Convergence results for thick graphs

Olaf Post

November 14, 2010

Many physical systems have branching structure of thin transversal diameter. One can name for instance quantum wire circuits, thin branching waveguides, or carbon nano-structures. In applications, such systems are often approximated by the underlying one-dimensional graph structure, a so-called "quantum graph". In this way, many properties of the system like conductance can be calculated easier (sometimes even explicitly). We give an overview of convergence results obtained so far, such as convergence of Schrödinger operators, Laplacians and their spectra.

1 Introduction

In this note, we give an overview on convergence results of Laplace-like operators on shrinking tubular neighbourhoods of a metric graph. We start with defining the notion "graph-like manifold" and "thick graph", as well as the associated Laplace-like operators on these spaces. We also review some applications in Physics and Mathematics in Section 2. The current state of art of quantum graph models is described in the recent proceedings volume [11] to which we refer for an extensive bibliography. Section 3 is devoted to convergence results for the Neumann Lalacian as well as a general convergence scheme for operators acting in different Hilbert spaces. Section 4 contains results for the Dirichlet Laplacian on thick graphs, as well as operators on thick graphs converging to delta-couplings on the underlying metric graph. Finally, in Section 5 we comment on some work in progress and open problems.

2 Thick graphs

Roughly speaking, a *thick graph* is a family of neighbourhoods $\{X_\varepsilon\}_{\varepsilon>0}$ of a metric graph X_0 embedded in $\mathbb{R}^d$, which shrinks to X_0 if $\varepsilon \to 0$. Sometimes, we also refer to a single member X_ε of the family $\{X_\varepsilon\}_\varepsilon$ for a suitably small $\varepsilon > 0$ as a thick graph. We give a more formal definition below.

Thick graphs have a lot of different names in the literature, basically due to the intended application. Thick graphs are also called *fat graphs* (cf. [9]), *mesoscopic systems collapsing onto a graph* (cf. [23]), *graph neighbourhoods* (cf. [18]), *graph-like (thin) manifolds* (cf. [12], *quasi-one-dimensional spaces* (cf. [31]), *thin branched (quantum) waveguides* (cf. [13]) or *quantum networks modelled by graphs* (cf. [15]).

2.1 Definition of a thick graph

2.1.1 Metric and quantum graphs

We give here a brief outline of the concept of metric and quantum graphs. We refer to [22] for more details and further references.

Assume that X_0 is a *metric graph*, i.e., a topological graph X_0 with vertices V and edges E such that each edge $e \in E$ is associated a *length* $\ell_e > 0$. In this way, we can identify an edge e with the interval $I_e := [0, \ell_e]$ and the adjacent initial and terminal vertices $\partial_- e \in V$ and $\partial_+ e \in V$ with $0 \in I_e$ and $\ell_e \in I_e$. Note that we can view $s \in I_e$ as a *coordinate* on the edge e, which introduces an orientation on the graph X_0. Moreover, the coordinate allows to *integrate* and *differentiate* a function on the edge. We also allow edges of infinite length (so-called *infinite leads*), this edge is assumed to have only one adjacent initial vertex $\partial_- e \in I_e = [0, \infty)$.

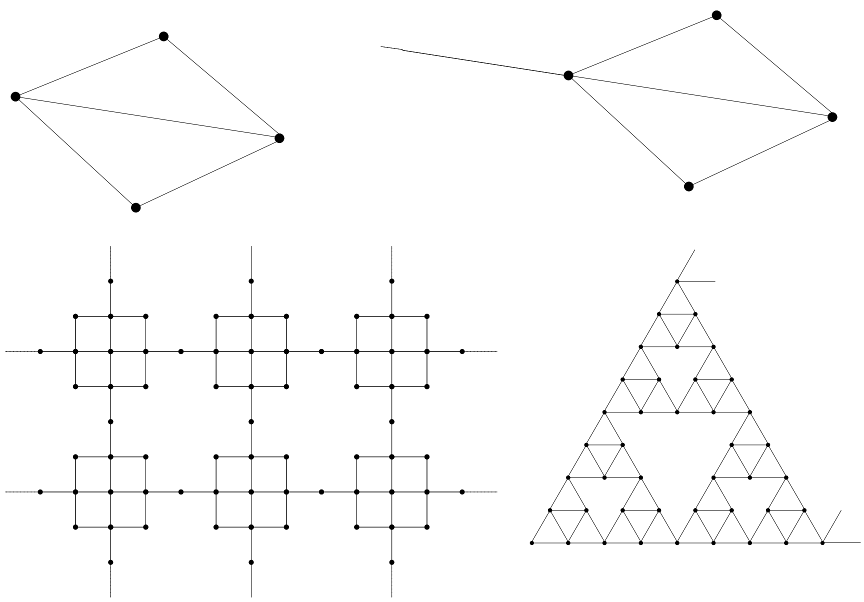

Figure 1: Four examples of metric graphs: a *compact* one, a *non-compact* one with compact interior part and one infinite lead, a non-compact $\mathbb{Z}^2$-periodic metric graph and a self-similar non-compact example, the *Sierpiński* graph.

Thus, the topological graph X_0 can be turned into a *metric measure* space, by defining the distance of two points $x, y \in X_0$ to be the shortest distance of all Lipschitz continuous paths joining x and y, where the length of a path is defined in the obvious way. The measure on X_0 is determined by the Lebesque measure on each edge I_e.

Associated to a metric graph, we have a natural Hilbert space, namely

$$\mathscr{H}_0 := \mathsf{L}_2(X_0) := \bigoplus_e \mathsf{L}_2(I_e). \tag{2.1}$$

Moreover, we can naturally define differential operators, like e.g. a Laplace-type operator $(\Delta f)_e = -f_e''$ for a function $f = \{f_e\}_e \in \mathsf{H}^2_{\max}(X_0)$, where $\mathsf{H}^k_{\max}(X_0) := \bigoplus_e \mathsf{H}^k(I_e)$. In order to turn Δ into a self-adjoint operator, we have to fix vertex conditions on the boundary values

$$f_e(v) := \begin{cases} f_e(0), & v = \partial_- e \\ f_e(\ell_e), & v = \partial_+ e \end{cases} \quad \text{and} \quad f_e'(v) := \begin{cases} -f_e'(0), & v = \partial_- e \\ f_e'(\ell_e), & v = \partial_+ e \end{cases} \tag{2.2}$$

of the function at a vertex v and its adjacent edges $e \in E_v$, the *neighbouring edges of* v. A prominent example is given by the so-called *free* or *Kirchhoff* vertex conditions

$$f_{e_1}(v) = f_{e_2}(v) \quad \forall e_1, e_2 \in E_v \quad \text{and} \quad \sum_{e \in E_v} f'_e(v) = 0. \tag{2.3}$$

The first condition in (2.3) is referred to as *continuity* of the function f viewed as function on the topological space X_0, the second is a flux condition on the derivative viewed as vector field on X_0. The *free* or *Kirchhoff Laplacian* Δ_{X_0} on the metric graph X_0 is now the operator acting as $(\Delta f)_e = -f''_e$ for functions $f \in \mathsf{H}^2_{\max}(X_0)$ fulfilling (2.3). We will see in a moment that if we have a uniform lower *positive* bound on the edge length, i.e.,

$$\inf_{e \in E} \ell_e > 0, \tag{2.4}$$

then Δ_{X_0} is self-adjoint. We will give further examples of self-adjoint vertex conditions in Section 4.3.

Let us remark that the Kirchhoff Laplacian Δ_{X_0} is associated with the quadratic form

$$\mathfrak{d}_{X_0}(f) := \sum_e \int_0^{\ell_e} |f'_e|^2 \, \mathrm{d}s, \quad \operatorname{dom} \mathfrak{d}_{X_0} := \mathsf{H}^1(X_0), \tag{2.5}$$

where $\mathsf{H}^1(X_0)$ is the subspace of those functions $f \in \mathsf{H}^1_{\max}(X_0)$ such that f is continuous at each vertex. It follows from (2.4) that $\mathsf{H}^1(X_0)$ is a *closed* subspace in $\mathsf{H}^1_{\max}(X_0)$, and that $\mathfrak{d}_{X_0}$ is a *closed* non-negative quadratic form. Moreover, the associated operator is precisely Δ_{X_0}, which shows in particular that Δ_{X_0} is self-adjoint (see [22] for details and further references).

