# EXISTENCE OF SPECTRAL GAPS, COVERING MANIFOLDS AND RESIDUALLY FINITE GROUPS

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*Dedicated to Volker Enß on his 65th birthday*

Abstract. In the present paper we consider Riemannian coverings $(X, g) \to (M, g)$ with residually finite covering group $\Gamma$ and compact base space $(M, g)$. In particular, we give two general procedures resulting in a family of deformed coverings $(X, g_\varepsilon) \to (M, g_\varepsilon)$ such that the spectrum of the Laplacian $\Delta_{(X_\varepsilon, g_\varepsilon)}$ has at least a prescribed finite number of spectral gaps provided $\varepsilon$ is small enough.

If $\Gamma$ has a positive Kadison constant, then we can apply results by Brüning and Sunada to deduce that $\operatorname{spec} \Delta_{(X,g_\varepsilon)}$ has, in addition, band-structure and there is an asymptotic estimate for the number $\mathcal{N}(\lambda)$ of components of $\operatorname{spec} \Delta_{(X,g_\varepsilon)}$ that intersect the interval $[0, \lambda]$. We also present several classes of examples of residually finite groups that fit with our construction and study their interrelations. Finally, we mention several possible applications for our results.

## 1. Introduction

Spectral properties of the Laplacian on a compact manifold is a well-established and still active field of research. Much less is known on the spectrum of *non-compact* manifolds. We restrict ourselves here to the class of non-compact *covering* manifolds $X \to M$ with compact quotient $M$, in which the covering group $\Gamma$ plays an important role. In the open problem section of [ScY94, Ch. IX, Problem 37], Yau posed the question about the nature and the stability of the (purely essential) spectrum of such a covering $X \to M$.

The aim of this paper is to provide a large class of examples of Riemannian coverings $X \to M$ having spectral gaps in the essential spectrum of its Laplacian $\Delta_X$. Here, a spectral gap is a non-void open interval $(\alpha, \beta)$ with $(\alpha, \beta) \cap \operatorname{spec} \Delta_X = \emptyset$ and $\alpha, \beta \in \operatorname{spec} \Delta_X$. The manifolds $X$ and $M$ are $d$-dimensional, $d \geq 2$, and we denote by $D$ a fundamental domain associated to this covering. The main idea for producing spectral gaps is to construct a family of Riemannian metrics $(g_\varepsilon)_{\varepsilon>0}$ on $X$ such that the length scale w.r.t. the metric $g_\varepsilon$ is of order $\varepsilon$ at the boundary of a fundamental domain $D$ and unchanged elsewhere (cf. Figure 1). If such a fundamental domain exists, we say that the family of metrics $(g_\varepsilon)$ *decouples* the manifold $X$. The covering $X \to M$ with a decoupling family of metrics $(g_\varepsilon)$ "converges" in a sense to be specified below to a limit covering consisting of the infinite disjoint ("decoupled") union of the limit quotient manifold $N$ which are again $d$-dimensional (see Subsection 1.3 and Section 3 for details). We stress that the curvature does not remain bounded as $\varepsilon \to 0$; in contrast to degeneration of Riemannian metrics under curvature bounds developed e.g. in [Ch01]. All groups $\Gamma$ are assumed to be discrete and finitely generated throughout the present article.

### 1.1. Statement of the main results.

**Main Theorem 1** (cf. Theorem 6.8). *Suppose that $X \to M$ is a Riemannian covering with residually finite covering group $\Gamma$ and metric $g$. Then by a local deformation of $g$*

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*we construct a family of metrics $(g_\varepsilon)$ decoupling $X$, such that for each $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ where* $\operatorname{spec}\Delta_{(X,g_{\varepsilon_n})}$ *has at least $n$ gaps, i.e. $n+1$ components as subset of* $[0,\infty)$.

Basically, we will give two different constructions for the family of manifolds $(X, g_\varepsilon)$: first, "adding small handles" to a given manifold $(N, g)$ and second, a conformal perturbation of $g$. As a set, $(X, g_\varepsilon)$ converges to a limit manifold consisting of infinitely many disjoint copies of the limit quotient manifold $N$ as $\varepsilon \to 0$.

A *residually finite* group is a countable discrete group such that the intersection of all its normal subgroups of finite index is trivial. Roughly speaking, a residually finite group has many normal subgroups of finite index. Geometrically, a covering with a residually finite covering group can be approximated by a sequence of finite coverings $M_i \to M$ (a *tower of coverings*). The class of residually finite groups is very large, containing e.g. finitely generated abelian groups, type I groups (i.e. finite extensions of $\mathbb{Z}^r$), free groups or finitely generated subgroups of the isometries of the $d$-dimensional hyperbolic space $\mathbb{H}^d$ (cf. Section 6).

Denote by $\mathcal{N}(g, \lambda)$ the number of components of $\operatorname{spec}\Delta_{(X,g)}$ which intersect the interval $[0, \lambda]$. Our result gives a *lower* bound on $\mathcal{N}(g, \lambda)$, in particular, we can reformulate the Main Theorem 1 as follows: *For each $n \in \mathbb{N}$ there exists $g = g_{\varepsilon_n}$ such that $\mathcal{N}(g, \lambda) \ge n+1$.*

Using the Weyl eigenvalue asymptotic on the limit $d$-dimensional manifold $(N, g)$ associated to the decoupling family $(g_\varepsilon)$ on $X \to M$, we obtain the following asymptotic lower bound on the number of gaps (where $\omega_d$ denotes the volume of the $d$-dimensional Euclidean unit ball):

**Main Theorem 2** (cf. Theorem 7.5)**.** *Assume that the covering group is residually finite and that the spectrum of the Laplacian on the limit manifold $(N, g)$ is simple, i.e. all eigenvalues have multiplicity one. Then for each $\lambda \ge 0$ there exists $\varepsilon(\lambda) > 0$ such that*

$$\liminf_{\lambda\to\infty} \frac{\mathcal{N}(g_{\varepsilon(\lambda)}, \lambda)}{(2\pi)^{-d}\omega_d \operatorname{vol}(N, g)\lambda^{d/2}} \ge 1.$$

The assumption on the spectrum of $(N, g)$ is natural since $\mathcal{N}(g, \lambda)$ counts components in the spectrum *without* multiplicity.

A priori, the number of gaps $\mathcal{N}(g, \lambda)$ could be infinite, e.g. if $\operatorname{spec}\Delta_{(X,g)}$ contains a Cantor set. But Brüning and Sunada showed in [BS92] that for covering groups $\Gamma$ with positive *Kadison constant* $C(\Gamma) > 0$ (cf. Section 7) asymptotic upper bound

$$\limsup_{\lambda\to\infty} \frac{\mathcal{N}(g, \lambda)}{(2\pi)^{-d}\omega_d \operatorname{vol}(M, g)\lambda^{d/2}} \le \frac{1}{C(\Gamma)}$$

holds. In particular, $\mathcal{N}(g, \lambda)$ is finite, and the spectrum of $\Delta_{(X,g)}$ does not contain Cantor-like subsets. Applying these results to our situation we give a partial answer on the question of Yau of the nature of the spectrum:

**Main Theorem 3** (cf. Theorem 7.5)**.** *Suppose that $X \to M$ is a Riemannian $\Gamma$-covering with decoupling family of metrics $(g_\varepsilon)$, where $\Gamma$ is a residually finite group that has positive Kadison constant $C(\Gamma) > 0$. Then* $\operatorname{spec}\Delta_{(X,g_\varepsilon)}$ *has band-structure, i.e. $\mathcal{N}(g_\varepsilon, \lambda) < \infty$ for all $\lambda \ge 0$ and $\mathcal{N}(g_\varepsilon, \lambda)$ can be made arbitrary large provided $\varepsilon$ is small and $\lambda$ is large enough.*

Some examples of groups with positive Kadison constant and which are residually finite are finitely generated, abelian groups, the free (non-abelian) group in $r \ge 2$ generators or fundamental groups of compact, orientable surfaces (see also Section 8).

1.2. **Motivation and related work.** A main motivation for our work comes from the spectral theory of Schrödinger operators $H = -\Delta + V$ on $\mathbb{R}^d$, $d \ge 2$, with $V$ periodic w.r.t. the action of a discrete abelian group $\Gamma_{\text{ab}} = \mathbb{Z}^d$ on $\mathbb{R}^d$. For such operators, it is a well known fact that if $V$ has high barriers near the boundary of a fundamental domain $D$, then gaps appear in the spectrum of $H$. In this way, the potential $V$ essentially decouples the fundamental domain $D$ from its neighbouring domains (see [HP03] for an overview on this subject).

A natural generalisation into a geometric context is to replace the periodic structure $(\mathbb{R}^d, \mathbb{Z}^d)$ by a Riemannian covering $X \to M$ with a discrete (in general non-abelian) group $\Gamma$. Our work shows that the decoupling effect of the potential $V$ can be replaced purely by geometry, in particular by the decoupling family of metrics $(g_\varepsilon)$ on $X \to M$. From a quantum mechanical or probabilistic point of view, the correspondence seems to be natural: One has a small probability to find a particle (with low energy) in a region with a high potential barrier or where the manifold $(X, g_\varepsilon)$ is very thin and the absolute value of the curvature is very large.

It was already observed by, e.g., Brüning, Gruber, Kobayashi, Ono and Sunada [BS92, Gr01, S90, KOS89] that many properties of the spectrum of a periodic Schrödinger operator (e.g. band-structure, Bloch's property etc.) generalise to the context of Riemannian coverings. An important difference is the existence of $\mathsf{L}_2$-eigenvalues in the context of manifolds (cf. [KOS89]). Such eigenvalues cannot occur in the spectrum of a periodic Schrödinger operator on $\mathbb{R}^d$ (cf. [S90]).

The existence of (covering) manifolds with spectral gaps has also been established by Brüning, Exner, Geyler and Lobanov in [BEG03, BGL05]. They couple compact manifolds by points or line-segments with certain boundary condition at the coupling points; the point coupling corresponds to the case $\varepsilon = 0$ in our situation (with decoupled boundary condition). The case of abelian *smooth* coverings has been established in [P03] (cf. also the references therein). Spectral gaps of Schrödinger operators on the hyperbolic space have been analysed in [KaPe00]. For other manifolds with spectral gaps (not necessarily periodic), we refer to [EP05, P06]. Under certain topological restrictions on the middle degree homology group one can show the existence of spectral gaps also for the differential form Laplacian on a $\mathbb{Z}$-covering (see [ACP07]).

Some further interesting results on the group $\Gamma$ and spectral properties of a Riemannian $\Gamma$-covering were shown by Brooks [Br81], e.g. that $\Gamma$ is amenable iff $0 \in \operatorname{spec} \Delta_X$. Moreover, Brooks [Bro86] provided a combinatorial criterion whether the second eigenvalue of $\Delta_{M_i}$ is bounded from below as $i \to \infty$, where $M_i \to M$ is a tower of coverings.

For physical applications of our results we refer to Section 9. Let us finish with two consequences of our result giving partial answers to the question of Yau on the nature and stability of the spectrum of $\Delta_X$:

*Consequence* 1 (Manifold with given spectrum). First, we can solve the following inverse spectral problem: Given a compact (connected) manifold $N$ of dimension $d \geq 3$ and a sequence of numbers $0 = \lambda_1(0) < \ldots < \lambda_n(0)$ it is possible to construct a metric $g$ on $N$ having exactly the numbers $\lambda_k(0)$ as first $n$ eigenvalues with multiplicity 1 (cf. [CdV87]). Then, applying our Main Theorem 3 and using the relation between $\operatorname{spec} \Delta_{(X, g_\varepsilon)}$ and $\operatorname{spec} \Delta_{(N,g)}$ we can construct a covering $X \to M$ with decoupling family $(g_\varepsilon)$ having band spectrum close to the given points $\{\lambda_k(0)\}$, $k = 1, \ldots, n$. The covering $(X, g_\varepsilon) \to (M, g_\varepsilon)$ is obtained roughly by joining copies of $N$ through small, thin cylinders (see first construction mentioned below). In particular, we have constructed a covering manifold with approximatively given spectrum in a finite spectral interval $[0, \lambda]$, *independently* of the covering group!

*Consequence* 2 (Instability of gaps). Suppose $X = \mathbb{H}^d$ is the $d$-dimensional ($d \geq 3$) hyperbolic space (or more generally, a simply connected, complete, symmetric space of non-compact type) with its natural metric $g$. It is known, that $\Delta_{(X,g)}$ has no spectral gaps, in particular $\operatorname{spec} \Delta_{(X,g)} = [\lambda_0, \infty)$ for some constant $\lambda_0 \geq 0$ (see e.g. [Don79]).

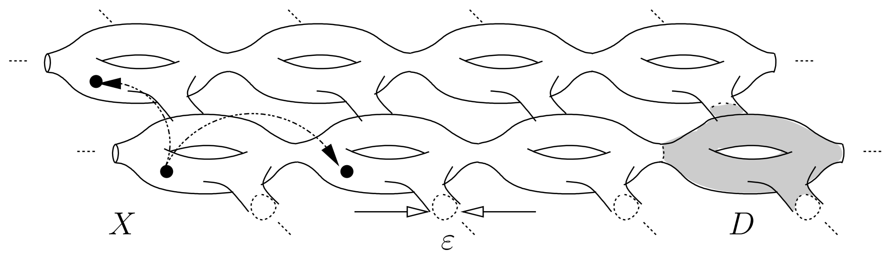



FIGURE 1. A covering manifold $X$ with fundamental domain $D$. The junctions between different translates of $D$ are of order $\varepsilon$.

