max}\psi = z\psi$, $\psi(0) = 1$ and $\psi \in \mathsf{L}_2(L)$.*

*Proof.* We use an argument similar to the definition of the solution map in (D.3). Let $\widetilde{\psi}$ be a function in $\mathsf{H}^2(L)$ with compact support such that $\psi(0) = 1$. We set

$$\psi_z := \widetilde{\psi} - (H^{\mathrm{D}} - z)^{-1}(H^{\max} - z)\widetilde{\psi}.$$

Here, $H^{\mathrm{D}}$ is the *Dirichlet* Hamiltonian as defined in Definition 4.7. The function $\psi_z$ is well-defined, in $\mathsf{L}_2(L)$ and satisfies the eigenvalue equation. Furthermore, since functions in the domain of the Dirichlet Hamiltonian vanish at 0, we also have $\psi_z(0) = \widetilde{\psi}(0) = 1$. This proves the existence of $\psi_z$.

Uniqueness follows from the fact that $H^{\mathrm{D}}$ is self-adjoint: Suppose there is another function $\hat{\psi}_z \in \mathsf{H}^2(L)$ solving the eigenvalue equation. Then $u := \hat{\psi}_z - \psi_z \in \mathsf{L}_2(L)$ is also a nontrivial solution of (D.18) and $u(0) = 0$. In particular, $u \in \operatorname{dom} H^{\mathrm{D}}$. Since a self-adjoint operator cannot have a nonreal eigenvalue, we have $u = 0$, i.e, $\hat{\psi}_z = \psi_z$. $\square$

We can now define the *Weyl-Titchmarsh* function as[6]

$$m(z) := \frac{\psi_z^\dagger(0)}{\psi_z(0)} = \psi_z^\dagger(0), \tag{D.19}$$

due to normalization. Note that $m$ is an analytic function on $\mathbb{C}_+$. The next lemmas will show that $m$ maps $\mathbb{C}_+$ into $\mathbb{C}_+$, i.e., that $m$ is a *Herglotz* function.[7]

**Lemma D.14.** *We have $m(z) = \operatorname{Im} z \|\psi_z\|_L^2$. In particular, $m$ is a Herglotz function.*

*Proof.* We have

$$\langle \psi, H^{\max}\psi \rangle_{L_{0,n}} - \langle H^{\max}\psi, \psi \rangle_{L_{0,n}} = W(\overline{\psi}, \psi)(n_-) - W(\overline{\psi}, \psi)(0)$$

since all other boundary terms vanish due to the inner vertex boundary conditions. Now, the left hand side equals $2\mathrm{i} \operatorname{Im} z \|\psi\|_{L_{0,n}}^2$, and $-W(\overline{\psi}, \psi)(0) = 2\mathrm{i}\, m(z)$ and

$$|W(\overline{\psi}, \psi)(n_-)| \leq 2|(\psi^\dagger\psi)(n_-)| = 2|(\psi^\dagger\psi)(n_+)| \to 0$$

using the boundary condition at $n$ and Lemma C.11. The result follows as $n \to \infty$. $\square$

[6] For the notation $(\cdot)^\dagger$ see (4.7).

[7] A *Herglotz* or *Nevanlinna* function $m$ is an analytic function on the upper half-plane $\mathbb{C}_+$ such that $\operatorname{Im} m(z) > 0$ for all $z \in \mathbb{C}_+$, or, equivalently, an analytic function $m\colon \mathbb{C}_+ \longrightarrow \mathbb{C}_+$.

We now want to relate the Weyl-Titchmarsh function with (a component of) the spectral measure $\rho$ associated to $H := H^{\mathrm{D}}$. To do so we need the Green's function near the connecting vertices $n$. Let

$$L_{\mathrm{split}} := \{\, x \in L \,|\, L \setminus \{x\} \text{ has two disjoint components } L_{0,x} \text{ and } L_{x,\infty} \,\}. \tag{D.20}$$

In particular, points on a loop of the graph do not lie in $L_{\mathrm{split}}$. From Assumption (4.13c) it follows that $n \in L_{\mathrm{split}}$ is *not* an isolated point. In particular, $n_+$ is always succeeded by an interval contained in $L_{\mathrm{split}}$.