A *quantum graph* is a metric graph X_0 together with a self-adjoint differential operator H_0 acting on X_0. The most prominent example is a metric graph X_0 together with its Kirchhoff Laplacian Δ_{X_0} just defined.

2.1.2 Graph-like manifolds and thick graphs

Let us now give an abstract definition of — what we call in this review — a *graph-like manifold*. Let $\varepsilon > 0$ and let X_0 be a metric graph.

A *graph-like manifold* (associated to X_0) is a family of d-dimensional manifolds X_ε ($d \geq 2$) which can be decomposed into

$$X_\varepsilon = \bigcup_{e \in E} X_{\varepsilon,e} \cup \bigcup_{v \in V} X_{\varepsilon,v} \tag{2.6}$$

such that the closed sets $X_{\varepsilon,e}$ and $X_{\varepsilon,v}$ are disjoint or intersect only in submanifolds of dimension $d-1$. The so-called *edge* and *vertex neighbourhoods* $X_{\varepsilon,e}$ and $X_{\varepsilon,v}$ are supposed to have the following structure (cf. Figure 2):

- The edge neighbourhood $X_{\varepsilon,e}$ is a cylinder, i.e., $X_{\varepsilon,e} := I_e \times \varepsilon Y_e$, where Y_e is a compact Riemannian manifold (with or without boundary) with metric h_e, called *transversal manifold*, and where εY_e denotes the *ε-homothetically scaled* Riemannian manifold, i.e., the manifold Y_e with metric $h_{\varepsilon,e} := \varepsilon^2 h_e$. In particular, X_ε carries the metric $g_e = \mathrm{d}s^2 + \varepsilon^2 h_e$.

- The vertex neighbourhood $X_{\varepsilon,v}$ is ε-homothetic to a fixed Riemannian manifold X_v with metric g_v, i.e., $X_{\varepsilon,v}$ carries the metric $g_{\varepsilon,v} = \varepsilon^2 g_v$. Moreover, we assume that the boundary ∂X_v of X_v contains a subset $\mathring{\partial} X_v$ which is isometric to the disjoint union of Y_e, e adjacent to v.

Let us give now an important example, which is a graph-like manifold in the sense above only in an *approximate* sense: Let $\widetilde{X}_\varepsilon$ denote the (closed) ε-neighbourhood of a metric graph X_0 embedded in $\mathbb{R}^d$ (such that the edges in a vertex meet non-tangentially), then a decomposition similar to (2.6) yields

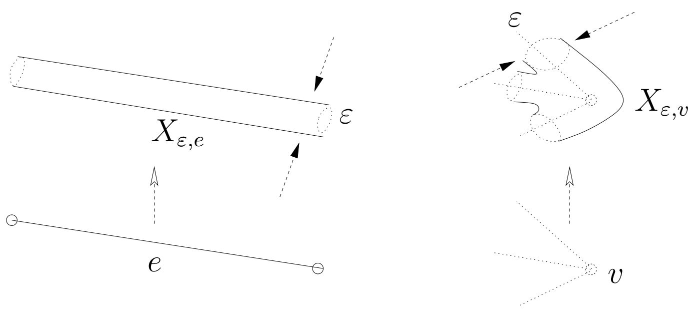

Figure 2: An edge and an vertex neighbourhood associated to an edge e and a vertex v with three adjacent edges.

edge neighbourhoods $\widetilde{X}_{\varepsilon,e}$ with are only *approximate* cylinders of length ℓ_e, and vertex neighbourhoods $\widetilde{X}_{\varepsilon,v}$, which are only *approximately* ε-homothetic to a fixed manifold X_v.

First, this is due to the fact that the vertex neighbourhood needs some space, so that the length of the cylinder is only $\ell_e - 2\varepsilon$. Second, the embedded edge need not to be straight. In both cases, one can show that the error made by introducing the coordinates $(s, y) \in I_e \times Y_e$ on the approximate cylinder $\widetilde{X}_{\varepsilon,e}$ yields a metric $\widetilde{g}_{\varepsilon,e}$ which is *close* to $g_{\varepsilon,e}$ up to some ε-depending errors. We call a space $\widetilde{X}_\varepsilon$ which is a graph-like manifold only up to small ε-depending errors a *thick graph.* A more detailed discussion of these errors can be found e.g. in [34, Secs. 5.3–5.6 and 6.7]. For simplicity, we call graph-like manifolds also *thick graphs.*

At first sight, the definition of a graph-like manifold looks pretty abstract in comparison with the concrete definition of a thick graph. The main reason for using the spaces $X_{\varepsilon,e}$ and $X_{\varepsilon,v}$ is to have ε*-independent* coordinates $(s, y) \in X_e = I_e \times Y_e$ and $x \in X_v$, and to put the ε-dependence only in the *metric* of the Riemannian manifold. This is a significant simplification in the reduction to a graph model; the particular error estimates coming from a concrete embedding of the metric graph X_0 into some ambient space do not enter into this reduction step.

For other shrinking behaviour at the vertices, we refer to [24, 12, 34].

2.1.3 Operators on thick graphs

On a thick graph, we typically consider a Laplace-like operator, e.g., the Neumann-Laplacian (if X_ε has boundary) or the Laplacian on X_ε (if X_ε has no boundary) defined via its quadratic form

$$\mathfrak{d}_{X_\varepsilon}(u) := \int_{X_\varepsilon} |\mathrm{d}u|^2_{g_\varepsilon}, \quad \mathscr{H}_\varepsilon := \mathsf{H}^1(X_\varepsilon), \tag{2.7}$$

in the Hilbert space $\mathscr{H} := \mathsf{L}_2(X_\varepsilon)$. Note that the Neumann boundary condition $\partial_{\mathrm{n}} u = 0$ on ∂X_ε only enters in the corresponding *operator* domain via a partial integration formula.

We will see below that the Neumann case and the Laplacian on a manifold without boundary can be treated in the same way. The main reason for this fact is that on the transversal manifolds Y_e, the lowest eigenfunction is *constant* in both cases with corresponding eigenvalue 0.

If $\partial X_\varepsilon \neq \emptyset$, then we also consider the Dirichlet-Laplacian $\Delta^{\mathrm{D}}_{X_\varepsilon}$ on X_ε defined via the quadratic form $\mathring{\mathfrak{d}}_{X_\varepsilon}$ defined as above, but with domain $\operatorname{dom} \mathring{\mathfrak{d}}_{X_\varepsilon} := \mathring{\mathsf{H}}^1(X_\varepsilon)$, the closure of $\mathsf{C}^\infty_{\mathrm{c}}(X_\varepsilon)$ of the space of smooth functions with compact support *away* from ∂X_ε in $\mathsf{H}^1(X_\varepsilon)$.

2.2 Examples of thick graphs

The thick graph X_ε may have boundary or not, depending on whether the transversal manifolds Y_e have boundary or not.

2.2.1 An abstract example

We can construct a graph-like manifold X_ε according to a given metric graph X_0 by associating appropriate manifolds Y_e and X_v to each edge e and vertex v as in Section 2.1.2. It is not difficult to see that one can define a globally smooth metric g_ε on the underlying manifold such that the decomposition (2.6) holds with $X_{\varepsilon,e} = I_e \times \varepsilon Y_e$ and $X_{\varepsilon,v} = \varepsilon X_v$ (see also Figure 2).