Let $\Gamma$ be a finitely generated subgroup of the isometries of $X$ such that $M = X/\Gamma$ is compact. Note that such groups are residually finite. The second construction described below allows us to find a decoupling family $(g_\varepsilon)$ on $X$ where $g_\varepsilon = \rho_\varepsilon^2 g$ is conformally equivalent to $g$. We then apply Main Theorem 1 and obtain for each $n \in \mathbb{N}$ a metric $g_{\varepsilon_n}$ such that the corresponding Laplacian has at least $n$ gaps. In particular, the number of gaps is *not* stable, even under uniform conformal changes of the metric. Note that the conformal factor $\rho_\varepsilon$ can be chosen in such a way that $\rho_\varepsilon \to \rho_{\varepsilon_0}$ uniformly as $\varepsilon \to \varepsilon_0$ provided $\varepsilon_0 > 0$. Nevertheless, the band-gap structure remains invariant due to Main Theorem 3, once $\Gamma$ has a positive Kadison constant.

1.3. **An outline of the argument.** In the rest of the introduction we will present the main ideas of the construction of the decoupling metrics and mention the strategy for showing the existence of spectral gaps.

The first construction starts from a compact Riemannian manifold $N$ of dimension $d \geq 2$ (for simplicity without boundary) and a group $\Gamma$ with generators $\gamma_1, \dots, \gamma_r$. We choose $2r$ different points $x_1, y_1, \dots, x_r, y_r$. For each generator, we endow $x_i$ and $y_i$ with a cylindrical end of radius and length of order $\varepsilon > 0$ (by changing the metric appropriately on $D := N \setminus \{x_1, y_1, \dots, x_r, y_r\}$). If we join $\Gamma$ copies of these decorated manifolds $(D, g_\varepsilon)$ according to the Cayley graph of $\Gamma$ associated to $\gamma_1, \dots, \gamma_r$, we obtain a $\Gamma$-covering $X \to M$ with a decoupling family of metrics $(g_\varepsilon)$ (cf. Figure 1).

The second construction starts with an arbitrary covering $(X, g) \to (M, g)$ (with compact quotient) of dimension $d \geq 3$ and changes the metric conformally, i.e. $g_\varepsilon := \rho_\varepsilon^2 g$, in such a way, that $\rho_\varepsilon$ is still periodic and of order $\varepsilon$ close to the boundary of a fundamental domain $D$; more details can be found in Section 3. In the case of abelian coverings these constructions have already been used in [P03].

Once the construction of the family of decoupling metrics $(g_\varepsilon)$ has been done, the strategy to show the existence of spectral gaps goes as follows. We consider first the Dirichlet $(+)$ and Neumann $(-)$ eigenvalues $\lambda_k^\pm(\varepsilon)$ of the Laplacian on the fundamental domain $(D, g_\varepsilon)$. One can show that $\lambda_k^\pm(\varepsilon)$ converges to the eigenvalues $\lambda_k(0)$ of the Laplacian on the limit manifold $(N, g)$ (see [P03] and references therein). In other words, the Dirichlet-Neumann intervals

$$I_k(\varepsilon) := [\lambda_k^-(\varepsilon), \lambda_k^+(\varepsilon)]$$

converge to a point as $\varepsilon \to 0$. Therefore, if $\varepsilon$ is small enough, the union

$$I(\varepsilon) := \bigcup_{k \in \mathbb{N}} I_k(\varepsilon)$$

is a closed set having at least $n$ gaps, i.e. $n + 1$ components as a subset of $[0, \infty)$.

The rest of the argument depends on the properties of the covering group $\Gamma$:

(i) For abelian groups $\Gamma_{\rm ab}$, the inclusion $\operatorname{spec}\Delta_{(X,g_\varepsilon)} \subset I(\varepsilon)$ is given by the Floquet theory (cf. Section 4 or [K93, S88]). Basically, one shows that $\Delta_{(X,g_\varepsilon)}$ is unitary equivalent to a direct integral of operators on $(D, g_\varepsilon)$ acting on $\rho$-equivariant functions, where $\rho$ runs through the set of irreducible unitary representations $\widehat{\Gamma}_{\rm ab}$ (characters). Note that in the abelian case all $\rho$ are one-dimensional and $\widehat{\Gamma}_{\rm ab}$ is homeomorphic to (disjoint copies of) the torus $\mathbb{T}^r$. The Min-max principle ensures that the $k$-th eigenvalue of the equivariant operator lies in $I_k(\varepsilon)$.

(ii) If the group is non-abelian but still has only finite-dimensional irreducible representations, then one can show that the spectrum of the $\rho$-equivariant Laplacian is still included in $I(\varepsilon)$. In this case the (non-abelian) Floquet theory guarantees again that $\operatorname{spec}\Delta_{(X,g_\varepsilon)} \subset I(\varepsilon)$. The class of groups which satisfy the previous condition are type I groups, i.e finite extensions of abelian groups. These groups have a dual object $\widehat{\Gamma}$ which is a nice measure space (*smooth* in the terminology of [Mac76, Chapter 2]).

(iii) If the group is *residually finite* (a much wider class of groups including type I groups), then one can construct a so-called *tower of coverings* consisting of finite coverings $M_i \to M$ "converging" to the original covering $X \to M$. The inclusion of the spectrum of $\Delta_{(X,g_\varepsilon)}$ in the closure of the union over all spectra of $\Delta_{(M_i,g_\varepsilon)}$ was shown in [AdSS94, Ad95]. For the *finite* coverings $M_i \to M$ we again have the inclusion $\operatorname{spec}\Delta_{(M_i,g_\varepsilon)} \subset I(\varepsilon)$.

(iv) For non-amenable groups (i.e. groups, for which $\operatorname{spec}\Delta_{(M,g_\varepsilon)}$ is not included in $\operatorname{spec}\Delta_{(X,g_\varepsilon)}$), cf. Remark 5.3, we have to assure that any of the intervals $I_k(\varepsilon)$ intersects $\operatorname{spec}\Delta_X$ non-trivially. This will be done in Theorem 3.3.

**Organisation of the paper.** In the following section we set up the problem, present the geometrical context and state some results and conventions that will be needed later. In Section 3 we present in detail the two procedures for constructing covering manifolds with a decoupling family of metrics. In this case the set $I(\varepsilon)$ defined above will have at least a prescribed finite number of spectral gaps. Each procedure is well adapted to a given initial geometrical context (cf. Remark 3.8 as well as Examples 8.3 and 8.4). In Section 4 we show the inclusion of the spectrum of equivariant Laplacians into the union of the Dirichlet-Neumann intervals $I_k(\varepsilon)$ and review briefly the Floquet theory for non-abelian groups. The Floquet theory is applied in Section 5 for coverings with type I groups. In Section 6 we study a class of covering manifolds with residually finite groups. In Section 7 we consider residually finite groups $\Gamma$ that in addition have a positive Kadison constant. In Section 8 we illustrate the results obtained with some classes of examples and point out their mutual relations. Subsection 8.3 contains an interesting example of a covering with an amenable, *not* residually finite group which cannot be treated with our methods. We expect though that in this case one can still generate spectral gaps by the construction presented in Section 3. Finally, we conclude mentioning several possible applications for our results.

## 2. Geometrical preliminaries: covering manifolds and Laplacians

We begin fixing our geometrical context and recalling some results that will be useful later on. We denote by $X$ a *non-compact* Riemannian manifold of dimension $d \geq 2$ with a metric $g$. We also assume the existence of a finitely generated (infinite) discrete group $\Gamma$ of isometries acting *properly discontinuously* and *cocompactly* on $X$, i.e. for each $x \in X$ there is a neighbourhood $U$ of $x$ such that the sets $\gamma U$ and $\gamma' U$ are disjoint if $\gamma \neq \gamma'$ and $M := X/\Gamma$ is compact. Moreover, the quotient $M$ is a Riemannian manifold which also has dimension $d$ and is locally isometric to $X$. In other words, $\pi\colon X \longrightarrow M$ is a *Riemannian covering space* with covering group $\Gamma$. We call such a manifold $\Gamma$-*periodic* or simply *periodic*. All groups $\Gamma$ appearing in this paper will satisfy the preceding properties.

We also fix a *fundamental domain* $D$, i.e. an open set $D \subset X$ such that $\gamma D$ and $\gamma' D$ are disjoint for all $\gamma \neq \gamma'$ and $\bigcup_{\gamma\in\Gamma} \gamma\overline{D} = X$. We always assume that $\overline{D}$ is compact and that $\partial D$ is piecewise smooth. If not otherwise stated we also assume that $D$ is connected. Note that we can embed $D \subset X$ isometrically into the quotient $M$. In the sequel, we will not always distinguish between $D$ as a subset of $X$ or $M$ since they are isometric. For details we refer to [Ra94, §6.5].

As a prototype for an elliptic operator we consider the Laplacian $\Delta_X$ on a Riemannian manifold $(X, g)$ acting on a dense subspace of the Hilbert space $\mathsf{L}_2(X)$ with norm $\|\cdot\|_X$. For the formulation of the Theorems 5.4 and 6.8 and at other places, it is useful to denote explicitly the dependence on the metric, since we deform the manifold by changing the metric. In this case we will write $\Delta_{(X,g)}$ for $\Delta_X$ or $\mathsf{L}_2(X, g)$ for $\mathsf{L}_2(X)$.

The positive self-adjoint operator $\Delta_X$ can be defined in terms of a suitable quadratic form $q_X$ (see e.g. [K95, Chapter VI], [RS80] or [Dav96]). Concretely we have

$$q_X(u) := \|du\|_X^2 = \int_X |du|^2, \quad u \in \mathsf{C}_\mathrm{c}^\infty(X) \tag{2.1}$$

where the integral is taken with respect to the volume density measure of $(X, g)$. In coordinates we write the pointwise norm of the 1-form $du$ as

$$|du|^2(x) = \sum_{i,j} g^{ij}(x)\partial_i u(x)\, \partial_j \overline{u(x)},$$

where $(g^{ij})$ is the inverse of the metric tensor $(g_{ij})$ in a chart. Taking the closure of the quadratic form we can extend $q_X$ onto the Sobolev space

$$\mathsf{H}^1(X) = \mathsf{H}^1(X, g) = \{\, u \in \mathsf{L}_2(X) \,|\, q_X(u) < \infty \,\}.$$

As usual the operator $\Delta_X$ is related with the quadratic form by the formula $\langle \Delta_X u, u\rangle = q_X(u)$, $u \in \mathsf{C}_\mathrm{c}^\infty(X)$. Since the metric on $X$ is $\Gamma$-invariant, the Laplacian $\Delta_X$ (i.e. its resolvent) commutes with the translation on $X$ given by

$$(T_\gamma u)(x) := u(\gamma^{-1}x), \quad u \in \mathsf{L}_2(X), \gamma \in \Gamma. \tag{2.2}$$

Operators with this property are called *periodic*.

For an open, relatively compact subset $D \subset X$ with sufficiently smooth boundary $\partial D$ (e.g. Lipschitz) we define the Dirichlet (respectively, Neumann) Laplacian $\Delta_D^+$ (resp., $\Delta_D^-$) via its quadratic form $q_D^+$ (resp., $q_D^-$) associated to the closure of $q_D$ on $\mathsf{C}_\mathrm{c}^\infty(D)$, the space of smooth functions with compact support, (resp., $\mathsf{C}^\infty(\overline{D})$, the space of smooth functions with continuous derivatives up to the boundary). We also use the notation $\mathsf{H}_\circ^1(D) = \operatorname{dom} q_D^+$ (resp., $\mathsf{H}^1(D) = \operatorname{dom} q_D^-$). Note that the usual boundary condition of the Neumann Laplacian occurs only in the *operator* domain via the Gauß-Green formula. Since $\overline{D}$ is compact, $\Delta_D^+$ has purely discrete spectrum $\lambda_k^+$, $k \in \mathbb{N}$. It is written in ascending order and repeated according to multiplicity. The same is true for the Neumann Laplacian and we denote the corresponding purely discrete spectrum by $\lambda_k^-$, $k \in \mathbb{N}$.

One of the advantages of the quadratic form approach is that one can easily read off from the inclusion of domains an order relation for the eigenvalues. In fact, by the the *min-max principle* we have

$$\lambda_k^\pm = \inf_{L_k} \sup_{u\in L_k\setminus\{0\}} \frac{q_D^\pm(u)}{\|u\|^2}, \tag{2.3}$$

where the infimum is taken over all $k$-dimensional subspaces $L_k$ of the corresponding *quadratic* form domain $\operatorname{dom} q_D^{\pm}$, cf. e.g. [Dav96]. Then the inclusion

$$\operatorname{dom} q_D^{+} = \mathsf{H}^1_\circ(D) \subset \mathsf{H}^1(D) = \operatorname{dom} q_D^{-} \tag{2.4}$$

implies the following important relation between the corresponding eigenvalues

$$\lambda_k^+ \geq \lambda_k^- . \tag{2.5}$$

This means, that the Dirichlet $k$-th eigenvalue is in general larger than the $k$-th Neumann eigenvalue and this justifies the choice of the labels $+$, respectively, $-$.

## 3. Construction of periodic manifolds

In the present section we will give two different construction procedures (labelled by the letters 'A' and 'B') for covering manifolds, such that the corresponding Laplacian will have a prescribed finite number of spectral gaps. In contrast with [P03] (where only abelian groups were considered) we will base the construction on the specification of the quotient space $M = X/\Gamma$. By doing this, the spectral convergence result in Theorem 3.1 becomes manifestly independent of the fact whether $\Gamma$ is abelian or not.

Both constructions are done in two steps: first, we specify in two ways the quotient $M$ together with a family of metrics $g_\varepsilon$. Second, we construct in either case the covering manifold with covering group $\Gamma$ which has $r$ generators. In the last section we will localise the spectrum of the covering Laplacian in certain intervals given by an associated Dirichlet, respectively, Neumann eigenvalue problem. Some reasons for presenting two different methods (A) and (B) are formulated in a final remark of this section.