**Lemma D.15.** *For* nonisolated *points $x, y$ in $L_{\mathrm{split}}$ we have*

$$G_z(x,y) = (s_z \wedge \psi_z)(x,y) \tag{D.21}$$

*where*

$$(f \wedge g)(x,y) := \begin{cases} f(x)g(y) & \text{if } y \in L_{x,\infty}, \\ f(y)g(x) & \text{if } y \in L_{0,x}, \end{cases}$$

*$s = s_z$ is the (unique)*[8] *generalized eigenfunction with $s_z(0) = 0$ and $s_z^\dagger(0) = 1$, and $G_z(x,y)$ is the Green's function for $z \in \mathbb{C}_+$, i.e., the kernel of $(H-z)^{-1}$.*

*Proof.* Let $f \in \mathsf{L}_2(L)$ and

$$g(x) := \int_L (s \wedge \psi)(x,y) f(y)\,\mathrm{d}y = \int_{L_{0,x}} s(y) f(y)\,\mathrm{d}y\, \psi(x) + \int_{L_{x,\infty}} \psi(y) f(y)\,\mathrm{d}y\, s(x).$$

It is easy to see that $g$ is smooth on the interior of $L_{\mathrm{split}}$, that $g$ satisfies the boundary conditions at those vertices in $L_{\mathrm{split}}$ (here, we need the fact that $x$ is not isolated in $L_{\mathrm{split}}$ in order to apply a limit argument). Furthermore, a simple calculation shows that $-g''(x) - zg(x) = W(s,\psi)(x) f(x) = f(x)$ since the Wronskian is constant on $L_{\mathrm{split}}$ (see Lemma D.12) and equals 1 due to our boundary condition at $x = 0$. The Green's function is pointwise defined, smooth away from the vertices and satisfies the boundary conditions at each vertex in $x$ and $y$ separately (cf. Corollary C.16). In particular, $g(x) = \int_L G_z(x,y) f(x)\,\mathrm{d}y$ for all $x \in L$. Since a continuous kernel is uniquely defined, (D.21) follows for nonisolated points in $L_{\mathrm{split}}$. □

In the general graph-decorated case, it may happen that $\delta_0'$ is not a cyclic vector. In this case, one has to assure that the spectral measure on the complement can only be pure point:

**Lemma D.16.** *We have*

$$m(z) = \partial_{xy} G_z(0,0) = \int_{\mathbb{R}} \frac{1}{\lambda - z}\,\mathrm{d}\hat\rho(\lambda) \tag{D.22}$$

*where $\mathrm{d}\hat\rho(\lambda) := \sum_j |\varphi_{\lambda,j}'(0)|^2\,\mathrm{d}\rho(\lambda)$. In addition, there is a countable set $E_{\mathrm{pp}} \subset \mathbb{R}$ such that the measure $\hat\rho + \rho_{\mathrm{pp}}$ is a spectral measure for $H$ and $\rho_{\mathrm{pp}}$ is pure point and a spectral measure for $H\mathbb{1}_{E_{\mathrm{pp}}}(H)$. In particular, on a tree graph, $E_{\mathrm{pp}} = \emptyset$, i.e., $\hat\rho = \rho$ itself is a spectral measure for $H$.*

*Proof.* The first equality follows from (D.21), the fact that 0 is not isolated in $L_{\mathrm{split}}$ (cf. (4.13c)) and the definition of $m(z)$ in (D.19).

For the second equality, we use the pointwise representation of Corollary C.16 and (C.16) for $x, y = 0$. We set $E_{\mathrm{pp}} := \{\, \lambda \,|\, \partial_{xy} \widetilde e_\lambda(0,0) = \sum_j |\varphi_{\lambda_j}^\dagger(0)|^2 = 0 \,\}$. In Lemma C.15 we have seen that $E_{\mathrm{pp}} \subset E(L)$, where $E(L)$ is the *countable* set of exceptional energy values (see Lemma D.6).

[8]Note that $z \in \mathbb{C}_+$ is always a *nonexceptional energy (cf. Definition D.5).*

Since the derivative of the kernel in (C.16) is also defined at $x, y = 0$, we have

$$\hat{\rho}(I) = (\delta_0', \mathbb{1}_I(H)\delta_0') = \int_I \partial_{xy}\widetilde{e}_\lambda(0,0)\,\mathrm{d}\rho(\lambda) \tag{D.23}$$

which shows that $\hat{\rho}$ is a spectral measure for the part of the operator $H$ on the complement of $E_{\mathrm{pp}}$. Clearly, the countable set $E_{\mathrm{pp}}$ can only support a pure point measure, i.e., $\rho_{\mathrm{pp}}(I) := \rho(I \cap E_{\mathrm{pp}})$ is a pure point measure and $\hat{\rho} + \rho_{\mathrm{pp}}$ is a spectral measure for the whole operator $H$. Here, $(\cdot,\cdot)$ is the dual pairing of the Hilbert scales $\mathcal{H}_{-2}$ and $\mathcal{H}_2 := \operatorname{dom} H$. In addition, we set $\delta_0' f := f^\dagger(0)$. Note that $\delta_0' \in \mathcal{H}_{-2}$ due to (C.2) and (C.9). □

## E. Spectral averaging

Using the Weyl-Titchmarsh function, we want to prove a spectral averaging formula in the sense that integrating the spectral measure of $H = H(\omega)$ on $L_0$ with respect to the first random variable $\omega_1 \in \Omega_1$ yields in a measure *absolutely continuous* with respect to the Lebesgue measure. The Weyl-Titchmarsh function associated to $H = H(\omega)$ (see Section D.2) has the advantage that there is a formula (cf. (E.4)) separating the first random variable $\omega_1 \in \Omega_1$ from the other ones $\hat{\omega} = (\omega_2, \dots) \in \hat{\Omega}$ where $\omega = (\omega_1, \hat{\omega}) \in \Omega = \Omega_1 \times \hat{\Omega}$.