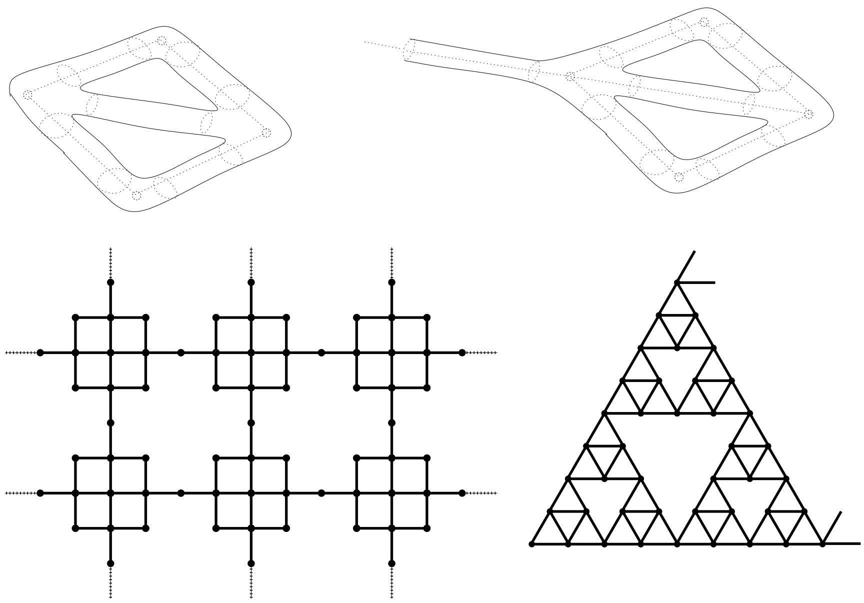

Figure 3: Four examples of thick graphs: the first two are examples of thick graphs, viewed either as 2-dimensional manifold without boundary (the surface of the pipeline network) or as 3-dimensional manifold with boundary. The examples in the second row correspond to the periodic and self-similar metric graphs of Figure 1.

2.2.2 Examples with boundary from embedded graphs

Let X_0 be a metric graph embedded in $\mathbb{R}^d$ such that the angle between two edges meet in a vertex always with a non-zero angle. Let $\widetilde{X}_\varepsilon$ be the ε-neighbourhood, then $\widetilde{X}_\varepsilon$ is close to a thick graph X_ε as discussed above. This example has boundary, and corresponds to the case with transversal manifold Y_e being a ball in $\mathbb{R}^d$. In this situation, the boundary ∂X_ε may have corners (but with non-zero angle). This does not bother us, inasmuch as we can define a Neumann Laplacian with compact resolvent (if X_ε is compact).

2.2.3 Examples without boundary

If we choose $\widetilde{X}_\varepsilon$ to be (a smoothed version of) the boundary of the ε-neighbourhood of an embedded graph in $\mathbb{R}^{d+1}$ as above, then we obtain an example which is close to a thick graph X_ε (a d-dimensional manifold without boundary), with transversal manifold being a $(d-1)$-dimensional in $\mathbb{R}^d$ (see also Figure 3).

2.3 Appearance of thick graphs in Physics and Mathematics

2.3.1 Physical models

Possibly the first time thick graphs appeared is in [36], where Ruedenberg and Scherr used thick graphs as justification of quantum graph models for the spectra of aromatic hydrocarbons. Although the justification is mathematically not correct (the limit of a shifted Dirichlet Laplacian on a thick graph is generally *not* the Kirchhoff Laplacian, see Section 4), the quantum graph models lead to a good approximation of the spectra.

Since quantum graphs are generally believed to provide good models for electronic and optical nano-structures, a natural question arises:

Is the quantum graph model a good approximation of a physically more realistic system with finite, but non-vanishing thickness $\varepsilon > 0$?

Moreover, one is interested in modelling devices with certain properties like a quantum switch modelled by a certain vertex condition in a quantum graph.

Is it possible to find a thick graph model converging to a prescribed quantum graph vertex coupling?

On a quantum graph, many physical properties like the conductance or existence of bound states can be calculated explicitly. Such models are called *solvable models*, since mathematically, the calculation on a quantum graph mostly reduces to a system of coupled ODEs. Note that the conductance of a periodic medium (a periodic semi-conductor or a photonic crystal transmitting light) is guaranteed if the spectrum of the corresponding operator has *band structure* and is absolutely continuous. If the energy of a particle lies in such a band, then it can "travel" through the medium; if the energy lies outside the bands (i.e., in a *spectral gap*, then no transport is possible.

Thick graph models are also used in other areas; e.g. thick graphs as models for proteins have been analysed recently in [29].

2.3.2 Thick graphs in Mathematics: Spectral geometry

In Spectral Geometry, one investigates relations of the spectrum of the Laplacian (or related operators) on a Riemannian manifold to its geometry. Graph-like manifolds may serve as toy models in order to show certain properties, or to disprove a conjecture. Maybe the first mathematical treatment of convergence results for thick graphs is provided by Colin de Verdière [7]:

Theorem 2.8. *Given a compact oriented manifold M of dimension $d \geq 3$ without boundary and a natural number $n \geq 1$, then there exists a metric g_n such that the first non-zero eigenvalue of the associated Laplacian has multiplicity n.*

In dimension 2, the multiplicity of the non-zero eigenvalues is bounded from above by the genus of the surface (see [4]). If $d \geq 3$, Colin de Verdière embeds a complete metric graph X_0 with $n+1$ vertices in M. Such an embedding is possible, since $\dim M \geq 3$. Then he deforms a given metric g on M into a family of metrics $\{g_\varepsilon\}$ such that g_ε equals g on a small ε–neighbourhood X_ε of X_0 and which is small outside. He then shows that the eigenvalues of $\Delta_{(M,g_\varepsilon)}$ are close to the eigenvalues of the Neumann Laplacian on $(X_\varepsilon, g_\varepsilon)$. In a second step, it can be seen that these Neumann eigenvalues converge to the eigenvalues of the Kirchhoff Laplacian on X_0, using methods discussed below (cf. Theorem 3.2). If all lengths of the metric graph are the same, then the first non-zero eigenvalue of the Kirchhoff Laplacian has the desired multiplicity n. The use of enough parameters (the lengths of the edges in the metric graph X_0) allows to find a path in this parameter space such that the multiplicity is preserved. A similar construction is used in [8] in order to show the following more general result: Let $d \geq 3$, and let $\lambda_1 = 0 < \lambda_2 \leq \cdots \leq \lambda_n$ be a sequence of n numbers. Then there exists a metric g such that the corresponding Laplacian has $\lambda_1, \ldots, \lambda_n$ as its first n eigenvalues.

2.3.3 Thick graphs in Mathematics: Global analysis

The heat kernel of a Riemannian manifold X is the smallest positive fundamental solution to the heat equation $-\partial_t u = \Delta_X u$ (recall that $\Delta_X \geq 0$).

On a complete Riemannian manifold X with non-negative Ricci curvature, Li and Yau [25] showed that the heat kernel $p_t(x,y)$ has the asymptotic behaviour (2.9) with $\beta = 2$ for all times t, i.e., the heat kernel behaves very similar as the heat kernel of the Laplacian on $X = \mathbb{R}^d$, namely

$$p_t(x,y) = \frac{1}{(4\pi t)^{d/2}} \exp\Big(-\frac{|x-y|^2}{4t}\Big)$$

for $t > 0$ and $x, y \in \mathbb{R}^d$. In contrast, on self-similar graphs X_0 (like the Sierpiński graph, see Figure 1), in general, a different asymptotic behaviour occurs, e.g.,

$$p_t(x,y) \sim \frac{1}{\operatorname{vol} B_x(\sqrt{t})} \exp\Big(-\frac{d(x,y)^\beta}{ct}\Big) \tag{2.9}$$

for some $c > 0$ and $\beta = \log 5/\log 2 > 2$ for the Sierpiński graph, where $d(x,y)$ denotes the geodesic distance between the points $x, y \in X$ and $\operatorname{vol} B_x(r)$ denotes the volume of a geodesic ball $B_x(r)$.

Up to recent time it was believed that Gaussian estimates with $\beta > 2$ are typical only for such self-similar spaces Surprisingly, one can construct a fractal-like Riemannian manifold X according to the metric graph X_0 having the Gaussian estimate with $\beta = \log 5/\log 2 > 2$ for *large* times t, and the classical Gaussian estimate $\beta = 2$ for *short* times (see [2, 1] and references therein).