### 3.1. Construction of the quotient.

In the following two methods we define a family of Riemannian manifolds $(M, g_\varepsilon)$ that converge to a Riemannian manifold $(N, g)$ of the same dimension (cf. Figure 2). In each case we will also specify a domain $D \subset M$ (in the following section $D$ will become a fundamental domain of the corresponding covering):

(1A) **Attaching $r$ handles:** We construct the manifold $M$ by attaching $r$ handles diffeomorphic with $C := (0, 1) \times \mathbb{S}^{d-1}$ to a given $d$-dimensional compact orientable manifold $N$ with metric $g$. For simplicity we assume that $N$ has no boundary. Concretely, for each handle we remove two small discs of radius $\varepsilon > 0$ from $N$, denote the remaining set by $R_\varepsilon$ and identify $\{0\} \times \mathbb{S}^{d-1}$ with the boundary of the first hole and $\{1\} \times \mathbb{S}^{d-1}$ with the boundary of the second hole. We denote by $D$ the open subset of $M$ where the mid section $\{1/2\} \times \mathbb{S}^{d-1}$ of each handle is removed.

One can finally define a family of metrics $(g_\varepsilon)_\varepsilon$, $\varepsilon > 0$, on $M$ such that the diameter and length of the handle is of order $\varepsilon$ (see e.g. [P03, ChF81]). In this situation the handles shrink to a point as $\varepsilon \to 0$. Note that $(R_\varepsilon, g)$ can be embedded isometrically into $(N, g)$, resp., $(M, g_\varepsilon)$. This fact will we useful for proving Theorem 3.3.

(1B) **Conformal change of metric:** In the second construction, we start with an arbitrary compact $d$-dimensional Riemannian manifold $M$ with metric $g$. We consider only the case $d \geq 3$ (for a discussion of some two-dimensional examples see [P03]). Moreover, we assume that $N$ and $D$ are two open subsets of $M$ such that (i) $\partial N$ is smooth, (ii) $\overline{N} \subset D$, (iii) $\overline{D} = M$ and (iv) $D \setminus N$ can completely be described by Fermi coordinates (i.e. coordinates $(r, y)$, $r$ being the distance from $N$ and $y \in \partial N$) up to a set of measure 0 (cf. Figure 2 (B)). The last assumption assures that $N$ is in some sense large in $D$.

Suppose in addition, that $\rho_\varepsilon \colon M \longrightarrow (0, 1]$, $\varepsilon > 0$, is a family of smooth functions such that $\rho_\varepsilon(x) = 1$ if $x \in N$ and $\rho_\varepsilon(x) = \varepsilon$ if $x \in M \setminus N$ and

$\operatorname{dist}(x, \partial N) \ge \varepsilon^d$. Then $\rho_\varepsilon$ converges pointwise to the characteristic function of $N$. Furthermore, the Riemannian manifold $(M, g_\varepsilon)$ with $g_\varepsilon := \rho_\varepsilon^2 g$ converges to $(N, g)$ in the sense that $M \setminus N$ shrinks to a point in the metric $g_\varepsilon$.

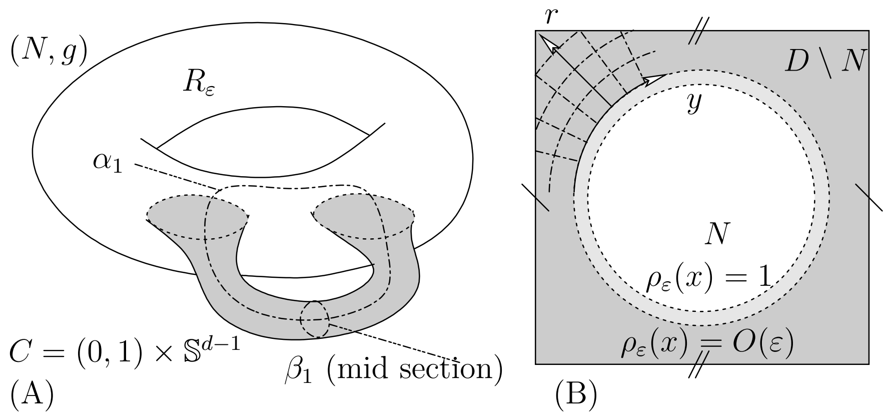



FIGURE 2. Two constructions of a family of manifold $(M, g_\varepsilon)$, $\varepsilon > 0$: In both cases, the grey area has a length scale of order $\varepsilon$ in all directions. (A) We attach $r$ handles (here $r = 1$) of diameter and length of order $\varepsilon$ to the manifold $(N, g)$. We also denoted the two cycles $\alpha_1$ and $\beta_1$. (B) We change the metric conformally to $g_\varepsilon = \rho_\varepsilon^2 g$. The grey area $D \setminus N$ (with Fermi coordinates in the upper left corner) shrinks conformally to a point as $\varepsilon \to 0$ whereas $N$ remains fixed. Note that the opposite sides of the square are identified (to obtain a torus as manifold $M$).

Now we can formulate the following spectral convergence result which was proven in [P03]:

**Theorem 3.1.** *Suppose $(M, g_\varepsilon)$ and $D \subset M$ are constructed as in parts (1A) or (1B) above. In Case (1B) we assume in addition that $d \ge 3$. Then*

$$\lambda_k^\pm(\varepsilon) \to \lambda_k(0)$$

*as $\varepsilon \to 0$ for each $k$. Here, $\lambda_k^\pm(\varepsilon)$ denotes the $k$-th Dirichlet, resp., Neumann eigenvalue of the Laplacian on $(D, g_\varepsilon)$ whereas $\lambda_k(0)$ is the $k$-th eigenvalue of $(N, g)$ (with Neumann boundary conditions at $\partial N$ in Case (1B)).*

3.2. **Construction of the covering spaces.** Given $(M, g_\varepsilon)$ and $D$ as in the previous subsection, we will associate a Riemannian covering $\pi\colon (X, g_\varepsilon) \longrightarrow (M, g_\varepsilon)$ with covering group $\Gamma$ such that $D$ is a fundamental domain. Note that we identify $D \subset M$ with a component of the lift $\widetilde{D} := \pi^{-1}(D)$. Moreover, $\Gamma$ is isomorphic to a normal subgroup of the fundamental group $\pi_1(M)$.

(2A) Suppose that $\Gamma$ is a discrete group with $r$ generators $\gamma_1, \dots, \gamma_r$. We will construct a $\Gamma$-covering $(X, g_\varepsilon) \to (M, g_\varepsilon)$ with fundamental domain $D$ where $D$ and $(M, g_\varepsilon)$ are given as in Part (1A) of the previous subsection. Roughly speaking, we glue together $\Gamma$ copies of $D$ along the handles according to the Cayley graph of $\Gamma$ w.r.t. the generators $\gamma_1, \dots, \gamma_r$. For convenience of the reader, we specify the construction:

The fundamental group of $M$ is given by $\pi_1(M) = \pi_1(N) * \mathbb{Z}^{*r}$ in the case $d \ge 3$. Here, $G_1 * G_2$ denotes the free product of $G_1$ and $G_2$, and $\mathbb{Z}^{*r}$ is the free group in $r$ generators $\alpha_1, \dots, \alpha_r$. If $d = 2$ we know from the classification result for 2-dimensional orientable manifolds that $N$ is diffeomorphic to an $s$-holed torus. In this case the fundamental group is given by

$$\pi_1(M) = \langle \alpha_1, \beta_1, \dots, \alpha_{r+s}, \beta_{r+s} \mid [\alpha_1, \beta_1] \cdot \dots \cdot [\alpha_{r+s}, \beta_{r+s}] = e \rangle, \tag{3.1}$$

where $[\alpha, \beta] := \alpha\beta\alpha^{-1}\beta^{-1}$ is the usual commutator. We may assume that $\alpha_i$ represents the homotopy class of the cycle *transversal* to the section of the $i$-th handle and that $\beta_i$ represents the section itself ($i = 1, \dots, r$) (cf. Figure 2 (A)).

One easily sees that there exists an epimorphism $\varphi\colon \pi_1(M) \longrightarrow \Gamma$ which maps $\alpha_i \in \pi_1(M)$ to $\gamma_i \in \Gamma$ ($i = 1, \dots, r$) and all other generators to the unit element $e \in \Gamma$. Note that this map is also well-defined in the case $d = 2$, since the relation in (3.1) is trivially satisfied in the case when the $\beta_i$'s are mapped to $e$.

Finally, $\Gamma \cong \pi_1(M)/\ker\varphi$, and $X \to M$ is the associated covering with respect to the universal covering $\widetilde{M} \to M$ (considered as a principal bundle with discrete fibre $\Gamma$) and the natural action of $\Gamma$ on $\pi_1(M)$.

Then $X \to M$ is a normal $\Gamma$-covering with fundamental domain $D$ constructed as in (1A) of the preceding subsection. Here we use the fact that $\alpha_i$ is *transversal* to the section of the handle in dimension 2.

(2B) Suppose $(X, g) \to (M, g)$ is a Riemannian covering with fundamental domain $D$ such that $\partial D$ is piecewise smooth. Then $\overline{D} = M$, where we have embedded $D$ into the quotient, cf. [Ra94, Theorem 6.5.8]. According to (1B) we can conformally change the metric on $M$, to produce a new covering $(X, g_\varepsilon) \to (M, g_\varepsilon)$ that satisfies the required properties.

In both cases, we lift for each $\varepsilon > 0$ the metric $g_\varepsilon$ from $M$ to $X$ and obtain a Riemannian covering $(X, g_\varepsilon) \to (M, g_\varepsilon)$. Note that the set $D$ specified in the first step of the previous construction becomes a fundamental domain after the specification of the covering in the second step.

The following statement is a direct consequence of the spectral convergence result in Theorem 3.1:

**Theorem 3.2.** *Suppose $(X, g_\varepsilon) \to (M, g_\varepsilon)$ ($\varepsilon > 0$) is a family of Riemannian coverings with fundamental domain $D$ constructed as in the previous parts (2A) or (2B). Then for each $n \in \mathbb{N}$ there exists $\varepsilon = \varepsilon_n > 0$ such that*

$$I(\varepsilon) := \bigcup_{k \in \mathbb{N}} I_k(\varepsilon), \qquad \textit{with} \qquad I_k(\varepsilon) := [\lambda_k^-(\varepsilon), \lambda_k^+(\varepsilon)], \tag{3.2}$$

*is a closed set having at least $n$ gaps, i.e. $n + 1$ components as subset of $[0, \infty)$. Here, $\lambda_k^\pm(\varepsilon)$ denotes the $k$-th Dirichlet, resp., Neumann eigenvalue of the Laplacian on $(D, g_\varepsilon)$.*

*Proof.* First, note that $\{\, \lambda_k^\pm(\varepsilon) \mid k \in \mathbb{N} \,\}$, $\varepsilon \geq 0$, has no finite accumulation point, since the spectrum is discrete. Second, Theorem 3.1 shows that the intervals $I_k(\varepsilon)$ reduce to the point $\{\lambda_k(0)\}$ as $\varepsilon \to 0$. Therefore, $I(\varepsilon)$ is a locally finite union of compact intervals, hence closed. □

3.3. **Existence of spectrum outside the gaps.** In the following subsection we will assure that each Neumann-Dirichlet interval $I_k(\varepsilon)$ contains at least one point of $\operatorname{spec} \Delta_{(X, g_\varepsilon)}$ provided $\varepsilon$ is small enough. In our general setting described below (cf. Theorems 5.4 and 6.8) we will show the inclusion

$$\operatorname{spec} \Delta_{(X, g_\varepsilon)} \subset \bigcup_{k \in \mathbb{N}} I_k(\varepsilon). \tag{3.3}$$

It is a priori not clear that each $I_k(\varepsilon)$ intersects the spectrum of the Laplacian on $(X, g_\varepsilon)$, i.e. that gaps in $\bigcup_{k \in \mathbb{N}} I_k(\varepsilon)$ are also gaps in $\operatorname{spec} \Delta_{(X, g_\varepsilon)}$. If the covering group is amenable, the $k$-th eigenvalue of the Laplacian on the quotient $(M, g_\varepsilon)$ is always an element of $I_k(\varepsilon) \cap \operatorname{spec}(\Delta_X, g_\varepsilon)$ (cf. the argument in the proof of Theorem 5.4). In general, this need not to be true. Therefore, we need the following theorem which will be used in Theorems 6.8 and 7.5:

**Theorem 3.3.** *With the notation of the previous theorem, we have*

$$I_k(\varepsilon) \cap \operatorname{spec} \Delta_{(X,g_\varepsilon)} \neq \emptyset \tag{3.4}$$

*for all $k \in \mathbb{N}$.*

We begin with a general criterion which will be useful to detect points in the spectra of a parameter-dependent family of operators using only its sesquilinear form. A similar result is also stated in [KK05, Lemma 5.1].

Suppose that $H_\varepsilon$ is a self-adjoint, non-negative, unbounded operator in a Hilbert space $\mathcal{H}_\varepsilon$ for each $\varepsilon > 0$. Denote by $\mathcal{H}^1_\varepsilon := \operatorname{dom} h_\varepsilon$ the Hilbert space of the corresponding quadratic form $h_\varepsilon$ associated to $H_\varepsilon$ with norm $\|u\|_1 := (h_\varepsilon(u) + \|u\|_{\mathcal{H}_\varepsilon})^{1/2}$ and by $\mathcal{H}^{-1}_\varepsilon$ the dual of $\mathcal{H}^1_\varepsilon$. Note that $H_\varepsilon \colon \mathcal{H}^1_\varepsilon \longrightarrow \mathcal{H}^{-1}_\varepsilon$ is continuous. In the next lemma we characterise for each $\varepsilon$ certain spectral points of $H_\varepsilon$.