Let $L = L(\omega)$ be a random line-like graph as defined in Section 4.2. For a unimodular matrix $A \in \mathrm{SL}_2(\mathbb{C})$ we denote by $[A] \colon \mathbb{C} \longrightarrow \mathbb{C}$ the corresponding Möbius transformation associated to $A$, i.e., if

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \qquad \text{then} \qquad [A]m := \frac{c + dm}{a + bm}$$

for $m \neq -a/b$. Our definition of $[A]$ differs from the standard one due to the fact that the projection of $\vec{\Psi} = (\psi, \psi') \in \mathbb{C}^2$ ($\vec{\Psi} \neq 0$) onto the complex projective line is $[\vec{\Psi}] := \psi'/\psi$. We use this convention since the transfer matrix $A$ acts on $\vec{\Psi} := (\psi, \psi^\dagger)$ and the Weyl-Titchmarsh function associated to $H = H(\omega)$ on the line-like graph $L = L(\omega)$ is given by $m(z) = [\vec{\Psi}_z] = (\psi_z^\dagger/\psi_z)(0)$.

We denote by $T_z(\omega_1)$ the transfer matrix of a single graph decoration (see Lemma D.6). Note that the transfer matrix is defined for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Our main tool in this section will be the following estimate. Let $\lambda_\pm \in \mathbb{R}$.

**Definition E.1.** We say that spectral averaging holds in the compact set $\Sigma_0 \subset [\lambda_-, \lambda_+]$ if for $C_4 > 0$ and $\varepsilon_0 > 0$ there exists a constant $C_3 = C_3(\lambda_\pm, \varepsilon_0, \Omega_1, C_4)$ such that

$$\int_{\Omega_1} \operatorname{Im}\left(\left[T_z(\omega_1)^{-1}\right]m\right) \mathrm{d}\mathbb{P}_1(\omega_1) \leq C_3 \tag{E.1}$$

for all $z = \lambda + \mathrm{i}\varepsilon \in \Sigma_0 \times \mathrm{i}(0, \varepsilon_0]$ and all $m \in \mathbb{C}_+$ such that $\varepsilon|m| \leq C_4$ and $\operatorname{Im}([T_z(\omega_1)^{-1}]m) > 0$.

We will see in (E.9)–(E.10) that the Möbius transformation in (E.1) has no poles in $\mathbb{C}_+$. Furthermore, we will relate this estimate to the Weyl-Titchmarsh function $m(z)$ for the line-like graph $L = L_0$. Here and in the sequel, $\psi = \psi_z$ is the unique eigenfunction $H\psi = z\psi$ with $\psi(0) = 1$ and $\psi \in \mathsf{L}_2(L)$. If $L = L(\omega)$, then $\psi_z$ also depends on $\omega$. More generally, we define the Weyl-Titchmarsh function for the line-like *subgraph* $L_n$ (see Section 4.2) as

$$m_n(z) := (\psi_z^\dagger/\psi_z)(n_+). \tag{E.2}$$

**Lemma E.2.** *The function $m_n$ is a Herglotz function, i.e., $m_n$ is the Weyl-Titchmarsh function for $L_n$. In addition, $m_n(z)$ only depends on the random variables $\hat{\omega}_n := (\omega_{n+1}, \omega_{n+2}, \dots)$, and is given explicitly by*

$$m_n(z, \hat{\omega}_n) = [T_z(\omega_n)] m_{n-1}(z, \hat{\omega}_{n-1}) \tag{E.3}$$

*Proof.* The proof that the solution space of $Hu = zu$ on $L_n$ is one-dimensional is the same as the proof of Lemma D.13. In particular, the solution is determined by its value at $\psi_z(n_+)$ and $m_n(z)$ only depends on the data of $L_n$, i.e., $m_n(z)$ only depends on $\hat{\omega}_n$. From Lemma D.14 applied to $L_n$, we see that also $\operatorname{Im} m_n(z) > 0$ for $z \in \mathbb{C}_+$. The last equality follows from the definition of the transfer matrix (cf. Section D.1). □

In particular, we have

$$m_0(\omega, z) = [T_z(\omega_1)^{-1}] m_1(\hat{\omega}, z) \tag{E.4}$$

where $m_0$ is the Weyl-Titchmarsh function on $L = L_0(\omega)$. In addition, $m_1$ is the Weyl-Titchmarsh function on $L_1 = L_1(\hat{\omega})$ and $\hat{\omega} := \hat{\omega}_1$, i.e., $\omega = (\omega_1, \hat{\omega}) \in \Omega = \Omega_1 \times \hat{\Omega}$.