From a probabilistic point of view, this behaviour can be understand as follows: $p_t(x,y)$ is the probability density that a particle starting at the point x is at the point y in time t. A particle moving on a fractal-like manifold sees the smooth structure for short times, but for large times, the fractal nature becomes apparent.

3 Neumann Laplacians on thick graphs

Let X_ε be a thick graph constructed from the building blocks $X_{\varepsilon,v} = \varepsilon X_v$ and $X_{\varepsilon,e} = I_e \times \varepsilon Y_e$ with transversal manifolds εY_e. Let H_ε be the Laplacian on X_ε (in our notation, $H_\varepsilon \geq 0$) associated to the quadratic form $\mathfrak{d}_{X_\varepsilon}$, cf. (2.7). If $\partial X_\varepsilon \neq \emptyset$ we assume Neumann boundary conditions.

On the limit space, the metric graph, we consider a *weighted* Kirchhoff Laplacian, namely, $(H_0 f)_e = -f_e''$ for $f \in \mathsf{H}^2_{\max}(X_0)$ fulfilling

$$f \text{ continuous}, \qquad \sum_{e \in E_v} (\operatorname{vol}_{d-1} Y_e) f_e'(v) = 0. \tag{3.1}$$

3.1 Convergence results for Neumann Laplacians

Let us first assume that the thick graph and the metric graph are compact. In this case, H_ε and H_0 have purely discrete spectrum, denoted by $\lambda_k(H_\varepsilon)$ and $\lambda_k(H_0)$, written in increasing order and repeated according to their multiplicity.

The following convergence result on the discrete spectrum shows that the Kirchhoff Laplacian on the metric graph is natural in the sense that it occurs as a limit of an ε-neighbourhood of the graph. It was proven for the first time by Colin de Verdière, where he showed Theorem 2.8 above. Since this convergence result is used as a technical step only and presented in a brief way, the paper seemed to be overlooked in much of the mathematical physics community until recently. Later on, Rubinstein-Schatzman [35] proved it in a concrete embedded situation, and Kuchment-Zeng [23] simplified some arguments. In [12] we introduced *graph-like manifolds* and stressed the geometric point of view of the analysis.

Theorem 3.2. *Let X_ε be a compact thick graph with underlying compact metric graph X_0, and let H_ε be the (Neumann-)Laplacian on X_ε, and let H_0 be the weighted Kirchhoff Laplacian on X_0, then*

$$\lambda_k(H_\varepsilon) - \lambda_k(H_0) = \mathrm{O}(\varepsilon^{1/2}) \qquad \text{as } \varepsilon \to 0.$$

Idea of the proof: The proof uses a variational characterisation of eigenvalues (the *Min-max principle*) and identification operators for quadratic form domains $J^1 : \mathsf{H}^1(X_0) \longrightarrow \mathsf{H}^1(X_\varepsilon)$ and $J'^1 : \mathsf{H}^1(X_\varepsilon) \longrightarrow \mathsf{H}^1(X_0)$. The main step in the proof is then to compare the Rayleigh quotients

$$\frac{\|f'\|^2_{\mathsf{L}_2(X_0)}}{\|f\|^2_{\mathsf{L}_2(X_0)}} \qquad \text{and} \qquad \frac{\|\mathrm{d}u\|^2_{\mathsf{L}_2(X_\varepsilon)}}{\|u\|^2_{\mathsf{L}_2(X_\varepsilon)}}. \qquad \square$$

In all of the above-cited papers, the underlying spaces are assumed to be compact, and therefore, only the discrete spectrum was considered, and the spectral convergence does not (directly) imply the convergence of eigenfunctions. We introduce the following notion of *convergence of operators acting in different Hilbert spaces*, developed (to our knowledge) for the first time in [31]; implying in particular the convergence of the discrete and essential spectrum for non-compact graph-like spaces (in the fast decaying case).

Definition 3.3. For each $\varepsilon \ge 0$, let H_ε be a non-negative operator acting in a Hilbert space $\mathscr{H}_\varepsilon$. We say that $H_\varepsilon \xrightarrow{\text{gnr}} H_0$ *in the generalised norm resolvent sense* of order $\mathrm{O}(\varepsilon^{1/2})$ iff there is a bounded operator $J : \mathscr{H}_0 \longrightarrow \mathscr{H}_\varepsilon$ such that

$$J^*J = \mathrm{id}_0, \quad \|(\mathrm{id}_\varepsilon - JJ^*)R_\varepsilon\| = \mathrm{O}(\varepsilon^{1/2}) \quad \text{and} \quad \|JR_0 - R_\varepsilon J\| = \mathrm{O}(\varepsilon^{1/2}),$$

where $R_\varepsilon := (H_\varepsilon + 1)^{-1}$ denotes the resolvent for $\varepsilon \ge 0$.

This is not the most general condition, more details can be found in [34].

For the following result, we need some uniformity conditions on the metric and thick graph: We say that a metric graph X_0 is *uniform* iff there is a positive lower bound on the edge lengths, cf. (2.4).[1] We say that a graph-like manifold X_ε is *uniform* iff

$$\inf_{e \in E} \lambda_2^{\mathrm{N}}(Y_e) > 0, \quad \inf_{v \in V} \lambda_2^{\mathrm{N}}(X_v) > 0 \quad \text{and} \quad \sup_{v \in V} \frac{\mathrm{vol}_d X_v}{\mathrm{vol}_{d-1} \mathring{\partial} X_v} < \infty. \tag{3.4}$$

Here, $\lambda_2^{\mathrm{N}}(M)$ denotes the second (first non-zero) eigenvalue of the Neumann Laplacian on the manifold M. Recall that $\mathring{\partial} X_v$ is the part of the boundary of X_v where the edge neighbourhoods are attached.

For a thick graph (in our notation, a graph-like manifold up to some error terms), one needs in general more assumptions on the embedding, e.g., one needs a lower bound on the angles of two adjacent edges at a vertex, and upper bounds on the curvature of an edge embedded in $\mathbb{R}^d$, cf. [31] and [34, Sec. 6.7].

The following result was first proven in [31], see also [34]:

Theorem 3.5. *Let X_ε be a uniform thick graph with underlying uniform metric graph X_0, and let H_ε be the (Neumann-)Laplacian on X_ε, and let H_0 be the weighted Kirchhoff Laplacian on X_0 defined in (3.1), then $H_\varepsilon \xrightarrow{\text{gnr}} H_0$ of order $\mathrm{O}(\varepsilon^{1/2})$. Moreover, the error depends only on the bounds in (2.4) and (3.4).*

Idea of the proof. Let us motivate why a condition like $\|(\mathrm{id}_\varepsilon - JJ^*)R_\varepsilon\| = \mathrm{O}(\varepsilon^{1/2})$ should be true:

[1] In [31], we assumed additionally that the graph has uniformly bounded vertex degrees. Actually, this is not needed, cf. [34].

- JJ^* is the projection onto *transversally constant* functions, and functions vanishing on vertex neighbourhoods $X_{\varepsilon,v}$;
- functions in $(\operatorname{ran} JJ^*)^\perp$ have high energy (spectral parameter of H_ε);
- the resolvent "cuts off" high energies.

In other words: eigenfunctions u_ε of H_ε with bounded eigenvalues $\lambda_\varepsilon \le \text{const}$ *do not concentrate* on $X_{\varepsilon,v}$ and are almost *transversally constant*. Actually, the arguments for a rigorous proof of $H_\varepsilon \xrightarrow{\text{gnr}} H_0$ are very similar to the arguments for the proof of Theorem 3.2. □

Freidlin and Wentzell consider the problem from a probabilistic point of view in [16]. They show that a suitable Markov process on a thin graph neighbourhood converges to a Markov process on the metric graph. In essence, they prove strong resolvent convergence of the Laplacian with Neumann boundary conditions on the graph neighbourhood to a Laplace-type operator on the metric graph. A similar result for tree graphs is proven by Saito in [37].