**Lemma 3.4.** *Suppose there exist a family $(u_\varepsilon) \subset \mathcal{H}^1_\varepsilon$ and constants $\lambda \geq 0$, $c > 0$ such that*

$$\|(H_\varepsilon - \lambda)u_\varepsilon\|_{-1} \to 0 \qquad \text{as} \qquad \varepsilon \to 0 \tag{3.5}$$

*and $\|u_\varepsilon\| \geq c > 0$ for all $\varepsilon > 0$, then there exists $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that*

$$\lambda + \delta(\varepsilon) \in \operatorname{spec} H_\varepsilon.$$

*Proof.* Suppose that the conclusion is false. Then there exist a sequence $\varepsilon_n \to 0$ and a constant $\delta_0 > 0$ such that

$$I_\lambda \cap \operatorname{spec} H_{\varepsilon_n} = \emptyset \qquad \text{with} \qquad I_\lambda := (\lambda - \delta_0, \lambda + \delta_0)$$

for all $n \in \mathbb{N}$. Denote by $E_t$ the spectral resolution of $H_\varepsilon$. Then

$$\begin{aligned} \|(H_\varepsilon - \lambda)u_\varepsilon\|^2_{-1} &= \int_{\mathbb{R}_+ \setminus I_\lambda} \frac{(t-\lambda)^2}{(t+1)} \, \mathrm{d}\langle E_t u_\varepsilon, u_\varepsilon\rangle \\ &\geq \frac{\delta_0^2}{\lambda + \delta_0 + 1} \int_{\mathbb{R}_+ \setminus I_\lambda} \mathrm{d}\langle E_t u_\varepsilon, u_\varepsilon\rangle \geq \frac{c\delta_0^2}{\lambda + \delta_0 + 1} \end{aligned}$$

since $I_\lambda$ does not lie in the support of the spectral measure. But this inequality contradicts (3.5). ☐

*Remark* 3.5. Eq. (3.5) is equivalent to the inequality

$$|h_\varepsilon(u_\varepsilon, v_\varepsilon) - \lambda\langle u_\varepsilon, v_\varepsilon\rangle| \leq o(1)\|v_\varepsilon\|_1 \qquad \text{for all } v_\varepsilon \in \mathcal{H}^1_\varepsilon \tag{3.6}$$

as $\varepsilon \to 0$. Note that $o(1)$ could depend on $u_\varepsilon$. The advantage of the criterion in the previous lemma is that one only needs to find a family $(u_\varepsilon)$ in the domain of the quadratic form $h_\varepsilon$.

We will need the following lemma in order to define a cut-off function with convergent $L_2$-integral of its derivative. Its proof is straightforward.

**Lemma 3.6.** *Denote by $h(r) := r^{-d+2}$ if $d \geq 3$ and $h(r) = \ln r$ if $d = 2$. For $\varepsilon \in (0,1)$ define*

$$\chi_\varepsilon(r) := \begin{cases} 0, & 0 < r \leq \varepsilon \\ \frac{h(r) - h(\varepsilon)}{h(\sqrt{\varepsilon}) - h(\varepsilon)}, & \varepsilon \leq r \leq \sqrt{\varepsilon} \\ 1, & \sqrt{\varepsilon} \leq r \end{cases} \tag{3.7}$$

*then $\chi_\varepsilon \in \mathsf{H}^1((0,1))$ and*

$$\|\chi'_\varepsilon\|^2 := \int_0^1 |\chi'_\varepsilon(r)|^2 r^{d-1} \, \mathrm{d}r = o(1)$$

*as* $\varepsilon \to 0$.

Remember that $(N, g)$ is the unperturbed manifold as in Figure 2. In Case A of Subsection 3.1, we denoted by $R_\varepsilon$ the manifold $N$ with a closed ball of radius $\varepsilon$ removed around each point where the handles have been attached (note that $R_\varepsilon$ is also contained in $D$) and denote by $(r, y)$ the polar coordinates around such a point ($r = \varepsilon$ corresponds to a component of $\partial R_\varepsilon$).

*Proof of Theorem 3.3.* Let $\varphi$ be the $k$-th eigenfunction of the limit operator $\Delta_N$ with eigenvalue $\lambda = \lambda_k(0)$. We will treat Cases A and B of Subsection 3.1 separately.
(3A) Set $u_\varepsilon(r, y) := \chi_\varepsilon(r)\varphi(r, y)$ in the polar coordinates described above and $u_\varepsilon := \varphi$ on $R_{\sqrt{\varepsilon}}$. Now, $\|\varphi\|^2_{R_{\sqrt{\varepsilon}}} \geq c$ since $\|\varphi\|^2_{R_{\sqrt{\varepsilon}}} \to \|\varphi\|^2_N > 0$ as $\varepsilon \to 0$. In addition, $u_\varepsilon \in \mathsf{H}^1_\circ(R_\varepsilon) \subset \mathsf{H}^1(X, g_\varepsilon)$ and

$$|\langle du_\varepsilon, dv_\varepsilon\rangle - \lambda\langle u_\varepsilon, v_\varepsilon\rangle| = \Big|\int_{R_\varepsilon} \big[\langle d\varphi, d(\chi_\varepsilon v_\varepsilon)\rangle - \lambda\varphi\overline{\chi_\varepsilon v}\big] + \int_{R_\varepsilon} \varphi\langle d\chi_\varepsilon, dv_\varepsilon\rangle - \int_{R_\varepsilon} \overline{v}\langle d\varphi, d\chi_\varepsilon\rangle\Big|$$

for all $v_\varepsilon \in \mathsf{H}^1(D_\varepsilon)$. Now the first integral vanishes since $\varphi$ is the eigenfunction with eigenvalue $\lambda$ on $N$. Note that $\chi_\varepsilon v \in \mathsf{H}^1_\circ(R_\varepsilon)$ can be interpreted as function in $\mathsf{H}^1(N)$. The second and third integral can be estimated from above by

$$\sup_{x \in N}\big[|\varphi(x)| + |d\varphi(x)|\big]\|\chi'_\varepsilon\|\|v_\varepsilon\|_1 = o(1)\|v_\varepsilon\|_1$$

since $\varphi$ is a smooth function on an $\varepsilon$-independent space and due to Lemma 3.6.
(3B) Set $u_\varepsilon := \varphi$ on $N$ and $u_\varepsilon(r, y) := \widetilde{\chi}_\varepsilon(r)\varphi(0, y)$, $r > 0$, i.e. on $D \setminus N$ with $\widetilde{\chi}_\varepsilon(r) := \chi_\varepsilon(\sqrt{\varepsilon}+\varepsilon^d-r)$, where $\chi_\varepsilon$ is defined in (3.7) with $d = 2$. Note that $\widetilde{\chi}'_\varepsilon(r) \neq 0$ only for those $r = \operatorname{dist}(x, \partial N)$ where the conformal factor $\rho_\varepsilon(x) = \varepsilon$. Now, $u_\varepsilon \in \mathsf{H}^1_\circ(D, g_\varepsilon) \subset \mathsf{H}^1(X, g_\varepsilon)$. Furthermore, for $v_\varepsilon \in \mathsf{H}^1(D, g_\varepsilon)$ we have

$$\begin{aligned}|\langle du_\varepsilon, dv_\varepsilon\rangle - \lambda\langle u_\varepsilon, v_\varepsilon\rangle| &\leq \int_{D\setminus N}\Big[\big|\widetilde{\chi}'_\varepsilon(r)\varphi(0, y)\partial_r v_\varepsilon\big|\rho_\varepsilon^{d-2} \\ &\quad + \big|\widetilde{\chi}_\varepsilon(r)\langle d_y\varphi(0, y), d_y v_\varepsilon\rangle\big|\rho_\varepsilon^{d-2} + \lambda\widetilde{\chi}_\varepsilon(r)|\varphi(0, y)v_\varepsilon|\rho_\varepsilon^d\Big]\,\mathrm{d}r\,\mathrm{d}y \\ &\leq C\Big[\Big(\int_{\varepsilon^d}^{\sqrt{\varepsilon}+\varepsilon^d-\varepsilon} |\widetilde{\chi}'_\varepsilon(r)|^2\varepsilon^{d-2}\,\mathrm{d}r\Big)^{\frac{1}{2}} \\ &\quad + \Big(\int_0^{\sqrt{\varepsilon}} |\widetilde{\chi}_\varepsilon(r)|^2\rho_\varepsilon^{d-2}\,\mathrm{d}r\Big)^{\frac{1}{2}} + \Big(\int_0^{\sqrt{\varepsilon}} |\widetilde{\chi}_\varepsilon(r)|^2\rho_\varepsilon^{d}\,\mathrm{d}r\Big)^{\frac{1}{2}}\Big]\|v_\varepsilon\|_1\end{aligned}$$

where we have used that $\varphi$ is the Neumann eigenfunction on $N$. Furthermore, $C$ depends on the supremum of $\varphi$ and $d\varphi$ and on $\lambda$. Note that the conformal factor $\rho_\varepsilon$ equals $\varepsilon$ on the support of $\widetilde{\chi}'_\varepsilon$, therefore, the first integral converges to 0 since $d \geq 3$. Finally, estimating $\widetilde{\chi}_\varepsilon$ and $\rho_\varepsilon$ by 1, the second and third integral are bounded by $\varepsilon^{1/4}$. □

We finally can define formally the meaning of "decoupling":

**Definition 3.7.** We call a family of metrics $(g_\varepsilon)_\varepsilon$ on $X \to M$ *decoupling*, if the conclusions of Theorems 3.2 and 3.3 hold, i.e., if there exists a fundamental domain $D$ such that for each $n$ there exists $\varepsilon_n > 0$ such that $I(\varepsilon_n)$ in (3.2) has at least $n+1$ components and if (3.4) holds for all $k \in \mathbb{N}$.

*Remark* 3.8. In the present section we have specified two constructions of decoupling families of metrics on covering manifolds, such that the corresponding Laplacians will have at least a prescribed number of spectral gaps (cf. Sections 5 and 6). The construction specified in method (A) is feasible for every given covering group $\Gamma$ with $r$ generators. Note that this method produces fundamental domains that have smooth boundaries (see e.g. Example 8.3 below).

The construction in (B) applies for every given Riemannian covering $(X,g) \to (M,g)$, since, by the procedure described, one can modify conformally this covering in order to satisfy the spectral convergence result of Theorem 3.1 (cf. Example 8.4).

## 4. Floquet theory for non-abelian groups

The aim of the present section is to state a spectral inclusion result (cf. Theorem 4.3) and the direct integral decomposition of $\Delta_X$ (cf. Theorem 4.5) for certain *non-abelian* discrete groups $\Gamma$. These results will be used to prove the existence of spectral gaps in the situations analysed in the next two sections. A more detailed presentation of the results in this section may be found in [LP07].

4.1. **Equivariant Laplacians.** We will introduce next a new operator that lies "between" the Dirichlet and Neumann Laplacians and that will play an important role in the following. Suppose $\rho$ is a unitary representation of the discrete group $\Gamma$ on the Hilbert space $\mathcal{H}$, i.e. $\rho\colon \Gamma \longrightarrow \mathcal{U}(\mathcal{H})$ is a homomorphism. We fix a fundamental domain $D$ for the $\Gamma$-covering $X \to M$.

We now introduce the space of smooth $\rho$-equivariant functions

$$\mathsf{C}^\infty_\rho(D,\mathcal{H}) := \{\, h\restriction_D \mid h \in \mathsf{C}^\infty(X,\mathcal{H}), \quad h(\gamma x) = \rho_\gamma h(x), \quad \gamma \in \Gamma, x \in X \,\}. \tag{4.1}$$

This definition coincides with the usual one for abelian groups, cf. [LP07]. Note that we need *vector-valued* functions $h\colon X \longrightarrow \mathcal{H}$ since the representation $\rho$ acts on the Hilbert space $\mathcal{H}$, which, in general, has dimension greater than 1.

We define next the so-called *equivariant Laplacian* (w.r.t. the representation $\rho$) on $\mathsf{L}_2(D,\mathcal{H}) \cong \mathsf{L}_2(D) \otimes \mathcal{H}$: Let a quadratic form be defined by

$$\|dh\|^2_D := \int_D \|dh(x)\|^2_{\mathcal{H}} \, \mathrm{d}X(x) \tag{4.2}$$

for $h \in \mathsf{C}^\infty_\rho(D,\mathcal{H})$, where the integrand is locally given by

$$\|dh(x)\|^2_{\mathcal{H}} = \sum_{i,j} g^{ij}(x) \, \langle \partial_i h(x), \partial_j h(x) \rangle_{\mathcal{H}}, \qquad x \in D.$$

This generalises Eq. (2.1) to the case of vector-valued functions. We denote the domain of the closure of the quadratic form by $\mathsf{H}^1_\rho(D,\mathcal{H})$. The corresponding non-negative, self-adjoint operator on $\mathsf{L}_2(D,\mathcal{H})$, the *$\rho$-equivariant Laplacian*, will be denoted by $\Delta^\rho_{D,\mathcal{H}}$ (cf. [K95, Chapter VI]).

4.2. **Dirichlet-Neumann bracketing.** We study in this section the spectrum of a $\rho$-equivariant Laplacian $\Delta^\rho$ associated with a finite-dimensional representation $\rho$. In particular, we show that $\operatorname{spec}\Delta^\rho$ is contained in a suitable set determined by the spectrum of the Dirichlet and Neumann Laplacians on $D$. The key ingredient in dealing with non-abelian groups is the observation that this set is *independent* of $\rho$.