*Remark* E.3. Note that $\operatorname{Im} m_0(\omega, z) > 0$ if $\operatorname{Im} m_1(\hat{\omega}, z) > 0$ although $T_z(\omega)$ generally has *complex* entries. It is the nontrivial dependence of $z$ entering in $m_1$ *and* the transfer matrix $T_z(\omega_1)$ which makes the quantum graph problem different from spectral averaging methods considered for other models before (see e.g. [GM03]) where usually only *real* entries are considered.

We can now prove the main result of this section:

**Theorem E.4.** *The spectral measure $\rho = \rho_\omega$ of $H = H(\omega)$ on the line-like graph $L = L(\omega)$ splits into two measures $\rho = \hat{\rho} + \rho_{\mathrm{pp}}$ where $\rho_{\mathrm{pp}}$ is pure point.*

*In addition, if* (E.1) *holds in $\Sigma_0 \subset [\lambda_-, \lambda_+]$, then the measure $\hat{\rho} = \hat{\rho}_\omega$ averaged over the first random variable $\omega_1$ is absolutely continuous w.r.t. the Lebesgue measure, i.e., there is a constant $C_5 = C_5(\lambda_\pm, \Omega_1) > 0$ uniform in $\hat{\omega}$ such that*

$$\int_{\Omega_1} \hat{\rho}_{(\omega_1, \hat{\omega})}(I) \, \mathrm{d}\mathbb{P}_1(\omega_1) \le C_5 \, \boldsymbol{\lambda}(I) \tag{E.5}$$

*for all measurable sets $I \subset \Sigma_0$, where $\boldsymbol{\lambda}$ denotes Lebesgue measure.*

*Proof.* From (D.22) and the theory of Herglotz functions (see e.g. [PF92, App. A]) we have

$$\hat{\rho}_\omega(I) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_I \operatorname{Im} m_\omega(\lambda + \mathrm{i}\varepsilon) \, \mathrm{d}\lambda$$

provided $\partial I$ does not contain eigenvalues of $H = H(\omega)$. Note that $\int_{\mathbb{R}} \frac{1}{1+|\lambda|} \, \mathrm{d}\rho(\lambda) < \infty$ by Corollary B.5 and Lemma C.12. Now,

$$\begin{aligned}
\pi \int_{\Omega_1} \hat{\rho}_{\omega_1, \hat{\omega}}(I) \, \mathrm{d}\mathbb{P}_1(\omega_1) &= \int_{\Omega_1} \Big( \lim_{\varepsilon \to 0} \int_I \operatorname{Im} m_0\big(\lambda + \mathrm{i}\varepsilon, (\omega_1, \hat{\omega})\big) \, \mathrm{d}\lambda \Big) \, \mathrm{d}\mathbb{P}_1(\omega_1) \\
&= \int_{\Omega_1} \Big( \lim_{\varepsilon \to 0} \int_I \operatorname{Im} [T_z(\omega_1)^{-1}] m_1(\lambda + \mathrm{i}\varepsilon, \hat{\omega}) \, \mathrm{d}\lambda \Big) \, \mathrm{d}\mathbb{P}_1(\omega_1).
\end{aligned}$$

Now $m_1(z, \hat{\omega})$ is a Herglotz function and all components of $\hat{\omega}$ are iid random variables. In particular, there exists a constant $C_4 = C_4(\lambda_\pm, \varepsilon_0, \Omega_1)$ such that $\varepsilon |m_1(\lambda + \mathrm{i}\varepsilon, \hat{\omega})| \le C_4$ for all $z \in \Sigma_0 + \mathrm{i}(0, \varepsilon_0) \subset \mathbb{C}_+$. Interchanging the first integral and the limit by Fatou's lemma, we use Fubini's theorem to exchange the order of integration and obtain from (E.1),

$$\pi \int_{\Omega_1} \hat{\rho}_{\omega_1, \hat{\omega}}(I) \, \mathrm{d}\mathbb{P}_1(\omega_1) \le C_3 \, \boldsymbol{\lambda}(I)$$

i.e., $C_5 = C_3 / \pi$. □

In the rest of this section we provide some criteria guaranteeing (E.1). In our applications, it will be more convenient to use $w = \sqrt{z}$ as spectral parameter where we choose the branch with $\operatorname{Im}\sqrt{z} > 0$, i.e., cut along $\mathbb{R}_+$. We write the transfer matrix as

$$T_z(t) = D(b)\hat{T}_z(t), \qquad \hat{T}_z(t) = \begin{pmatrix} t_{11}(t,w) & t_{12}(t,w) \\ t_{22}(t,w) & t_{22}(t,w) \end{pmatrix} \tag{E.6}$$

where $\hat{T}_z(t) = T_z(G_*(t))$ or $\hat{T}_z(t) = S(t)T_z(G_*)$ in a random length, respectively, Kirchhoff model, denotes the transfer matrix of the decoration graph $G_*(t)$ as defined in (D.12). We also assume that $\Omega_1 = [t_-, t_+]$ and often write $t = \omega_1$ for the integration parameter.