Results for certain classes of *compact* manifolds converging in the Gromov-Hausdorff distance are given in the works of Kasue [20, 21] (see also the references therein and [17, 3] for related results); in particular, the convergence of the discrete spectrum and *strong* convergence of resolvents is shown. Typically, these results need some uniform curvature bounds, which are in general not fulfilled for a family $\{X_\varepsilon\}_\varepsilon$ of graph-like manifolds, and imply only *strong* resolvent convergence.

For the convergence of resonances, we refer to [13] and the survey article [14].

3.2 Convergence of operators in different Hilbert spaces

Let us comment on the generalised norm resolvent convergence, cf. [31] and [34, Ch. 4] for more results):

Theorem 3.6. *Assume that $H_\varepsilon \xrightarrow{\text{gnr}} H_0$, then the following assertions hold:*

i. Convergence of operator functions: *We have*

$$\|\varphi(H_\varepsilon)J - J\varphi(H_0)\| \to 0 \quad \textit{and} \quad \|\varphi(H_\varepsilon) - J\varphi(H_0)J^*\| \to 0$$

for suitable functions φ (in particular, $\lim_{\lambda\to\infty}\varphi(\lambda)$ exists), e.g. $\varphi(\lambda) = \mathrm{e}^{-t\lambda}$ or $\varphi = \mathbb{1}_I$, $I \subset \mathbb{R}$ with $\partial I \cap \sigma(H_0) = \emptyset$.

ii. Convergence of discrete spectrum: *Let λ_0 be a (for simplicity) simple discrete eigenvalue of H_0 with corresponding normalised eigenfunction φ_0, then there exist simple discrete eigenvalues λ_ε of H_ε with corresponding eigenfunctions φ_ε such that $\lambda_\varepsilon \to \lambda_0$ and $\|J\varphi_0 - \varphi_\varepsilon\| \to 0$.*

iii. Convergence of essential spectrum: $\sigma_{\text{ess}}(H_\varepsilon) \to \sigma_{\text{ess}}(H_0)$ *converges uniformly in* $[0, \Lambda]$ *for all* $\Lambda > 0$. *In particular, H_ε has a spectral gap if H_0 has (provided $\varepsilon > 0$ is small enough).*

In particular, the convergence of all discrete eigenvalues Theorem 3.2 follows. Under certain additional assumptions (*positivity* and *contractivity*, fulfilled for the above example of the Neumann Laplacian on a thick graph, we also have convergence of $\varphi(H_\varepsilon) - J\varphi(H_0)J^* \to 0$ in the operator norm on $\mathscr{L}(\mathsf{L}_p(X_\varepsilon))$ (cf. [28]).

As a consequence for thick graphs, we know that a thick graph has spectral gaps once the corresponding metric graph has spectral gaps. A typical example of an operator having spectral gaps is given by a *periodic* operator; in [26], we showed, that the Kirchhoff Laplacian on the periodic graph of Figure 1 (lower left) has spectral gaps; so the same is true for a corresponding thick graph.

Another interesting example is given by the self-similar *Sierpiński graph*: Teplyaev showed in [38] that the spectrum of the discrete Laplacian on this graph is fractal, a simple argument shows that the same is true for a corresponding (equilateral) metric graph (cf. e.g. [32]). In particular, $\sigma(H_0)$ has *infinitely* many components in any *compact* spectral interval $I \subset [0, \infty)$. Therefore, Theorem 3.6 implies that the number of components of $\sigma(H_\varepsilon) \cap I$ tends to ∞ as $\varepsilon \to 0$. Note that our analysis is too weak in order to show that the number of components is actually infinite for a positive $\varepsilon > 0$.

4 Dirichlet and other Laplacians on thick graphs

4.1 Dirichlet Laplacians on thick graphs

Let us now review some results concerning the *Dirichlet* Laplacian on a thick graph. For a more detailed review especially on the Dirichlet case we refer to [19]. Related results are proven in [27, 10].

Let us assume (for simplicity) that X_ε is a *compact* graph-like manifold associated to a metric graph X_0. Moreover, we assume that X_ε has "straight" edge neighbourhoods $X_{\varepsilon,e} = I_e \times \varepsilon Y_e$ (the non-compact case and the case of curved embedded edges can be found in [33, Sec. 6.11].

We assume that each transversal manifold has non-empty boundary $\partial Y_e \neq \emptyset$. On a single tubular neighbourhood $X_{\varepsilon,e} = [0,1] \times \varepsilon Y_e$ with Dirichlet conditions on $[0, \ell_e] \times \varepsilon \partial Y_e$ and Neumann conditions on $\{0, \ell_e\} \times \varepsilon Y_e$, the spectrum is given by

$$\sigma(\Delta^{\mathrm{DN}}_{X_{\varepsilon,e}}) = \Big\{ \frac{p^2\pi^2}{\ell_e^2} + \frac{\lambda_q(\Delta^{\mathrm{D}}_{Y_e})}{\varepsilon^2} \,\Big|\, p = 0, 1, \dots, q = 1, 2, \dots \Big\}, \tag{4.1}$$

where $\lambda_q(\Delta^{\mathrm{D}}_{Y_e})$ denotes the q-th Dirichlet eigenvalue of Y_e. Since the first Dirichlet eigenvalue is non-negative, we have to consider a *shifted* operator in order to expect a convergence limit. Let $\lambda_1 := \min_e \lambda_1(\Delta^{\mathrm{D}}_{Y_e})$ and set

$$H_\varepsilon := \Delta^{\mathrm{D}}_{X_\varepsilon} - \frac{\lambda_1}{\varepsilon^2}.$$

Note that only the "thickest" edges (i.e., the edges with $\lambda_1 = \lambda_1(\Delta^{\mathrm{D}}_{Y_e})$ count. Let us assume for simplicity that $\lambda_1 = \lambda_1(\Delta^{\mathrm{D}}_{Y_e})$ for all edges $e \in E$.

A first result for the Dirichlet Laplacian is the following (cf. [30]):

Theorem 4.2. *Let $H_\varepsilon = \Delta^{\mathrm{D}}_\varepsilon - \lambda_1/\varepsilon^2$. If $\min_v \lambda_1(\Delta^{\mathrm{DN}}_{X_v}) > \lambda_1$ then*

$$\lambda_k(H_\varepsilon) \to \lambda_k(H_0),$$

where $H_0 = \bigoplus_e \Delta^{\mathrm{D}}_{I_e}$ is the decoupled *Dirichlet Laplacian on the metric graph X_0.*

A vertex neighbourhood X_v fulfilling the condition $\lambda_1(\Delta^{\mathrm{DN}}_{X_v}) > \lambda_1(Y_e)$ may look like in Figure 4. Here, $\Delta^{\mathrm{DN}}_{X_v}$ is the Laplacian on X_v with Neumann boundary conditions at the "inner" boundary $\mathring{\partial} X_v$ (where the edge neighbourhoods are attached) and with Dirichlet conditions on the remaining part. We call such manifolds X_v *spectrally small.* Note that this condition implies that $H_\varepsilon \geq 0$: Introducing additional Neumann boundary conditions at the junctions of $X_{\varepsilon,e}$ and $X_{\varepsilon,v}$ gives a lower bound on the shifted Dirichlet Laplacian, i.e.,

$$H_\varepsilon \geq \bigoplus_e \Big(\Delta^{\mathrm{DN}}_{X_{\varepsilon,e}} - \frac{\lambda_1}{\varepsilon^2} \Big) \oplus \bigoplus_v \Big(\Delta^{\mathrm{DN}}_{X_{\varepsilon,v}} - \frac{\lambda_1}{\varepsilon^2} \Big).$$

Now the shifted Laplacians on $X_{\varepsilon,e}$ are non-negative since λ_1 is the lowest transversal mode, see (4.1), and the shifted Laplacians on $X_{\varepsilon,v}$ are non-negative since X_v is spectrally small and since $\Delta^{\mathrm{DN}}_{X_{\varepsilon,v}} - \lambda_1/\varepsilon^2 = \varepsilon^{-2}(\Delta^{\mathrm{DN}}_{X_v} - \lambda_1)$.