We begin with the definition of certain operators acting in $\mathsf{L}_2(D,\mathcal{H})$ and its eigenvalues. We denote by $\lambda^-_m(\mathcal{H})$, $\lambda^\rho_m(\mathcal{H})$, resp., $\lambda^+_m(\mathcal{H})$ the $m$-th eigenvalue of the operator $\Delta^-_{D,\mathcal{H}}$, $\Delta^\rho_{D,\mathcal{H}}$, resp., $\Delta^+_{D,\mathcal{H}}$ corresponding to the quadratic form (4.2) on $\mathsf{H}^1_\circ(D,\mathcal{H})$, $\mathsf{H}^1_\rho(D,\mathcal{H})$, resp., $\mathsf{H}^1(D,\mathcal{H})$. Recall that $\mathsf{H}^1_\circ(D,\mathcal{H})$ is the $\mathsf{H}^1$-closure of the space of smooth functions

$h\colon D \longrightarrow \mathcal{H}$ with support away from $\partial D$ and $\mathsf{H}^1(D,\mathcal{H})$ is the closure of the space of smooth functions with derivatives continuous up to the boundary.

The proof of the next lemma follows, as in the abelian case (cf. Eqs. (2.4) and (2.5)), from the reverse inclusions of the quadratic form domains

$$\mathsf{H}^1(D,\mathcal{H}) \supset \mathsf{H}^1_\rho(D,\mathcal{H}) \supset \mathsf{H}^1_\circ(D,\mathcal{H}) \tag{4.3}$$

and the min-max principle (2.3).

**Lemma 4.1.** *We have*

$$\lambda_m^-(\mathcal{H}) \le \lambda_m^\rho(\mathcal{H}) \le \lambda_m^+(\mathcal{H})$$

*for all* $m \in \mathbb{N}$.

¿From the definition of the quadratic form in the Dirichlet, resp., Neumann case we have that the corresponding vector-valued Laplacians are a direct sum of the scalar operators. Therefore the eigenvalues of the corresponding vector-valued Laplace operators consist of repeated eigenvalues of the scalar Laplacian. We can arrange the former in the following way:

**Lemma 4.2.** *If* $n := \dim \mathcal{H} < \infty$ *then*

$$\lambda_m^\pm(\mathcal{H}) = \lambda_k^\pm, \qquad m = (k-1)n+1, \dots, kn,$$

*where* $\lambda_k^\pm$ *denotes the (scalar) k-th Dirichet/Neumann eigenvalue on* $D$.

*Proof.* Note that $\Delta^\pm_{D,\mathcal{H}}$ is unitarily equivalent to an $n$-fold direct sum of the scalar operator $\Delta^\pm_D$ on $\mathsf{L}_2(D)$ since there is no coupling between the components on the boundary. □

Recall the definition of the intervals $I_k := [\lambda_k^-, \lambda_k^+]$ in Eq. (3.2) (for simplicity, we omit in the following the index $\varepsilon$). From the preceding two lemmas we may collect the $n$ eigenvalues of $\Delta^\rho_{D,\mathcal{H}}$ which lie in $I_k$:

$$B_k(\rho) := \{\, \lambda_m^\rho(\mathcal{H}) \,|\, m = (k-1)n+1, \dots, kn \,\} \subset I_k, \qquad n := \dim \mathcal{H}. \tag{4.4}$$

Therefore, we obtain the following spectral inclusion for equivariant Laplacians. This result will be applied in Theorems 5.4 and 6.8 below.

**Theorem 4.3.** *If* $\rho$ *is a unitary representation on a finite-dimensional Hilbert space* $\mathcal{H}$ *then*

$$\operatorname{spec} \Delta^\rho_{D,\mathcal{H}} = \bigcup_{k\in\mathbb{N}} B_k(\rho) \subseteq \bigcup_{k\in\mathbb{N}} I_k$$

*where* $\Delta^\rho_{D,\mathcal{H}}$ *denotes the* $\rho$*-equivariant Laplacian.*

4.3. **Non-abelian Floquet transformation.** Consider first the right, respectively, left regular representation $R$, resp., $L$ on the Hilbert space $\ell_2(\Gamma)$:

$$(R_\gamma a)_{\widetilde\gamma} = a_{\widetilde\gamma\gamma}, \qquad (L_\gamma a)_{\widetilde\gamma} = a_{\gamma^{-1}\widetilde\gamma}, \qquad a = (a_\gamma)_\gamma \in \ell_2(\Gamma), \quad \gamma, \widetilde\gamma \in \Gamma. \tag{4.5}$$

Using standard results we introduce the following unitary map (see e.g., [LP07, Section 3 and the appendix] and references cited therein)

$$F\colon \ell_2(\Gamma) \longrightarrow \int_Z^\oplus \mathcal{H}(z)\,\mathrm{d}z \tag{4.6}$$

for a suitable measure space $(Z, \mathrm{d}z)$. The map $F$ is a generalisation of the Fourier transformation in the abelian case. Moreover, it transforms the right regular representation $R$ into the following direct integral representation

$$\widehat R_\gamma = F R_\gamma F^* = \int_Z^\oplus R_\gamma(z)\,\mathrm{d}z, \qquad \gamma \in \Gamma. \tag{4.7}$$

*Remark* 4.4. Let $\mathcal{R}$ be the von Neumann algebra generated by all unitaries $R_\gamma$, $\gamma \in \Gamma$, i.e.

$$\mathcal{R} = \{ R_\gamma \,|\, \gamma \in \Gamma \}'', \tag{4.8}$$

where $\mathcal{R}'$ denotes the commutant of $\mathcal{R}$ in $\mathcal{L}(\ell_2(\Gamma))$. Then we decompose $\mathcal{R}$ with respect to a maximal abelian von Neumann subalgebra $\mathcal{A} \subset \mathcal{R}'$ (for a concrete example see Example 4.6). The space $Z$ is the compact Hausdorff space associated, by Gelfand's isomorphism, to a *separable* $C^*$-algebra $\mathcal{C}$, which is strongly dense in $\mathcal{A}$. Furthermore, $\mathrm{d}z$ is a regular Borel measure on $Z$. We may identify the algebra $\mathcal{A}$ with $\mathsf{L}_\infty(Z, \mathrm{d}z)$ and since it is maximal abelian, the fibre representations $R(z)$ are irreducible a.e. (see [W92, Section 14.8 ff.]).

The generalised Fourier transformation introduced in Eq. (4.6) can be used to decompose $\mathsf{L}_2(X)$ into a direct integral. In particular, we define for a.e. $z \in Z$:

$$(Uu)(z)(x) := \sum_{\gamma \in \Gamma} u(\gamma x) R_{\gamma^{-1}}(z) v(z), \tag{4.9}$$

where $v := F\delta_e \in \ell_2(\Gamma)$, $u \in \mathsf{C}_\mathrm{c}^\infty(X)$ and $x \in D$. The map $U$ extends to a unitary map

$$U \colon \mathsf{L}_2(X) \longrightarrow \int_Z^\oplus \mathsf{L}_2(D, \mathcal{H}(z))\,\mathrm{d}z \cong \int_Z^\oplus \mathcal{H}(z)\,\mathrm{d}z \otimes \mathsf{L}_2(D),$$

the so-called *Floquet* or *partial Fourier transformation*. Moreover, operators commuting with the translation $T$ on $\mathsf{L}_2(X)$ are decomposable, in particular, we can decompose $\Delta_X$ since its resolvent commutes with all translations (2.2).

We denote by $\mathsf{C}_\mathrm{eq}^\infty(D, \mathcal{H}(z))$ the set of smooth $R(z)$-equivariant functions defined in (4.1) and $\Delta_D^\mathrm{eq}(z)$ is the $R(z)$-equivariant Laplacian in $\mathsf{L}_2(D, \mathcal{H}(z))$. One can show in this context (cf. [S88, LP07]):

**Theorem 4.5.** *The operator $U$ maps $\mathsf{C}_\mathrm{c}^\infty(X)$ into $\int_Z^\oplus \mathsf{C}_\mathrm{eq}^\infty(D, \mathcal{H}(z))\,\mathrm{d}z$. Moreover, $\Delta_X$ is unitary equivalent to $\int_Z^\oplus \Delta_D^\mathrm{eq}(z)\,\mathrm{d}z$ and*

$$\operatorname{spec} \Delta_X \subseteq \overline{\bigcup_{z \in Z} \operatorname{spec} \Delta_D^\mathrm{eq}(z)}. \tag{4.10}$$

*If $\Gamma$ is amenable (cf. Remark 5.3), then we have equality in* (4.10).

*Example* 4.6. Let us illustrate the above direct integral decomposition in the case of the free group $\Gamma = \mathbb{Z} * \mathbb{Z}$ generated by $\alpha$ and $\beta$. Let $A \cong \mathbb{Z}$ be the cyclic subgroup generated by $\alpha$. We can decompose the algebra $\mathcal{R}$ given in (4.8) w.r.t. the abelian algebra $\mathcal{A} := \{ L_a \in \mathcal{L}(\ell_2(\Gamma)) \,|\, a \in A \} \subset \mathcal{R}'$, and, in this case, we have $Z = \mathbb{S}^1$. Since the set $\{ a\gamma a^{-1} \,|\, a \in A \}$ is infinite provided $\gamma \notin A$, the algebra is *maximal* abelian in $\mathcal{R}'$ (i.e. $\mathcal{A} = \mathcal{A}' \cap \mathcal{R}'$), and therefore, each fibre representation $R(z)$ is irreducible in $\mathcal{H}(z)$. Moreover, since $L_a \in \mathcal{A}'$ ($a \in A$) we can also decompose these operators w.r.t the previous direct integral.

We can give a more concrete realisation of the abstract Fourier transformation $F = F_\Gamma$ (see e.g. [Ro83, Section 19]): We interprete $\Gamma \to A \setminus \Gamma$ as covering space with abelian covering group $A$ acting on $\Gamma$ from the left; the corresponding translation action $T_a$ on $\ell_2(\Gamma)$ coincides with the left regular representation $L_a$ ($a \in A$). The (abelian) Floquet transformation $U = U_A$ gives a direct integral decomposition

$$F_\Gamma = U_A \colon \ell_2(\Gamma) \longrightarrow \int_{\widehat{A}}^\oplus \mathcal{H}(\chi)\,\mathrm{d}\chi,$$

where $\mathcal{H}(\chi) \cong \ell_2(A \setminus \Gamma)$ is the space of $\chi$-equivariant sequences in $\ell_2(\Gamma)$. Note that $\mathcal{H}(\chi)$ is infinite dimensional. A straightforward calculation shows that

$$R_\gamma \cong \int_{\widehat{A}}^{\oplus} R_\gamma(\chi)\,\mathrm{d}\chi \quad \text{and} \quad L_a \cong \int_{\widehat{A}}^{\oplus} L_a(\chi)\,\mathrm{d}\chi,$$

where $R_\gamma(\chi)u(\widetilde{\gamma}) = u(\widetilde{\gamma}\gamma)$ and $L_a(\chi)u(\widetilde{\gamma}) = \overline{\chi}(a)u(\widetilde{\gamma})$ for $u \in \mathcal{H}(\chi)$. Note that $L_\gamma$, $\gamma \notin A$, does not decompose into a direct integral over $Z$ since it mixes the fibres. Furthermore, one sees that $v = (U\delta_e)(\chi)$ is the *unique* normalised eigenvector of $R_a(\chi)$ with eigenvalue $\chi(a)$. This follows from the fact that the set of cosets $\{\, A\gamma a \,|\, a \in A \,\} \subset A \setminus \Gamma$ is infinite provided $\gamma \notin A$. From the previous facts one can directly check that each $R(\chi)$ is an irreducible representation of $\Gamma$ in $\mathcal{H}(\chi)$ and that these representations are mutually inequivalent. Finally, $R(\chi)$ is also inequivalent to any irreducible component of the direct integral decomposition obtained from a different maximal abelian subgroup $B \neq A$.

## 5. Spectral gaps for type I groups

We will present in this section the first method to show that the Laplacian of the manifolds constructed in Section 3 with (in general *non-abelian*) type I covering groups have an arbitrary finite number of spectral gaps. We begin recalling the definition of type I groups in the context of discrete groups.

**Definition 5.1.** A discrete group $\Gamma$ is of *type I* if $\Gamma$ is a finite extension of an abelian group, i.e. if there is an exact sequence

$$0 \longrightarrow A \longrightarrow \Gamma \longrightarrow \Gamma_0 \longrightarrow 0,$$

where $A \lhd \Gamma$ is abelian and $\Gamma_0 \cong \Gamma/A$ is a finite group.

*Remark* 5.2. (i) In the previous definition we have used a simple characterisation of countable, *discrete* groups of type I due to Thoma, cf. [Th64]. Moreover, all irreducible representations of a type I group $\Gamma$ are finite-dimensional and have a uniform bound on the dimension (see [Th64, Mo72]). Therefore, the following properties are all equivalent: (a) there is a uniform bound on the dimensions of irreducible representations of $\Gamma$, (b) all irreducible representations of $\Gamma$ are finite-dimensional, (c) $\Gamma$ is a finite extension of an abelian group, (d) $\Gamma$ is CCR (completely continuous representation, cf. [W92, Ch. 14]), (e) $\Gamma$ is of type I. Recall also that $\Gamma$ is of type I iff the von Neumann algebra $\mathcal{R}$ generated by $\Gamma$ (cf. Eq. (4.8)) is of *type I* (cf. [Kan69]).