Let Ln be the complex logarithm on the infinite sheeted Riemann surface $\widetilde{\mathbb{C}}^*$ with branching points at 0 and $\infty$. For a map $t \to a(t) \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ we denote by $t \to \widetilde{a}(t)$ the lift of $t \mapsto a(t)$ onto $\widetilde{\mathbb{C}}^*$ such that $\widetilde{a}(0)$ lies in the first sheet (given by the argument $0 \le \varphi < 2\pi$). Note that if $a(t) = r(t)\mathrm{e}^{\mathrm{i}\varphi(t)}$ for *continuous* functions $r(\cdot)$ and $\varphi(\cdot)$, then $\operatorname{Ln}\widetilde{a}(t) = \ln r(t) + \mathrm{i}\varphi(t)$. In particular, we decompose the denominator of the Möbius transformation $[\hat{T}_z(t)^{-1}]m$, namely

$$f_{w,m}(t) := t_{22}(t,w) - t_{12}(t,w)m = r_{w,m}(t)\mathrm{e}^{\mathrm{i}\varphi_{w,m}(t)}, \tag{E.7}$$

into its polar decomposition with continuous functions $r_{w,m}$ and $\varphi_{w,m}$.

**Lemma E.5.** *Suppose that $\mathbb{P}_1$ has a bounded density on $\Omega_1 := [t_-, t_+]$ with respect to the Lebesgue measure, i.e., $\mathrm{d}\mathbb{P}_1(t) = \eta(t)\,\mathrm{d}t$ and $0 \le \eta(t) \le \|\eta\|_\infty$ for almost all $t$. Suppose in addition, that there are complex constants $A_w$, $B_w \in \mathbb{C}$ such that*

$$[\hat{T}_z(t)^{-1}]m = -A_w \frac{f'_{w,m}(t)}{f_{w,m}(t)} + B_w \tag{E.8}$$

*and measurable subsets $\Sigma_j \subset [\lambda_-, \lambda_+]$ with $\bigcup \Sigma_j = [\lambda_-, \lambda_+]$ up to a discrete set such that for all $w = \sqrt{\lambda + \mathrm{i}\varepsilon}$ with $\lambda \in \Sigma_j$, $0 < \varepsilon < \varepsilon_0$ we have*

(i) *for each $j \in \mathbb{N}$, there is a constant $C_6 = C_6(j, \lambda_\pm, \varepsilon_0) > 0$ such that*

$$|\operatorname{Re} A_w| \le C_6, \quad |B_w| \le C_6 \quad \text{and} \quad |\operatorname{Im} A_w| \le C_6\,\varepsilon;$$

(ii) *the winding number is bounded, i.e., there exists $N > 0$ such that $|\varphi_{w,m}(t_+) - \varphi_{w,m}(t_-)| \le N$ for all $m \in \mathbb{C}_+$.*

*All constants and error estimates are supposed to depend only on $\Sigma_j$ and $\varepsilon_0$. It suffices to choose $m \in \mathbb{C}_+$ such that $\varepsilon|m| \le C_4$. Here, $C_6$ and $N$ may depend on $C_4$ but* not *on $m$ directly. Then there exist $\Sigma'_j \subset \Sigma_j$ such that $\bigcup_j \Sigma'_j = \Sigma_0$ up to a discrete set such that* (E.1) *is fulfilled in $\Sigma_j$ with*

$$C_3 = C_3(j) = C_6\|\eta\|_\infty\big(\varepsilon O(|\ln\varepsilon|) + N + (t_+ - t_-)\big).$$

*If* $\operatorname{Im} A_w = 0$ *we can choose* $\Sigma'_j = \Sigma_j$.

*Proof.* From (E.8) we obtain

$$\begin{aligned}
\operatorname{Im} \int_{t_-}^{t_+} [\hat{T}_z(t)^{-1}]m\,\eta(t)\,\mathrm{d}t &\le \|\eta\|_\infty \operatorname{Im}\big[-A_w \operatorname{Ln}\widetilde{f}_{w,m}(t) + B_w t\big]_{t_-}^{t_+} \\
&= \|\eta\|_\infty \Big[-\operatorname{Im} A_w \ln\frac{r_{w,m}(t_+)}{r_{w,m}(t_-)} - \operatorname{Re} A_w\big(\varphi_{w,m}(t_+) - \varphi_{w,m}(t_-)\big) + B_w(t_+ - t_-)\Big] \\
&\le \|\eta\|_\infty \Big[|\operatorname{Im} A_w|\Big|\ln\frac{r_{w,m}(t_+)}{r_{w,m}(t_-)}\Big| + |\operatorname{Re} A_w|\big|\varphi_{w,m}(t_+) - \varphi_{w,m}(t_-)\big| + |B_w||t_+ - t_-|\Big]
\end{aligned}$$