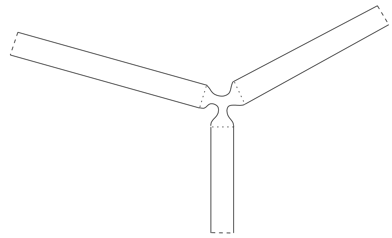

Figure 4: A vertex neighbourhood X_v in the centre which is *spectrally small*. For the graph-like manifold, the whole space is scaled by ε.

Note that the usual ε-neighbourhood is *not* spectrally small, as one can easily see by inserting test functions in the Rayleigh quotient of the quadratic form associated to H_ε. One obtains that there are eigenvalues $\tau_k(\varepsilon)$ of H_ε with associated eigenfunctions localised near $X_{\varepsilon,v}$ such that $\tau_k(\varepsilon) \to -\infty$ as $\varepsilon \to 0$ (from Theorem 4.3 below we actually conclude $\tau_k(\varepsilon) = (\tau_k - \lambda_1)/\varepsilon^2 < 0$).

A full description of the asymptotic behaviour of the Dirichlet spectrum (and other boundary conditions) was first given in [27, 18]. The main observation is to consider the rescaled space

$$\varepsilon^{-1} X_\varepsilon \to \dot{\bigcup_v} X_v^\infty =: X^\infty \qquad \text{as } \varepsilon \to 0,$$

i.e., to turn the rescaled compact space $\varepsilon^{-1} X_\varepsilon$ into a disjoint union of the star graph neighbourhoods X_v^∞ with infinite edges attached.

Denote by $\tau_1, \dots, \tau_{k_0}$ the L_2-eigenvalues of the Dirichlet Laplacian $\Delta^{\mathrm{D}}_{X^\infty}$ on X^∞ *below* the threshold λ_1. The first result is the following (see [18, 27]):

Theorem 4.3. *The eigenvalues of the Dirichlet Laplacian on the graph-like manifold X_ε below the threshold λ_1 have the asymptotic expansion $\lambda_k(\Delta^{\mathrm{D}}_{X_\varepsilon}) = \dfrac{\tau_k}{\varepsilon^2} + \mathrm{O}(\mathrm{e}^{-c/\varepsilon})$ for $k = 1, \dots, k_0$ as $\varepsilon \to 0$.*

Note that the spectral smallness assumption $\lambda_1(\Delta^{\mathrm{DN}}_{X_v}) > \lambda_1$ implies that there are no such eigenvalues (i.e., $k_0 = 0$).

Let us now treat the spectrum without these low-lying eigenvalues located at the vertex neighbourhoods. Denote by $H_\varepsilon := (\Delta^{\mathrm{D}}_{X_\varepsilon} - \lambda_1/\varepsilon^2)_+$ the non-negative part of the shifted Laplacian, where $A_+ := \mathbb{1}_{[0,\infty)}(A)A$.

It turns out that the asymptotic behaviour of the Dirichlet eigenvalues is determined by a *scattering problem* on the star graph neighbourhood X_v^∞. Denote by $S_v(\lambda)$ the *scattering matrix* of $\Delta^{\mathrm{D}}_{X_v^\infty}$ at the energy $\lambda \ge \lambda_1$, which is a $(\deg v \times \deg v)$-matrix.

Let $\mathscr{V}_v := \ker(S_v(\lambda_1) - 1)$, and let H_0 be the Laplacian on the underlying metric graph X_0 with vertex conditions

$$\{f_e(v)\}_{e \in E_v} \in \mathscr{V}_v \quad \text{and} \quad \{f_e'(v)\}_{e \in E_v} \in \mathscr{V}_v^\perp \subset \mathbb{C}^{E_v}. \tag{4.4}$$

Note that this vertex condition turns out to be the (unweighted) Kirchhoff condition if $\mathscr{V}_v = \mathbb{C}(1, \dots, 1)$.

Grieser [18] proved the following result (see also the results of Molchanov and Vainberg [27]):

Theorem 4.5. *Let $H_\varepsilon := (\Delta^{\mathrm{D}}_{X_\varepsilon} - \lambda_1/\varepsilon^2)_+$ be the non-negative part of the shifted Dirichlet Laplacian on a compact graph-like manifold X_ε, and let H_0 be the metric graph Laplacian with vertex condition as in (4.4), then the k-th eigenvalue has the asymptotics $\lambda_k(H_\varepsilon) - \lambda_k(H_0) = \mathrm{O}(\varepsilon)$ as $\varepsilon \to 0$.*

Note that Grieser's also gives an asymptotic expansion of the eigenvalues. This method also applies to other boundary conditions. In the Neumann case, this method gives the right error term of order ε instead of $\varepsilon^{1/2}$ as obtained by the simpler eigenvalue comparison techniques of Theorem 3.2.

4.2 Difference between Neumann and Dirichlet case

Let us make some comments why the case of Neumann boundary conditions on a thick graph is much easier to treat than the case of Dirichlet (or other) boundary conditions:

Let us first give an interpretation of the scattering matrix $S_v(\lambda)$ of the Dirichlet Laplacian on the star graph neighbourhood X_v^∞. One observes that the *vertex space* $\mathscr{V}_v$ (the range of the function values $\{f_e(v)\}_{e \in E_v}$ in a vertex condition), defined by $\mathscr{V}_v := \ker(S_v(\lambda_1) - 1)$ is *non-trivial* iff there exist *generalised (bounded) eigenfunctions* ψ_v such that $\psi_{v,e}(x,y) \sim \varphi_e(y)$ as $x \to \infty$ on the edge neighbourhood X_e ($e \in E_v$), where φ_e is the eigenfunction on Y_e associated to λ_1: Such functions ψ_v are called *energy resonance at* λ_1.

As a consequence, a *non-trivial* coupling at the vertex v, i.e., a vertex space $\mathscr{V}_v \neq 0$, is a "rare" event: generically, one only has a decoupling *Dirichlet* vertex condition as in Theorem 4.2. Actually, the spectral smallness condition ensures that there is no energy resonance at λ_1.

In the Neumann (or boundaryless) case, the threshold is $\lambda_1 = 0$, and the energy resonance at 0 is just given by the constant function $\psi_v = \mathbb{1}$, and the corresponding vertex space is $\mathscr{V}_v = \mathbb{C}(1, \dots, 1)$. Note that in this case, the energy resonance function $\psi_v = \mathbb{1}$ *exactly* matches with the lowest transversal eigenfunctions φ_e (appropriately scaled), which are also constant, i.e., we have $\psi_e(x,y) = \varphi_e(y)$ for *all* $(x,y) \in X_e$.

Since this energy resonance function at 0 does not see the geometry of X_v, it is not seen in the limit operator, the Kirchhoff Laplacian, either. Moreover, the embedding of the metric graph X_0 into an ambient space like $\mathbb{R}^2$ does not enter in the limit either. In contrast, in the Dirichlet case, a curved edge leads to an additional potential on the metric graph determined by the curvature of the edge, see [30], [34, Sec. 6.11] and references therein.

4.3 Other vertex conditions in the limit

We first give a non-existence result for certain vertex couplings. We claim that it is impossible to approximate a delta-coupling by a pure Laplacian, using a topological argument.

Let H_0 be the the Laplacian on a compact metric graph X_0 with delta-coupling of strength $q(v) > 0$ at each vertex v, i.e., $(H_0 f)_e = -f_e''$, and

$$f \text{ is continuous and } \quad \sum_{e \in E_v} f_e'(v) = q(v) f(v). \tag{4.6}$$

Note that $H_0 \geq 0$ iff $q(v) \geq 0$ for all $v \in V$. We formally write $H_0 = \Delta_{X_0} + \sum_v q(v)\delta_v$ for this vertex condition.

We now want to factorise H_0 as $H_0 = d_0^* d_0$. An easy calculation shows that this can be done by choosing $d_0 f := (f', (\sqrt{q(v)})_v) \in \mathsf{L}_2(X_0) \oplus \mathbb{C}^V =: \widehat{\mathscr{H}}_0$ with $\operatorname{dom} d_0 = \mathsf{H}^1(X_0)$, the Sobolev space of order 1 with *continuous* functions at the vertices. Similarly, we can factorise the Laplacian on a graph-like manifold as $H_\varepsilon = d_\varepsilon^* d_\varepsilon$ with $d_\varepsilon u \in \mathsf{L}_2^{\mathrm{exact}}(T^* X_\varepsilon) =: \widehat{\mathscr{H}}_\varepsilon$, the space of *exact* 1-forms.