Note that for our application it would be enough if $\Gamma$ has a decomposition over a measure space $(Z, \mathrm{d}z)$ as in Remark 4.4 such that *almost* every representation $\rho(z)$ is finite-dimensional. But such a group is already of type I: indeed, if the set $\{\, z \in Z \,|\, \dim \mathcal{H}(z) = \infty \,\}$ has measure 0, then it follows from [Dix81, Section II.3.5] that the von Neumann Algebra $\mathcal{R}$ (cf. Eq. (4.8)) is of type I. By the above equivalent characterisation this implies that $\Gamma$ is of type I.

(ii) The following criterion (cf. [Kan69, Kal70]) will be used in Examples 8.4 and 8.5 to decide that a group is not of type I: The von Neumann algebra $\mathcal{R}$ is of type $\mathrm{II}_1$ iff $\Gamma_{\mathrm{fcc}}$ has infinite index in $\Gamma$. Here,

$$\Gamma_{\mathrm{fcc}} := \{\, \gamma \in \Gamma \,|\, C_\gamma \text{ is finite} \,\} \qquad (5.1)$$

is the set of elements $\gamma \in \Gamma$ having finite conjugacy class $C_\gamma$. In particular such a group is not of type I. Even worse: Almost all representations in the direct integral decomposition (4.7) are of type $\mathrm{II}_1$ ([Dix81, Section II.3.5]) and therefore infinite-dimensional (see e.g. Example 4.6).

*Remark* 5.3. The notion of amenable discrete groups will be useful at different stages of our approach. For a definition of *amenability* of a discrete group $\Gamma$ see e.g. [Day57] or [Br81]. We will only need the following equivalent characterisations: (a) $\Gamma$ is amenable. (b) $0 \in \operatorname{spec} \Delta_X$ [Br81]. (c) $\operatorname{spec} \Delta_M \subset \operatorname{spec} \Delta_X$ [S88, Propositions 7–8]. Here, $X \to M$ is a covering with covering group $\Gamma$. Note that discrete type I groups are amenable since they are finite extensions of abelian groups (extensions of amenable groups are again amenable, cf. [Day57, Section 4]).

We want to stress that Theorem 3.3 is no contradiction to the fact that $\Gamma$ is amenable iff $0 \in \operatorname{spec} \Delta_{(X,g_\varepsilon)}$ although the first interval $I_1(g_\varepsilon) = [0, \lambda_k^+(g_\varepsilon)]$ tends to 0 as $\varepsilon \to 0$. Note that we have only shown that $I_1(g_\varepsilon) \cap \operatorname{spec} \Delta_{(X,g_\varepsilon)} \neq \emptyset$ and *not* $0 = \lambda_1(M, g_\varepsilon) \in \operatorname{spec} \Delta_{(X,g_\varepsilon)}$ which is only true in the amenable case.

The *dual of* $\Gamma$, which we denote by $\widehat{\Gamma}$, is the set of equivalence classes of unitary irreducible representations of $\Gamma$. We denote by $[\rho]$ the (unitary) equivalence class of a unitary representation $\rho$ on $\mathcal{H}$. Note that the spectrum of a $\rho$-equivariant Laplacian and $\dim \mathcal{H}$ only depend on the *equivalence class* of $\rho$.

If $\Gamma$ is of type I, then the dual $\widehat{\Gamma}$ becomes a nice measure space ("smooth" in the terminology of [Mac76, Chapter 2]). Furthermore, we can use $\widehat{\Gamma}$ as measure space in the direct integral decomposition defined in Subsection 4.3. In particular, combining the results of Section 2 and 4 we obtain the main result for type I groups:

**Theorem 5.4.** *Suppose $X \to M$ is a Riemannian $\Gamma$-covering with fundamental domain $D$, where $\Gamma$ is a type I group and denote by $g$ the Riemannian metric on $X$. Then*

$$\operatorname{spec} \Delta_{(X,g)} \subset \bigcup_{k \in \mathbb{N}} I_k(g), \qquad \text{and} \qquad I_k(g) \cap \operatorname{spec} \Delta_{(X,g)} \neq \emptyset, \quad k \in \mathbb{N},$$

*where $I_k(g) := [\lambda_k^-(D,g), \lambda_k^+(D,g)]$ is the Neumann-Dirichlet interval defined as in* (3.2)*. In particular, for each $n \in \mathbb{N}$ there exists a metric $g = g_{\varepsilon_n}$ constructed as in Subsection 3.2 such that $\operatorname{spec} \Delta_{(X,g)}$ has at least $n$ gaps, i.e. $n+1$ components as subset of $[0,\infty)$.*

*Proof.* We have

$$\operatorname{spec} \Delta_X = \overline{\bigcup_{[\rho] \in \widehat{\Gamma}} \operatorname{spec} \Delta^\rho_{D,\mathcal{H}}} \subseteq \overline{\bigcup_{k \in \mathbb{N}} I_k(g)} = \bigcup_{k \in \mathbb{N}} I_k(g),$$

where we used the Theorem 4.5 with $Z = \widehat{\Gamma}$ for the first equality and Theorem 4.3 for the inclusion. Note that $\Gamma$ is amenable and that the latter theorem applies since all (equivalence classes of) irreducible representations of a type I group are finite-dimensional (cf. Remark 5.2 (1)). The existence of gaps in $\bigcup_k I_k(g)$ follows from Theorem 3.2.

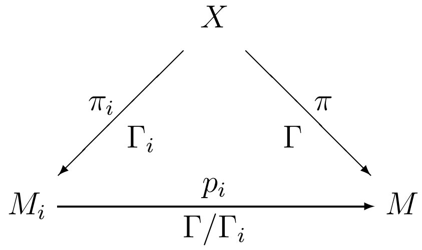
Since $\Gamma$ is amenable, $\operatorname{spec} \Delta_M \subset \operatorname{spec} \Delta_X$ (cf. (c) in Remark 5.3). Moreover, from Eq. (4.4) with $\rho$ the trivial representation on $\mathcal{H} = \mathbb{C}$, we have that $\lambda_k(M) \in I_k$. Note that functions on $M$ correspond to functions on $D$ with periodic boundary conditions. Therefore, we have shown that every gap of the union $\bigcup_k I_k(g)$ is also a gap of $\operatorname{spec} \Delta_X$. □

## 6. Spectral gaps for residually finite groups

In this section, we present a new method to prove the existence of a finite number of spectral gaps of $\Delta_X$. The present approach is applicable to so-called residually finite groups $\Gamma$, which is a much larger class of groups containing type I groups (cf. Section 8). Roughly speaking, residually finite means that $\Gamma$ has a lot of normal subgroups with finite index. Geometrically, this implies that one can approximate the covering $\pi \colon X \longrightarrow M$ with covering group $\Gamma$ by *finite* coverings $p_i \colon M_i \longrightarrow M$, where the $M_i$'s are compact.

Since the present section is central to the paper we will give for completeness proofs of known results, namely for Theorem 6.6 (see [AdSS94, Ad95]).

6.1. **Subcoverings and residually finite groups.** Suppose that $\pi\colon X \longrightarrow M$ is a covering with covering group $\Gamma$ (as in Section 2). Corresponding to a normal subgroup $\Gamma_i \lhd \Gamma$ we associate a covering $\pi_i\colon X \longrightarrow M_i$ such that




(6.1)

is a commutative diagram. The groups under the arrows denote the corresponding covering groups.

**Definition 6.1.** A (countable, infinite) discrete group $\Gamma$ is residually finite if there exists a monotonous decreasing sequence of normal subgroups $\Gamma_i \lhd \Gamma$ such that

$$\Gamma = \Gamma_0 \rhd \Gamma_1 \rhd \cdots \rhd \Gamma_i \rhd \cdots, \qquad \bigcap_{i \in \mathbb{N}} \Gamma_i = \{e\} \quad \text{and} \quad \Gamma/\Gamma_i \text{ is finite.} \tag{6.2}$$

Denote by $\mathfrak{R}\mathcal{F}$ the class of residually finite groups.

Suppose now that $\Gamma$ is residually finite. Then there exists a corresponding sequence of coverings $\pi_i\colon X \longrightarrow M_i$ such that $p_i\colon M_i \longrightarrow M$ is a *finite* covering (cf. Diagram (6.1)). Such a sequence of covering maps is also called *tower of coverings*.

*Remark* 6.2. We recall also the following equivalent definitions of residually finite groups (see e.g. [Mag69] or [Rob82, Section 2.3]).

(i) A group $\Gamma$ is called *residually finite* if for all $\gamma \in \Gamma \setminus \{e\}$ there is a group homomorphism $\Psi\colon \Gamma \longrightarrow G$ such that $\Psi(\gamma) \neq e$ and $\Psi(\Gamma)$ is a *finite* group.

(ii) Let $\mathcal{F}$ denote the class of finite groups. Then $\Gamma$ is residually finite, iff the so-called $\mathcal{F}$*-residual*

$$\mathfrak{R}_{\mathcal{F}}(\Gamma) := \bigcap_{\substack{N \lhd \Gamma \\ \Gamma/N \in \mathcal{F}}} N \tag{6.3}$$

is trivial, i.e. $\mathfrak{R}_{\mathcal{F}}(\Gamma) = \{e\}$.

Next we give some examples for residually finite groups (cf. the survey article [Mag69]):

*Example* 6.3. (i) Abelian and finite groups are residually finite. (ii) Free products of residually finite groups are residually finite, in particular, the free group in $r$ generators $\mathbb{Z}^{*r}$ is residually finite. (iii) Finitely generated linear groups are residually finite (for a simple proof of this fact cf. [Al87]; a group is called *linear* iff it is isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{C})$ for some $n \in \mathbb{N}$.) In particular, $\mathrm{SL}_n(\mathbb{Z})$, fundamental groups of closed, orientable surfaces of genus $g$ or, more generally, finitely generated subgroups of the isometry group on the hyperbolic space $\mathbb{H}^d$ are residually finite.

Next we need to introduce a metric on the discrete space $\Gamma$:

**Definition 6.4.** Let $G$ be a set which generates $\Gamma$. The *word metric* $d = d_G$ on $\Gamma$ is defined as follows: $d(\gamma, e)$ is the minimal number of elements in $G$ needed to express $\gamma$ as a word in the alphabet $G$; $d(e,e) := 0$ and $d(\gamma, \widetilde{\gamma}) := d(\gamma\widetilde{\gamma}^{-1}, e)$.

Geometrically, residually finiteness means that, given any compact set $K \subset X$, there exists a finite covering $p_i\colon M_i \longrightarrow M$ and a covering $\pi_i\colon X \longrightarrow M_i$ which is injective on $K$ (cf. [Bro86]). This idea is used in the following lemma:

**Lemma 6.5.** *Fix a fundamental domain $D$ for the covering $\pi\colon X \longrightarrow M$ and suppose that $\pi_i\colon X \longrightarrow M_i$ ($i \in \mathbb{N}$) is a tower of coverings as above. Then for each covering $\pi_i\colon X \longrightarrow M_i$ there is a fundamental domain $D_i$ (not necessarily connected) such that*

$$D_0 := D \subset D_1 \subset \cdots \subset D_i \subset \cdots \qquad \text{and} \qquad \bigcup_{i\in\mathbb{N}} D_i = X.$$

*Proof.* It is enough to show the existence of a family of representants $R_i \subset \Gamma$ of $\Gamma/\Gamma_i$, $i \in \mathbb{N}$, satisfying

$$R_0 := \{e\} \subset R_1 \subset \cdots \subset R_i \subset \cdots \qquad \text{and} \qquad \bigcup_{i\in\mathbb{N}} R_i = \Gamma.$$

In this case the fundamental domains are given explicitly by

$$D_i := \operatorname{int} \bigcup_{r\in R_i} r^{-1}\overline{D},$$

where int denotes the topological interior.

Let $d$ be the word metric on $\Gamma$ with respect to the set of generators $G := \{\, \gamma \in \Gamma \mid \gamma\overline{D} \cap \overline{D} \neq \emptyset \,\}$, which is naturally adapted to the fundamental domain $D$. Note that $G$ is finite and generates $\Gamma$ since $\overline{D}$ is compact (cf. [Ra94, Theorems 6.5.10 and 6.5.11]).