so that the estimate follows from the assumptions once we have shown that $r_{w,m}(t_+)$ is bounded from above by a polynomial in $\varepsilon^{-1}$ and that $r_{w,m}(t_-)$ is bounded from below by a polynomial in $\varepsilon$; uniformly for all $w = \sqrt{\lambda + \mathrm{i}\varepsilon}$, $\lambda \in \Sigma_j$, $0 < \varepsilon \le \varepsilon_0$ and for all $m \in \mathbb{C}_+$ such that $|m| \le C_4/\varepsilon$ (the polynomials may depend on $C_4$, but not on $m$ itself). Note that if $\operatorname{Im} A_w = 0$, we can skip the estimate on $r_{w,m}(t_\pm)$ and we are done.

To estimate $r_{w,m}(t_\pm)$, we write

$$f_{w,m}(t) = t_{12}(t,w)\Big(\frac{t_{22}(t,w)}{t_{12}(t,w)} - m\Big) = \frac{1}{\Lambda_{01}(t,z)}\big(\Lambda_{11}(t,z) - m\big) \tag{E.9}$$

for $z \notin E(G_*(t))$ using (D.12). Note that $t_{12}(t,w) = 1/\Lambda_{01}(t,z) \neq 0$ due to (D.12). The series representation (D.9) of the Dirichlet-to-Neumann map $\Lambda(t,z)$ of the decoration graph $G_*(t)$ yields

$$\operatorname{Im}\frac{t_{22}(t,w)}{t_{12}(t,w)} = \operatorname{Im}\Lambda_{11}(t,z) = -\varepsilon\sum_k \frac{|\varphi_k^\dagger(o_1)|^2}{|\lambda_k - z|^2} =: -\varepsilon\, C_7(z). \tag{E.10}$$

Now let $k_0$ be an index for which $\lambda_k > \lambda_+$ and $z_0 := \lambda_- + \mathrm{i}\varepsilon_0$. Then

$$C_7(z) \ge \sum_{k\ge k_0}\frac{|\varphi_k^\dagger(o_1)|^2}{|\lambda_k - z|^2} \ge \sum_{k\ge k_0}\frac{|\varphi_k^\dagger(o_1)|^2}{|\lambda_k - z_0|^2} =: C_8. \tag{E.11}$$

We also need a lower bound on the module of $t_{12} = 1/\Lambda_{01}$, i.e., an upper bound on $|\Lambda_{01}(t,z)|$, namely

$$|\Lambda_{01}(t_-,z)| \le \sum_k \frac{|z||\varphi_k^\dagger(o_0)\varphi_k^\dagger(o_1)|}{\lambda_k|\lambda_k - z|} + |\Lambda_{01}(t_-,0)|.$$

We restrict the values of $\lambda$ to the subset

$$\Sigma_j' := \big\{\,\lambda \in \Sigma_j \,\big|\, |\Lambda_{01}(t,\lambda)| \ge 1/j \text{ and } |\lambda - \lambda_k| \ge 1/j \text{ for all } k, t = t_\pm\,\big\}$$

and assume that $z = \lambda + \mathrm{i}\varepsilon$ with $\lambda \in \Sigma_j'$. A compactness argument yields the existence of a constant $C_9 > 0$ depending only on $j$, $t_-$ $\lambda_\pm$ and $\varepsilon_0$ such that $|\Lambda_{01}(t_-,z)| \le C_9$. Since $m \in \mathbb{C}_+$ and $\operatorname{Im} t_{22}/t_{11} \le -\varepsilon\, C_8$, we deduce $r_{w,m}(t_-) \ge \varepsilon C_8/C_9$.

The upper bound can be obtained similarly: Here, we need an upper bound on $|t_{12}|$ and $|t_{22}/t_{12}|$, i.e., a lower bound on $|\Lambda_{01}|$ and an upper bound on $|\Lambda_{11}|$. The upper bound $|\Lambda_{11}(t_+,z)| \le C_{10}$ for $z \in \Sigma_j' + \mathrm{i}(0,\varepsilon_0]$ can be obtained as above for $\Lambda_{01}$. For the global lower bound on $\Lambda_{01}(z) = \Lambda_{01}(t_+,z)$ we have

$$|\Lambda_{01}(z)| \ge |\Lambda_{01}(\lambda)| - \varepsilon|\Lambda_{01}'(\lambda + \mathrm{i}\tau\varepsilon)| \ge \frac{1}{j} - \varepsilon C_{11}$$