A simple example is given as follows: Let $X_0 = \mathbb{S}^1$ be a loop graph with only one vertex v, and let $X_\varepsilon = \mathbb{S}^1 \times_{r_\varepsilon} Y$ be a *warped product*, i.e., the product manifold with metric $g_\varepsilon = \mathrm{d}x^2 + r_\varepsilon(x)^2 h$, where (Y, h) is a closed manifold. Note that the warped product here corresponds to a manifold $\mathbb{S}^1 \times Y$ with variable radius given by the function $r_\varepsilon(x)$. It is an easy calculation that the index $\operatorname{ind} d_0$ (defined as $\operatorname{ind} d_0 := \dim \ker d_0 - \dim \ker d_0^*$) equals 0 on the loop graph, but $\operatorname{ind} d_\varepsilon = 1$ on the warped product.

As a consequence of the different indices we claim (work in progress with Claudio Cacciapuoti):

Conjecture 4.7. *It is impossible to approximate a (non-trivial) delta-coupling H_0 via a pure Laplacian $H_\varepsilon = \Delta_{X_\varepsilon}$ with $X_\varepsilon = \mathbb{S}^1 \times_{r_\varepsilon} Y$ such that $H_\varepsilon \xrightarrow{\mathrm{gnr}} H_0$.*

The arguments leading to this conjecture are rather simple: Since $H_\varepsilon \geq 0$ the limit operator also has to be non-negative, i.e., i.e., $q(v) \geq 0$. Moreover, H_ε is unitarily equivalent to the family of one-dimensional Schrödinger operators

$$\{H_{0,k}\}_k = \Big(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + K_\varepsilon + \frac{\lambda_k(Y)}{r_\varepsilon^2}\Big)_k, \quad \text{where} \quad K_\varepsilon = P_\varepsilon^2 + P_\varepsilon', \quad P_\varepsilon := \frac{m}{2} \cdot \frac{r_\varepsilon'}{r_\varepsilon}$$

and $m = \dim Y$. If one tries to approximate a delta-coupling by the lowest member of the family $H_{0,0} f = -f'' + K_\varepsilon f$ for appropriate radius functions r_ε, one *always* ends up with a trivial coupling strength $q(v) = 0$. Moreover, the above conjecture should also hold for more general spaces, since the approximation is *local*, i.e., depends only on the behaviour of r_ε near the vertex.

In order to obtain a delta-coupling we need to change either the operator or the topology of the approximating space. One result in this direction is to use *scaled Schrödinger operators* on the vertex neighbourhoods: Let X_ε be a thick graph associated to a metric graph X_0 (for simplicity with all transversal volumes being the same, e.g., $\mathrm{vol}_{d-1} Y_e = 1$). Set $H_\varepsilon := \Delta^{\mathrm{N}}_{X_\varepsilon} + Q_\varepsilon$ with Neumann boundary conditions (if $\partial X_\varepsilon \neq \emptyset$), where $Q_\varepsilon = \sum_v \varepsilon^{-1} Q_v$ is a potential supported on the vertex neighbourhood $X_{\varepsilon,v}$ only.

In the limit, we have the Laplacian with delta-couplings as in (4.6), i.e., $H_0 = \Delta_{X_0} + \sum_v q(v)\delta_v$, with coupling strengths $q(v) = \int_{X_v} Q_v$.

Under the same uniformity assumptions as in Theorem 3.5, we proved in [15]:

Theorem 4.8. *Let X_ε be a uniform thick graph with underlying uniform metric graph X_0, let $H_\varepsilon := \Delta^{\mathrm{N}}_{X_\varepsilon} + \sum_v \varepsilon^{-1} Q_v$ be the scaled Schrödinger operator, and let $H_0 := \Delta_{X_0} + \sum_v q(v)\delta_v$ be the Laplacian with delta-coupling with strengths $q(v) = \int_{X_v} Q_v$, then $H_\varepsilon \xrightarrow{\mathrm{gnr}} H_0$ of order $\mathrm{O}(\varepsilon^{1/2})$.*

The proof is very similar to the proof of Theorem 3.5, only the estimate of $\|(H_\varepsilon + 1)^{-1} J - J(H_0 + 1)^{-1}\|$ is slightly different (actually, in Theorem 3.5 and Theorem 4.8, we used the corresponding *quadratic forms* instead of the operators, making the verification of the estimates simpler, but the presentation a bit more technical).

Using arguments of [5, 6], one can now approximate a *general vertex condition* of a Laplacian H_0 on X_0 at a vertex v by the operator H^a on a metric graph X_0^a, where H^a and X_0^a are constructed from H_0 and X_0 using *properly scaled delta interactions* and *additional edges*, such that $X_0^a \to X_0$ as $a \to 0$.

If we now approximate H^a by H_ε with appropriate $a = a_\varepsilon$ and delta strengths, we can find a family of thick graphs X_ε and operators H_ε such that $H_\varepsilon \xrightarrow{\mathrm{gnr}} H_0$. The example of a delta'-coupling is presented in [15].

Another possibility of obtaining a delta coupling is to use *scaled Robin boundary conditions*: Let $H_\varepsilon := \Delta_{X_\varepsilon}$ with Robin boundary conditions

$$\partial_{\mathrm{n}_\varepsilon} u + \beta_\varepsilon u = 0 \quad \text{on} \quad \partial X_\varepsilon = \bigcup_v \Gamma_{\varepsilon,v} \cup \bigcup_e \Gamma_{\varepsilon,e},$$

i.e., we decompose the boundary of the thick graph ∂X_ε into the parts of the vertex neighbourhoods $\Gamma_{\varepsilon,v} = \varepsilon \partial X_v \cap \partial X_\varepsilon$ and the edge neighbourhoods $\Gamma_{\varepsilon,e} = I_e \times \varepsilon \partial Y_e$. Denote the corresponding restriction of β_ε by $\beta_{\varepsilon,v}$ and $\beta_{\varepsilon,e}$. Let $H_0 = \Delta_{X_0} + \sum_v q(v)\delta_v$ be the Laplacian on X_0 with delta interactions of strength $q(v)$. In [28], we proved the following:

Theorem 4.9. *Assume that $\beta_{\varepsilon,e} = \mathrm{O}(\varepsilon^{1+1/2})$, $\beta_{\varepsilon,v} = \beta_v$ and $q(v) = \int_{\Gamma_v} \beta_v$, then $H_\varepsilon \xrightarrow{\mathrm{gnr}} H_0$ of order $\mathrm{O}(\varepsilon^{1/2})$.*

Idea of proof. The lowest eigenvalue $\lambda_{1,e}(x,\varepsilon)$ on the scaled transversal manifold $\{x\} \times \varepsilon Y_e$ with Robin condition $\varepsilon^{-1}\partial_{\mathrm{n}} u(x,\cdot) + \beta_\varepsilon u(x,\cdot) = 0$ on $\{x\} \times \partial Y_e$ is of order $\lambda_{1,e}(x,\varepsilon) = \mathrm{O}(\varepsilon^{1/2}) \to 0$, i.e., this eigenvalue problem is *close* to the Neumann case. Therefore, we can use similar arguments as in the proof of Theorem 3.5. Moreover, the scaling behaviour of β_ε at the vertex neighbourhood is just the right one for a delta-coupling as $\varepsilon \to 0$. □

If we used scale invariant conditions, i.e., $\beta_\varepsilon = \beta/\varepsilon$, then the lowest transversal Robin eigenvalue would be of order $\lambda_{1,e}(\varepsilon) = \mathrm{O}(\varepsilon^{-2})$, and the arguments of Theorem 4.5 have to be applied.