We choose a set of representants $R_i$ of $\Gamma/\Gamma_i$ that have minimal distance in the word metric to the neutral element, i.e. if $r \in R_i$, then $d(r,e) \le d(r\Gamma_i, e)$. Note that since $\Gamma_{i+1} \subset \Gamma_i$ we have $R_{i+1} \supset R_i$. To conclude the proof we have to show that every $\gamma \in \Gamma$ is contained in some $R_i$, $i \in \mathbb{N}$. Since $\Gamma$ is finitely generated, there exists $n \in \mathbb{N}$ such that $\gamma \in B_n := \{\gamma \in \Gamma \mid d(\gamma, e) \le n\}$. Moreover, since $B_{2n}$ is finite and $\Gamma$ residually finite we also have $B_{2n} \cap \Gamma_i = \{e\}$ for $i$ large enough. Therefore, any other element $\widetilde{\gamma} = \gamma\gamma_i^{-1}$ in the class $\gamma\Gamma_i$ with $\gamma_i \in \Gamma_i \setminus \{e\}$ has a distance greater than $n$, since

$$d(\widetilde{\gamma}, e) = d(\gamma\gamma_i^{-1}, e) = d(\gamma, \gamma_i) \ge d(e, \gamma_i) - d(\gamma, e) > 2n - n = n.$$

This implies that $\gamma \in R_i$ by the minimality condition in the choice of the representants. □

**Theorem 6.6.** *Suppose $\Gamma$ is residually finite with the associated sequence of coverings $\pi_i\colon X \longrightarrow M_i$ and $p_i\colon M_i \longrightarrow M$ as in* (6.1)*. Then*

$$\operatorname{spec} \Delta_X \subseteq \overline{\bigcup_{i\in\mathbb{N}} \operatorname{spec} \Delta_{M_i}},$$

*and the Laplacian $\Delta_{M_i}$ w.r.t. the finite covering $p_i\colon M_i \longrightarrow M$ has discrete spectrum. Equality holds iff $\Gamma$ is amenable.*

*Proof.* (Cf. [Ad95]) If $\lambda \in \operatorname{spec} \Delta_X$, then for each $\varepsilon > 0$ there exists $u \in \mathsf{C}_c^\infty(X)$ such that

$$\frac{\|(\Delta_X - \lambda)u\|_X^2}{\|u\|_X^2} < \varepsilon.$$

Applying Lemma 6.5 there is an $i = i(\varepsilon)$ such that $\operatorname{supp} u \subset D_i$. Furthermore, since $D_i \hookrightarrow M_i = X/\Gamma_i$ is an isometry, $u$ can be written as the lift of a smooth $f$ on $M_i$, i.e. $f \circ \pi_i = u$. Therefore,

$$\frac{\|(\Delta_{M_i} - \lambda)f\|_{M_i}^2}{\|f\|_{M_i}^2} = \frac{\|(\Delta_X - \lambda)u\|_X^2}{\|u\|_X^2} < \varepsilon,$$

which implies $\lambda \in \overline{\bigcup_{i\in\mathbb{N}} \operatorname{spec} \Delta_{M_i}}$. Finally, since $M_i \to M$ is a finite covering and $M$ is compact, $\operatorname{spec} \Delta_{M_i}$ is discrete. For the second assertion cf. [Ad95] or [AdSS94]. One basically uses the characterisation due to [Br81] that $\Gamma$ is amenable iff $0 \in \operatorname{spec} \Delta_X$ (cf. Remark 5.3). □

Next we analyse the spectrum of the finite covering $M_i \to M$. Note that $D$ is also isometric to a fundamental domain for *each* finite covering $M_i \to M$, $i \in \mathbb{N}$.

**Lemma 6.7.** *We have*

$$\operatorname{spec} \Delta_{M_i} = \bigcup_{[\rho]\in\widehat{G_i}} \operatorname{spec} \Delta^{\rho}_{D,\mathcal{H}(\rho)},$$

*where $\Delta^\rho$ is the equivariant Laplacian introduced in Subsection 4.1 and $G_i := \Gamma/\Gamma_i$ is a finite group and $\widehat{G_i}$ its dual.*

*Proof.* Applying the results of Subsection 4.3 to the finite group $G_i$ and the finite measure space $Z := \widehat{G_i}$ with the counting measure all direct integrals become direct sums. By Peter-Weyl's theorem (see e.g. [HR70, §27.49]) we also have

$$F\colon \ell_2(G_i) \longrightarrow \bigoplus_{[\rho]\in\widehat{G_i}} n(\rho)\mathcal{H}(\rho),$$

where each multiplicity satisfies $n(\rho) = \dim \mathcal{H}(\rho) < \infty$. Finally,

$$\Delta_{M_i} \cong \bigoplus_{[\rho]\in\widehat{G_i}} \Delta^{\rho}_{D,\mathcal{H}(\rho)}$$

and the result follows. □

We now can formulate the main result of this section:

**Theorem 6.8.** *Suppose $X \to M$ is a Riemannian $\Gamma$-covering with fundamental domain $D$, where $\Gamma$ is a residually finite group and denote by $g$ the Riemannian metric on $X$. Then*

$$\operatorname{spec} \Delta_{(X,g)} \subset \bigcup_{k\in\mathbb{N}} I_k(g), \qquad I_k(g) \cap \operatorname{spec} \Delta_{(X,g)} \neq \emptyset, \quad k \in \mathbb{N},$$

*where $I_k(g) := [\lambda_k^-(D,g), \lambda_k^+(D,g)]$ is defined as in* (3.2)*. In particular, for each $n \in \mathbb{N}$ there exists a metric $g = g_{\varepsilon_n}$, constructed as in Subsection 3.2, such that* $\operatorname{spec} \Delta_{(X,g)}$ *has at least $n$ gaps, i.e. $n+1$ components as subset of* $[0,\infty)$*.*

*Proof.* We have

$$\operatorname{spec} \Delta_X \subseteq \overline{\bigcup_{i\in\mathbb{N}} \operatorname{spec} \Delta_{M_i}} = \overline{\bigcup_{\substack{i\in\mathbb{N}\\ [\rho]\in\widehat{G_i}}} \operatorname{spec} \Delta^{\rho}_{D,\mathcal{H}(\rho)}} \subseteq \overline{\bigcup_{k\in\mathbb{N}} I_k(g)} = \bigcup_{k\in\mathbb{N}} I_k(g),$$

where we used Theorem 6.6, Lemma 6.7 and Theorem 4.3. Note that the latter theorem applies since all (equivalence classes of) irreducible representations of the finite groups $G_i$, $i \in \mathbb{N}$, are finite-dimensional. The existence of gaps in $\bigcup_k I_k(g)$ follows from Theorem 3.2. Finally, by Theorem 3.3, a gap of $\bigcup_k I_k(g)$ is in fact a gap of $\operatorname{spec} \Delta_X$. □

## 7. Kadison constant and asymptotic behaviour

In the present section we will combine our main result stated in Theorem 6.8 with some results by Sunada and Brüning (cf. [S92, Theorem 1] or [BS92]), to give a more complete description of the spectrum of the Laplacian $\Delta_X$, where $X \to M$ is the $\Gamma$-covering constructed in Section 3. For this, we need a further definition:

**Definition 7.1.** Let $\Gamma$ be a finitely generated discrete group. The *Kadison constant* of $\Gamma$ is defined as

$$C(\Gamma) := \inf\{\, \operatorname{tr}_\Gamma(P) \mid P \text{ non-trivial projection in } C^*_{\mathrm{red}}(\Gamma,\mathcal{K}) \,\},$$

where $\operatorname{tr}_\Gamma(\cdot)$ is the canonical trace on $C^*_{\mathrm{red}}(\Gamma,\mathcal{K})$ , the tensor product of the reduced group $C^*$-algebra of $\Gamma$ and the algebra $\mathcal{K}$ of compact operators on a separable Hilbert space of infinite dimension (see [S92, Section 1] for more details.)

In this section, we assume that $\Gamma$ is is residually finite and has a strictly positive Kadison constant, i.e. $C(\Gamma) > 0$. For example, the free product $\mathbb{Z}^{*r} * \Gamma_1 * \cdots * \Gamma_a$ with finite groups $\Gamma_i$ satisfies both properties (cf. e.g. [Mag69], [S92, Appendix]). Another such group is the fundamental group (cf. Eq. (3.1)) of a (compact, orientable) surface of genus $g$ (see [MM99]).

*Remark* 7.2. Suppose that $K$ is an integral operator on $\mathsf{L}_2(X)$ commuting with the group action, having smooth kernel $k(x, y)$ and satisfying

$$k(x, y) = 0 \qquad \text{for all } x, y \in X \text{ with } d(x, y) \geq c$$

for some constant $c > 0$. Then $K$ can be interpreted as an element of $C^*_{\text{red}}(\Gamma, \mathcal{K})$ and one can write the $\Gamma$-trace as

$$\operatorname{tr}_\Gamma K = \int_D k(x, x)\, dx$$

(see [S92, Section 1] as well as [At76] for further details), where $D$ is a fundamental domain of $X \to M$.

If we consider the spectral resolution of the Laplacian $\Delta_X \cong \int^{\oplus} \lambda \, \mathrm{d}E(\lambda)$, then it follows that

$$E(\lambda_2) - E(\lambda_1) \in C^*_{\text{red}}(\Gamma, \mathcal{K})$$

if $\lambda_1 < \lambda_2$ and $\lambda_1, \lambda_2 \notin \operatorname{spec} \Delta_X$ (cf. [S92, Section 2]).

Denote by $\mathcal{N}(g, \lambda)$ the number of components of $\operatorname{spec} \Delta_{(X,g)} \cap [0, \lambda]$. From [BS92, S92] we obtain the following asymptotic estimate on $\mathcal{N}(g, \lambda)$:

**Theorem 7.3.** *Suppose $(X, g) \to (M, g)$ is a Riemannian $\Gamma$-covering where $\Gamma$ has a positive Kadison constant, i.e. $C(\Gamma) > 0$ then*

$$\limsup_{\lambda \to \infty} \frac{\mathcal{N}(g, \lambda)}{(2\pi)^{-d} \omega_d \operatorname{vol}(M, g) \lambda^{d/2}} \leq \frac{1}{C(\Gamma)}. \tag{7.1}$$

*In particular, the spectrum of $\Delta_X$ has band-structure, i.e. $\mathcal{N}(g, \lambda) < \infty$ for all $\lambda \geq 0$.*

*Remark* 7.4. Note that Theorem 7.3 only gives an *asymptotic* upper bound on the number of components of $\operatorname{spec} \Delta_X \cap [0, \lambda]$, not on the *whole* spectrum itself. Therefore, we have no assertion about the so-called *Bethe-Sommerfeld conjecture* stating that the number of spectral gaps for a periodic operator in dimensions $d \geq 2$ remains *finite*.

Combining Theorem 7.3 with our result on spectral gaps we obtain more information on the spectrum and a *lower* asymptotic bound on the number of components:

**Theorem 7.5.** *Suppose $(X, g) \to (M, g)$ is a Riemannian $\Gamma$-covering where $\Gamma$ is a residually finite group and where $g = g_\varepsilon$ is the family of decoupling metrics constructed in Section 3. Then we have:*

(i) *For each $n \in \mathbb{N}$ there exists $g = g_{\varepsilon_n}$ such that $\operatorname{spec} \Delta_{(X,g)}$ has at least $n$ gaps. If in addition $C(\Gamma) > 0$ then there exists $\lambda_0 > 0$ such that*

$$n + 1 \leq \mathcal{N}(g, \lambda) < \infty$$

*for all $\lambda \geq \lambda_0$, i.e. $\operatorname{spec} \Delta_{(X,g)}$ has band-structure.*

(ii) *Suppose in addition that the limit manifold $(N, g)$ has simple spectrum, i.e. all eigenvalues $\lambda_k(0)$ have multiplicity* 1 *(cf. Theorem 3.1). Then for each $\lambda \geq 0$ there exists $\varepsilon(\lambda) > 0$ such that*

$$\liminf_{\lambda \to \infty} \frac{\mathcal{N}(g_{\varepsilon(\lambda)}, \lambda)}{(2\pi)^{-d} \omega_d \operatorname{vol}(N, g) \lambda^{d/2}} \geq 1.$$

*Here, $g_\varepsilon$ denotes the metric constructed in Section 3.*

*Proof.* (i) follows immediately from Theorems 6.8 and 7.3. (ii) Suppose $\lambda \notin \operatorname{spec} \Delta_N$, then $\lambda_k(0) < \lambda < \lambda_{k+1}(0)$ for some $k \in \mathbb{N}$. Let $\varepsilon = \varepsilon(\lambda) \in (0,1]$ be the largest number such that $\mathcal{N}(\lambda, g_\varepsilon)$ is (at least) $k$, in other words, $\mathcal{N}(\lambda, g_\varepsilon) \geq k = \mathcal{N}(\lambda, \Delta_N)$ where the latter number denotes the number of eigenvalues of $\Delta_N$ below $\lambda$. We conclude with the Weyl theorem,

$$\lim_{\lambda\to\infty} \frac{\mathcal{N}(\lambda, \Delta_N)}{(2\pi)^{-d}\omega_d \operatorname{vol}(N,g)\lambda^{d/2}} = 1,$$

where $\omega_d$ denotes the volume of the $d$-dimensional Euclidean unit ball. □

To conclude the section we remark that generically, $\Delta_{(N,g)}$ has simple spectrum (cf. [U76]). The assumption on the spectrum of $(N,g)$ is natural since $\mathcal{N}(g,\lambda)$ counts the components without multiplicity.

## 8. Examples

8.1. **Relation between the approaches presented in Sections 5 and 6.** We begin comparing the two main approaches presented in this paper which assure the existence of spectral gaps (cf. Sections 5 and 6).

One easily sees from Definition 6.1 that a *finite* extension of a residually finite group is again residually finite. In particular, type I groups are residually finite as finite extensions of abelian groups (cf. Definition 5.1). Therefore, for type I groups one can also produce spectral gaps by the approximation method with finite coverings introduced in Section 6. Nevertheless we believe that the direct integral method will be useful when analysing further spectral properties:

*Example* 8.1. One of the advantages of the method described in Section 5 is that one has more information about the bands. Suppose $\Gamma$ is finitely generated and *abelian*, i.e. $\Gamma \cong \mathbb{Z}^r \oplus \Gamma_0$, where $\Gamma_0$ is the torsion subgroup of $\Gamma$. Then $\widehat{\Gamma}$ is the disjoint union of finitely many copies of $\mathbb{T}^r$. From the continuity of the map $\rho \to \lambda_k^\rho$ (cf. [BJR99] or [S90]), we can simplify the characterisation of the spectrum in Theorem 4.5 and obtain

$$\operatorname{spec} \Delta_X = \bigcup_{k\in\mathbb{N}} B_k, \quad \text{where} \quad B_k := \{\, \lambda_k^\rho \mid \rho \in \widehat{\Gamma} \,\} \subseteq I_k, \tag{8.1}$$

the $k$-th *band.* Since $\widehat{\Gamma}$ is compact, $B_k$ is also compact, but in general, $B_k$ need not to be connected (recall that $\widehat{\Gamma}$ is connected iff $\Gamma$ is torsion free, i.e. $\Gamma = \mathbb{Z}^r$). Note also that $B_k$ has only finitely many components. For non-abelian groups this approach may be generalised in the direction of Hilbert C*-modules (cf. [Gr01]).