where $z = \lambda + \mathrm{i}\varepsilon$, $\tau \in (0,1)$ and $C_{11}$ is the maximum of $\Lambda_{01}'(z)$ in a compact set avoiding the poles of $\Lambda_{01}$ (where $\Lambda_{01}(z)$ is large). Therefore, there exists $C_{12} = C_{12}(j)$ such that $|\Lambda_{01}(z)| \ge C_{12}$ for all $z \in \Sigma_j \times (0,\varepsilon_0]$ and for $\varepsilon_0 = \varepsilon_0(j)$ small enough. Note that still $\bigcup \Sigma_j' = [\lambda_-,\lambda_+]$ up to a discrete set since by Assumption 5.2 (iii), $\{\,\lambda \,|\, \Lambda_{01}(t_\pm,\lambda) = 0\,\}$ is discrete. Finally, we have shown $r_{w,m}(t_+) \le (C_{12})^{-1}(C_{10} + C_4/\varepsilon) = O(\varepsilon^{-1})$. $\square$

*Remark* E.6. Note that the constants defined in the proof below (for example $C_8$ in (E.11)) still depends on the decoration graph $G_*(t_\pm)$ via the eigenvalues and eigenvectors of $G_*(t_\pm)$. But here we see the advantage of the spectral averaging: After integrating, we only have to control the behavior at the points $t_\pm$ of the random space, *not* a uniform estimate over all $t = \omega_1 \in \Omega_1$ (which is in general not possible). In fact, even if we would have global lower bounds on the denominator of the Möbius transformation, we are usually not done, since the estimates are of order $\varepsilon^{-1}$ and therefore unbounded as in the proof above.

We will give two particular examples in which the spectral averaging estimate can be deduced from Lemma E.5.

**Random length models.** There is a particular simple form of the transfer matrix in certain random length models: Suppose that $T_z(\ell) = D(b)\hat{T}_z\widetilde{T}_z(\ell)$ where $\ell = \omega_1 \in \mathbb{R}$, $\hat{T}_z = (\hat{t}_{ij}(z)) \in \mathrm{SL}_2(\mathbb{C})$ and $\ell \mapsto \widetilde{T}_z(\ell) = \mathrm{e}^{-\ell X_z}$ is a one-parameter group in $\mathrm{SL}_2(\mathbb{C})$ with $X_z \in \mathrm{sl}_2(\mathbb{C})$, the Lie algebra of $\mathrm{SL}_2(\mathbb{C})$. We assume that

$$\widetilde{T}_z(\ell) = \begin{pmatrix} \widetilde{t}_{11}(\ell,w) & \widetilde{t}_{12}(\ell,w) \\ \widetilde{t}_{21}(\ell,w) & \widetilde{t}_{22}(\ell,w) \end{pmatrix} \qquad \text{and} \qquad X_z = \begin{pmatrix} \beta_z & \alpha_z \\ \gamma_z & -\beta_z \end{pmatrix}. \tag{E.12}$$

Using $\frac{\mathrm{d}}{\mathrm{d}\ell}T_z(\ell) = \widetilde{T}_z(\ell)X_z$ we obtain (denoting $(\cdot)'$ the derivative w.r.t. $\ell$)

$$\widetilde{t}_{12}' = \alpha\widetilde{t}_{11} - \beta\widetilde{t}_{12} \qquad \text{and} \qquad \widetilde{t}_{22}' = \alpha\widetilde{t}_{21} - \beta\widetilde{t}_{22}.$$

If $\alpha \neq 0$, we can decompose

$$\begin{aligned}
[\widetilde{T}_z(-\ell)\hat{T}_z^{-1}]m &= -\frac{\widetilde{t}_{21}(\hat{t}_{22} - \hat{t}_{21}m) + \widetilde{t}_{11}(\hat{t}_{21} - \hat{t}_{11}m)}{\widetilde{t}_{22}(\hat{t}_{22} - \hat{t}_{21}m) + \widetilde{t}_{12}(\hat{t}_{21} - \hat{t}_{11}m)} \\
&= -\frac{(\widetilde{t}_{22}' + \beta\widetilde{t}_{22})(\hat{t}_{22} - \hat{t}_{21}m) + (\widetilde{t}_{12}' + \beta\widetilde{t}_{12})(\hat{t}_{21} - \hat{t}_{11}m)}{\alpha\big(\widetilde{t}_{22}(\hat{t}_{22} - \hat{t}_{21}m) + \widetilde{t}_{12}(\hat{t}_{21} - \hat{t}_{11}m)\big)} = -\frac{f_{w,m}'(\ell)}{\alpha f_{w,m}(\ell)} - \frac{\beta}{\alpha}
\end{aligned}$$

where $f_{w,m}(\ell)$ denotes the denominator of the Möbius transformation so that $A_w := 1/\alpha$ and $B_w := -\beta/\alpha$ in the notation of (E.8).