5 Outlook and open problems

5.1 Work in progress and open problems

In a current project together with Jussi Behrndt, we show the convergence of the *Dirichlet-to-Neumann* operator on a graph-like manifold X_ε with cylindrical finite ends (which determine the boundary for the Dirichlet-to-Neumann operator) to a corresponding object on the underlying metric graph X_0. The main point here is to introduce Dirichlet-to-Neumann operators via *boundary triples* associated to quadratic forms.

There are still no conrete examples of vertex neighbourhoods known, such that the shifted and cut Dirichlet Laplacian $H_\varepsilon = (\Delta^{\mathrm{D}}_{X_\varepsilon} - \lambda_1/\varepsilon^2)_+$ converges to a Laplacian with *non-trivial* vertex couplings at the vertices. Moreover, the convergence in the generalised norm resolvent convergence in the Dirichlet case is not yet shown, although some work has been done in [10] in this direction.

5.2 Conclusion

Thick graphs (or fat graphs, graph-like manifolds . . .) provide an interesting class of *almost* solvable models and a *"construction kit"* for examples with special spectral behaviour. Neumann (and related) operators on thick graphs have a rather "simple" limit behaviour (Kirchhoff and related vertex conditions in the limit) independent of the vertex neighbourhoods, and can be treated with general (weak) methods. In contrast, Dirichlet and other operators with *non-zero* (large) first eigenvalue of order ε^{-2} are more complicated; the limit behaviour depends on the scattering matrix at the threshold, and the limit operator is *generically* decoupled. Finally, The generalised resolvent convergence is a *very general* scheme: it can be applied to many cases; a stronger version using quadratic forms allows results with *minimal* smoothness assumptions; a similar convergence scheme is also available for sectorial operators (see [28]).

Acknowledgements

The author would like to thank the organisers of the QMath11 held at Hradec Králové in September 2010 for the kind invitation.

References

[1] M. Barlow, T. Coulhon, and A. Grigor'yan, *Manifolds and graphs with slow heat kernel decay*, Invent. Math. **144** (2001), 609–649.

[2] M. T. Barlow, R. F. Bass, and T. Kumagai, *Stability of parabolic Harnack inequalities on metric measure spaces*, J. Math. Soc. Japan **58** (2006), 485–519.

[3] P. Bérard, G. Besson, and S. Gallot, *Embedding Riemannian manifolds by their heat kernel*, Geom. Funct. Anal. **4** (1994), 373–398.

[4] S. Y. Cheng, *Eigenfunctions and nodal sets*, Comment. Math. Helv. **51** (1976), 43–55.

[5] T. Cheon and P. Exner, *An approximation to δ' couplings on graphs*, J. Phys. A **37** (2004), L329–L335.

[6] T. Cheon, P. Exner, and O. Turek, *Approximation of a general singular vertex coupling in quantum graphs*, Ann. Physics **325** (2010), 548–578.

[7] Y. Colin de Verdière, *Sur la multiplicité de la première valeur propre non nulle du laplacien*, Comment. Math. Helv. **61** (1986), 254–270.

[8] ———, *Construction de laplaciens dont une partie finie du spectre est donnée*, Ann. Sci. École Norm. Sup. (4) **20** (1987), 599–615.

[9] G. F. Dell'Antonio and L. Tenuta, *Quantum graphs as holonomic constraints*, J. Math. Phys. **47** (2006), 072102, 21.

[10] G. Dell'Antonio and E. Costa, *Effective schroedinger dynamics on ε-thin dirichlet waveguides via quantum graphs i: star-shaped graphs*, Preprint (2010).

[11] P. Exner, J. P. Keating, P. Kuchment, T. Sunada, and A. Teplyaev (eds.), *Analysis on graphs and its applications*, Proc. Symp. Pure Math., vol. 77, Providence, R.I., Amer. Math. Soc., 2008.

[12] P. Exner and O. Post, *Convergence of spectra of graph-like thin manifolds*, Journal of Geometry and Physics **54** (2005), 77–115.

[13] ———, *Convergence of resonances on thin branched quantum wave guides*, J. Math. Phys. **48** (2007), 092104 (43pp).

[14] ———, *Quantum networks modelled by graphs*, Quantum Few-Body Systems, AIP Conf. Proc., vol. 998, Amer. Inst. Phys., Melville, NY, 2008, pp. 1–17.

[15] ———, *Approximation of quantum graph vertex couplings by scaled Schrödinger operators on thin branched manifolds*, J. Phys. A **42** (2009), 415305 (22pp).

[16] M. I. Freidlin and A. D. Wentzell, *Diffusion processes on graphs and the averaging principle*, Ann. Probab. **21** (1993), 2215–2245.

[17] K. Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator*, Invent. Math. **87** (1987), 517–547.

[18] D. Grieser, *Spectra of graph neighborhoods and scattering*, Proc. London Math. Soc. (3) **97** (2008), 718–752.

[19] D. Grieser, *Thin tubes in mathematical physics, global analysis and spectral geometry*, in [11] (2008), 565–593.

[20] A. Kasue, *Convergence of Riemannian manifolds and Laplace operators. I*, Ann. Inst. Fourier (Grenoble) **52** (2002), 1219–1257.

[21] ———, *Convergence of Riemannian manifolds and Laplace operators. II*, Potential Anal. **24** (2006), 137–194.

[22] P. Kuchment, *Quantum graphs: an introduction and a brief survey*, in [11] (2008), 291–312.

[23] P. Kuchment and H. Zeng, *Convergence of spectra of mesoscopic systems collapsing onto a graph*, J. Math. Anal. Appl. **258** (2001), 671–700.

[24] ______, *Asymptotics of spectra of Neumann Laplacians in thin domains*, Advances in differential equations and mathematical physics (Birmingham, AL, 2002), Contemp. Math., vol. 327, Amer. Math. Soc., Providence, RI, 2003, pp. 199–213.

[25] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153–201.

[26] F. Lledó and O. Post, *Eigenvalue bracketing for discrete and metric graphs*, J. Math. Anal. Appl. **348** (2008), 806–833.

[27] S. Molchanov and B. Vainberg, *Scattering solutions in networks of thin fibers: small diameter asymptotics*, Comm. Math. Phys. **273** (2007), 533–559.

[28] D. Mugnolo, R. Nittka, and O. Post, *Convergence of sectorial operators on varying Hilbert spaces*, arXiv:1007.3932 (2010).

[29] R. C. Penner, M. Knudsen, C. Wiuf, and J. E. Andersen, *Fatgraph models of proteins*, Comm. Pure Appl. Math. **63** (2010), 1249–1297.

[30] O. Post, *Branched quantum wave guides with Dirichlet boundary conditions: the decoupling case*, Journal of Physics A: Mathematical and General **38** (2005), 4917–4931.

[31] ______, *Spectral convergence of quasi-one-dimensional spaces*, Ann. Henri Poincaré **7** (2006), 933–973.

[32] ______, *Equilateral quantum graphs and boundary triples*, Analysis on Graphs and its Applications (Providence, R.I.) (P. Exner, J. P. Keating, P. Kuchment, T. Sunada, and A. Teplayaev, eds.), Proc. Symp. Pure Math., vol. 77, Amer. Math. Soc., 2008, pp. 469–490.

[33] ______, *Spectral analysis of metric graphs and related spaces*, in "Limits of graphs in group theory", eds. G. Arzhantseva and A. Valette, Presses Polytechniques et Universitaires Romandes, 109–140 (2009), 109–140.

[34] ______, *Spectral analysis on graph-like spaces*, Habilitation thesis, Humboldt University, Berlin, 2009.

[35] J. Rubinstein and M. Schatzman, *Variational problems on multiply connected thin strips. I. Basic estimates and convergence of the Laplacian spectrum*, Arch. Ration. Mech. Anal. **160** (2001), 271–308.

[36] K. Ruedenberg and C. W. Scherr, *Free–electron network model for conjugated systems, I. Theory*, J. Chem. Phys. **21** (1953), 1565–1581.

[37] Y. Saito, *The limiting equation for Neumann Laplacians on shrinking domains.*, Electron. J. Differ. Equ. **31** (2000), 25 p.

[38] A. Teplyaev, *Spectral analysis on infinite Sierpiński gaskets*, J. Funct. Anal. **159** (1998), 537–567.