In principle one could also consider a combination of the methods of Section 5 and 6: denote by $\mathcal{T}_1$ the class of type I groups and by $\mathfrak{R}\mathcal{T}_1$ the class of *residually type I* groups, i.e. $\Gamma \in \mathfrak{R}\mathcal{T}_1$ iff the $\mathcal{T}_1$-residual $\mathfrak{R}_{\mathcal{T}_1}(\Gamma)$ is trivial (cf. Eq. (6.3)). Similarly we denote by $\mathfrak{R}\mathcal{F}$ the class of residually finite groups (cf. Definition 6.1). If we consider a covering with a group $\Gamma \in \mathfrak{R}\mathcal{T}_1$, then instead of the *finite* covering $p_i \colon M_i \longrightarrow M$ considered in Eq. (6.1) we would have a covering with a type I group. For these groups, we can replace Lemma 6.7 by the direct integral decomposition of Theorem 4.5. Nevertheless the following lemma shows that the class of residually finite and residually type I groups coincide.

**Lemma 8.2.** *From the inclusion $\mathcal{F} \subset \mathcal{T}_1 \subset \mathfrak{R}\mathcal{F}$ it follows that the corresponding residuals for the group $\Gamma$ coincide, i.e. $\mathfrak{R}_{\mathcal{F}}(\Gamma) = \mathfrak{R}_{\mathcal{T}_1}(\Gamma)$. Moreover, $\mathfrak{R}\mathcal{F} = \mathfrak{R}\mathcal{T}_1$.*

*Proof.* From the inclusion $\mathcal{F} \subset \mathcal{T}_1$ it follows immediately that $\mathfrak{R}_{\mathcal{F}}(\Gamma) \supset \mathfrak{R}_{\mathcal{T}_1}(\Gamma)$. To show the reverse inclusion one uses the following characterisation: a group is residually $\mathcal{F}$ iff it is a subcartesian product of finite groups (cf. [Rob82, § 2.3.3]). Finally, from the equality of the residuals it follows that $\mathfrak{R}\mathcal{F} = \mathfrak{R}\mathcal{T}_1$. □

8.2. **Examples with residually finite groups.** In the rest of this subsection we present several examples of residually finite groups which are not type I. They show different aspects of our analysis.

For the next example recall the construction (A) described in Section 3.

*Example* 8.3 (Fundamental groups of oriented, closed surfaces). Suppose that $N := \mathbb{S}^2$ is the two-dimensional sphere with a metric such that $\Delta_N$ has simple spectrum (cf. [U76] for the existence of such metrics). Suppose, in addition, that $M$ is obtained by adding $r$ handles to $N$ as described in Section 3, Case A. The fundamental group $\Gamma$ of $M$ (cf. Eq. (3.1) with $s = 0$) is residually finite (recall Example 6.3 (iii)). Moreover, from the proof of Proposition 2.16 in [MM99], it follows that $\Gamma$ has a positive Kadison constant. Therefore, Theorem 7.5 applies to the the universal cover $X := \widetilde{M} \to M$ with the metric $g_\varepsilon$ specified in Section 3.

The following example uses the construction (B) in Section 3.

*Example* 8.4 (Heisenberg group). Let $\Gamma := H_3(\mathbb{Z})$ be the *discrete Heisenberg group*, where $H_3(R)$ denotes the set of matrices

$$A_{x,y,z} := \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \tag{8.2}$$

with coefficients $x, y, z$ in the ring $R$. A covering with group $\Gamma$ is given e.g. by $X := H_3(\mathbb{R})$ with compact quotient $M := H_3(\mathbb{R})/H_3(\mathbb{Z})$. Note that $X$ is diffeomorphic to $\mathbb{R}^3$. Clearly, $\Gamma$ is a finitely generated linear group and therefore residually finite (cf. Example 6.3 (iii)). Now, by Theorem 6.8 one can deform conformally a $\Gamma$-invariant metric $g$ as in Case (B) of Section 3, such that $\operatorname{spec} \Delta_X$ has at least $n$ spectral gaps, $n \in \mathbb{N}$.

In this case, $\Gamma$ is also amenable as an extension of amenable groups (cf. Remark 5.3). In fact, $\Gamma$ is isomorphic to the semi-direct product $\mathbb{Z} \ltimes \mathbb{Z}^2$, where $1 \in \mathbb{Z}$ acts on $\mathbb{Z}^2$ by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Therefore, we have equality in the characterisation of $\operatorname{spec} \Delta_X$ in Theorems 4.5 and 6.6.

Note finally that the group $\Gamma$ is not of type I since $\Gamma_{\text{fcc}} = \{\, A_{0,y,0} \,|\, y \in \mathbb{Z} \,\}$ has infinite index in $\Gamma$ (cf. Remark 5.2 (2)). Thus, our method in Section 5 does not apply since the measure $\mathrm{d}z$ in (4.6) is supported only on infinite-dimensional Hilbert spaces. Curiously, one can construct a *finitely* additive measure on the group dual $\widehat{\Gamma}$ supported by the set of finite-dimensional representations of $\widehat{\Gamma}$ (cf. [Py79]). The group dual $\widehat{\Gamma}$ is calculated e.g. in [Kan68, Beispiel 1].

*Example* 8.5 (Free groups). Let $\Gamma = \mathbb{Z}^{*r}$ be the free group with $r > 1$ generators. Then $\Gamma$ is residually finite (recall Example 6.3 (ii)) and has positive Kadison constant (cf. [S92, Appendix]). Therefore, Theorem 7.5 applies to the $\Gamma$-coverings $X \to M$ specified in Section 3.

Note that $\Gamma$ is not of type I since $\Gamma_{\text{fcc}} = \{e\}$ (cf. Remark 5.2 (2)). Such groups are called *ICC (infinite conjugacy class) groups*. Again, for any direct integral decomposition (4.6), almost all Hilbert spaces $\mathcal{H}(z)$ are infinite-dimensional. Finally, $\Gamma$ is not amenable.

8.3. **An example with an amenable, non-residually finite group.** Kirchberg mentioned in [Ki94, Section 5] an interesting example of a finitely generated *amenable* group which is not residually finite: Denote by $S_0$ the group of permutations of $\mathbb{Z}$ which leave unpermuted all but a finite number of integers. We call $A_0$ the normal subgroup of even permutations in $S_0$. Let $\mathbb{Z}$ act on $S_0$ as shift operator. Then the semi-direct product $\Gamma := \mathbb{Z} \ltimes S_0$ is (finitely) generated by the shift $n \mapsto n+1$ and the transposition interchanging 0 and 1. Note that $\Gamma$ and $S_0$ are ICC groups.

**Lemma 8.6.** *The group $\Gamma$ is amenable. Moreover, $\mathfrak{R}_{\mathcal{F}}(\Gamma) = A_0$, hence $\Gamma$ is not residually finite.*

*Proof.* The group $S_0$ is amenable as inductive limit of amenable groups; therefore, $\Gamma$ is amenable as semi-direct product of amenable groups (cf. [Day57, Section 4]).

The equality $\mathfrak{R}_{\mathcal{F}}(\Gamma) = A_0$ follows from the fact that $A_0$ is simple. □

**Proposition 8.7.** *Every finite-dimensional unitary representation $\rho$ of $\Gamma$ leaves $A_0$ elementwise invariant, i.e. $\rho(\gamma) = \mathbb{1}$ for all $\gamma \in A_0$.*

*Proof.* Let $\mathcal{E}$ be the class of countable subgroups of $\mathrm{U}(n)$, $n \in \mathbb{N}$, and $\mathcal{FG}$ the class of finitely generated groups. Note that $\mathcal{F} \subset \mathcal{E} \cap \mathcal{FG}$ and that finitely generated linear groups are residually finite (cf. Example 6.3 (iii)), i.e. $\mathcal{E} \cap \mathcal{FG} \subset \mathfrak{R}\mathcal{F}$. Arguing as in the proof of Lemma 8.2 we obtain from the inclusions $\mathcal{F} \subset \mathcal{E} \cap \mathcal{FG} \subset \mathfrak{R}\mathcal{F}$ that $\mathfrak{R}_{\mathcal{E}\cap\mathcal{FG}}(\Gamma) = \mathfrak{R}_{\mathcal{F}}(\Gamma)$. Now by Lemma 8.6 the $\mathcal{F}$-residual of $\Gamma$ is $A_0$. Finally, since $\Gamma$ itself is finitely generated (i.e. $\Gamma \in \mathcal{FG}$), we have

$$\mathfrak{R}_{\mathcal{E}}(\Gamma) = \mathfrak{R}_{\mathcal{E}\cap\mathcal{FG}}(\Gamma) = A_0.$$

This concludes the proof since $\rho$ is a finite-dimensional unitary representation iff $\mathrm{im}(\rho) \cong \Gamma/\ker\rho \in \mathcal{E}$, i.e. $\mathfrak{R}_{\mathcal{E}}(\Gamma)$ is the intersection of all $\ker\rho$, where $\rho$ are the finite-dimensional, unitary representations of $\Gamma$. □

In conclusion, we cannot analyse the spectrum of $\Delta_X$ by none of the above methods since $\Gamma$ is not residually finite (and therefore neither of type I). Nevertheless, equality holds in (4.10), but we would need infinite-dimensional Hilbert spaces $\mathcal{H}(z)$ in the direct integral decomposition in order to describe the spectrum of the whole covering $X \to M$ and not only of the subcovering $X/A_0 \to M$ (with covering group $\mathbb{Z} \times \mathbb{Z}_2$, cf. Diagram (6.1)).

*Remark* 8.8. Coverings with transformation groups as in the present subsection cannot be treated with the methods developed in this paper. It seems though reasonable that even for non-residually finite groups the construction specified in Section 3 still produces at least $n$ spectral gaps, $n \in \mathbb{N}$. To show this one needs to replace the techniques of Section 4 that use the min-max principle in order to prove the existence of spectral gaps for these types of covering manifolds.

## 9. Conclusions and applications

Given a Riemannian covering $(X, g) \to (M, g)$ with a residually finite transformation group $\Gamma$ we constructed a deformed $\Gamma$-covering $(X, g_\varepsilon) \to (M, g_\varepsilon)$ such that $\operatorname{spec}\Delta_{(X,g_\varepsilon)}$ has $n$ spectral gaps, $n \in \mathbb{N}$. Intuitively one decouples neighbouring fundamental domains by deforming the metric $g \to g_\varepsilon$ in such a way that the junctions of the fundamental domains are scaled down (cf. Figure 1). Therefore, our construction may serve as a model of how to use geometry to remove unwanted frequencies or energies in certain situations which may be relevant for technological applications.

For instance, the Laplacian on $(X, g_\varepsilon)$ may serve to give an approximate description of the energy operator of a quantum mechanical particle moving along the periodic space $X$. Usually, the energy operator contains additional potential terms coming form the curvature of the embedding in some ambient space, cf. [FH00], but, nevertheless, $\Delta_{(X,g_\varepsilon)}$ is still a good approximation for describing properties of the particle. A spectral gap in this context is related to the transport properties of the particle in the periodic medium, e.g., an insulator has a large first spectral gap.

Another application are photonic crystals, i.e. optical materials that allow only certain frequencies to propagate. Usually, one has to consider differential forms in order to describe the propagation of classical electromagnetic waves in a medium. Nevertheless, if we assume that the Riemannian density is related to the dielectric constant of the material, one can use the scalar Laplacian on a manifold as a simplified model. For more details, we refer to [K01, FK98] and the references therein.

A further interesting line of research would be to consider the opposite situation as in the present paper; that means the use of geometry to prevent the appearance of spectral gaps (cf. [Fr91, Maz91]). In fact, these authors proved that $\lambda_{k+1}^-(D) \le \lambda_k^+(D)$ for all $k \in \mathbb{N}$, i.e, that $I_k \cap I_{k+1} \neq \emptyset$ for all $k \in \mathbb{N}$ provided $D$ is an open subset of $\mathbb{R}^n$ or a Riemannian symmetric space of non-compact type. On such a space, we have a priory no information on the existence of gaps.

It would also be interesting to connect the number of gaps with geometric quantities, e.g., isoperimetric constants or the curvature. We want to stress that the curvature of $(X, g_\varepsilon)$ is *not* bounded as $\varepsilon \to 0$ (cf. [P03]) in contrast to the degeneration of Riemannian metrics under curvature bounds (cf. e.g. [Ch01]).

In the present paper we have considered $\Delta_X$ as a prototype of an elliptic operator and have avoided the use of a potential $V$. In this way we isolate the effect of geometry on $\operatorname{spec} \Delta_X$. Of course, our methods and results may also be extended to more general periodic structures that have a "reasonable" Neumann Laplacian as a lower bound and satisfy the spectral "localisation" result in Theorem 4.3. For example, one can also study periodic operators like $\Delta_X + V$, operators on quantum wave guides, more general periodic elliptic operators or operators on metric graphs (cf. e.g. [EP05] for examples of periodic metric graphs with spectral gaps).

Finally, we conclude mentioning that we can not apply directly our result to disprove the Bethe-Sommerfeld conjecture on manifolds, which says that the number of spectral gaps for a periodic operator in dimensions $d \ge 2$ remains *finite*. Even if we know that the spectrum of the Laplacian on $(X, g_\varepsilon)$ converges to the discrete set $\{ \lambda_k \mid k \in \mathbb{N} \}$ as $\varepsilon \to 0$, we cannot expect a *uniform* control of the spectral convergence on the whole interval $[0, \infty)$ since there are topological obstructions (cf. [ChF81]). Note that a uniform convergence would immediately imply that $\operatorname{spec} \Delta_{(X,g_\varepsilon)}$ would have an *infinite* number of spectral gaps. Nevertheless, we hope that our construction will contribute to the clarification of the status of this conjecture.

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