Typically, $\ell$ denotes the length and $\widetilde{T}_z(\ell) = R_{pw}(w\ell)$ where $R_w(\varphi)$ is defined in (2.14) and $p > 0$ is a fixed parameter, so in particular, $\beta = 0$, $\alpha = 1/p \in \mathbb{R}$ and $\gamma_z = pz \in \mathbb{C}$. Then

$$\widetilde{T}_z(\ell) = \begin{pmatrix} \cos w\ell & \dfrac{\sin w\ell}{pw} \\ -pw\sin w\ell & \cos w\ell \end{pmatrix} \tag{E.13}$$

where $w = \sqrt{z}$ and $\operatorname{Re} z, \operatorname{Im} z > 0$ (we choose the branch with $\operatorname{Im}\sqrt{z} > 0$). In this case, we obtain from the previous lemma:

**Corollary E.7.** *Assume that the single transformation matrix has the form*

$$T_z(\ell) = D(b)\hat{T}_z R_{pw}(w\ell).$$

*Suppose in addition that $\Omega_1 = [\ell_-, \ell_+]$ and that $\mathbb{P}_1$ has a bounded density with respect to the Lebesgue measure, i.e., $\mathrm{d}\mathbb{P}_1(\ell) = \eta(\ell)\,\mathrm{d}\ell$ and $0 \leq \eta(\ell) \leq \|\eta\|_\infty$ for almost all $\ell$. Then* (E.1) *is fulfilled for all $\lambda \in [\lambda_-, \lambda_+]$.*

*Proof.* In our case, we have $A_w = 1/\alpha = p$ and $B_w = 0$. Furthermore, the winding number of $\widetilde{f}_{w,m}$ can be estimated by a fixed number depending only on $\lambda_\pm$ and $\ell_\pm$. In particular, the assumptions of Lemma E.5 are fulfilled. Since $\operatorname{Im} A_w = 0$, we can skip the estimate on the real part of the logarithm and do not need the exceptional sets $\Sigma_j$. ∎

In general, changing the length of a subgraph $G_n$ does not yield a one-parameter group. For such general random length model the integrand is in general a very complicated rational function in $w, \sin w\ell$ and $\cos w\ell$.

**Random Kirchhoff models.** Suppose that $T_z(q) = D(b)S(q)\hat{T}_z$ where $S(q)$ is the shearing matrix as in (2.14), where $q = \omega_1 \in \Omega_1 = [q_-, q_+]$ and where $\hat{T}_z$ is the transition matrix for a (fixed) decoration graph, i.e., we assume a Kirchhoff model where the vertex potential is at the end point of the decoration graph. A simple calculation shows that

$$\begin{aligned}[T_z(q)^{-1}]m = [\hat{T}_z^{-1}](-q+bm) &= \frac{-t_{21}+t_{11}(-q+bm)}{t_{22}-t_{12}(-q+bm)} \\ &= \frac{1}{t_{12}}\Big(\frac{1}{t_{22}-t_{12}(-q+bm)} - t_{11}\Big) = \frac{1}{(t_{12})^2}\frac{f'_{w,m}(q)}{f_{w,m}(q)} - \frac{t_{11}}{t_{12}}\end{aligned}$$

with the notation of (E.6) ($t_{ij} = t_{ij}(q,w)$) and (E.7).

**Corollary E.8.** *Suppose that* $\Omega_1 = [q_-, q_+]$ *and that* $\mathbb{P}_1$ *has a bounded density with respect to the Lebesgue measure, i.e.,* $\mathrm{d}\mathbb{P}_1(q) = \eta(q)\,\mathrm{d}q$ *and* $0 \le \eta(q) \le \|\eta\|_\infty$ *for almost all* $q$. *Then there is a sequence* $\Sigma'_j \subset [\lambda_-, \lambda_+]$ *with* $\bigcup_j \Sigma_j = [\lambda_-, \lambda_+]$ *up to a discrete set such that* (E.1) *is fulfilled for all* $\lambda \in \Sigma_j$ *with a constant* $C_3$ *depending on* $j$.

*Proof.* Again, we use Lemma E.5. We have seen in the calculation above that $A_w := 1/(t_{12})^2 = (\Lambda_{01}(z))^2$ and $B_w := -t_{11}/t_{12} = \Lambda_{00}(z)$ (see (D.12)). The upper bounds on $|\mathrm{Re}\, A_w|$ and $|B_w|$ can be found as in the proof of Lemma E.5. Note in addition that $\mathrm{Im}\, A_w = 2\mathrm{Re}\,\Lambda_{01}(z)\mathrm{Im}\,\Lambda_{01}(z) = O(\varepsilon)$ using again the series representation of the Dirichlet-to-Neumann map (D.9). The winding number is bounded by $\pi$ since $q \mapsto f_{w,m}(q)$ describes a line in the complex plane. ☐

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