CONTINUITY PROPERTIES OF THE INTEGRATED DENSITY OF STATES ON MANIFOLDS

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Abstract. We first analyze the integrated density of states (IDS) of periodic Schrödinger operators on an amenable covering manifold. A criterion for the continuity of the IDS at a prescribed energy is given along with examples of operators with both continuous and discontinuous IDS'.

Subsequently, alloy-type perturbations of the periodic operator are considered. The randomness may enter both via the potential and the metric. A Wegner estimate is proven which implies the continuity of the corresponding IDS. This gives an example of a discontinuous "periodic" IDS which is regularized by a random perturbation.

1. Introduction

This paper is devoted to the study of continuity properties of the integrated density of states (IDS) of ergodic Schrödinger operators on manifolds. The IDS is a distribution function introduced in the quantum theory of solids which measures the number of electron levels per unit volume up to a given energy. It allows to calculate all basic thermodynamic properties of the corresponding non-interacting electron gas, like e.g. the free energy.

This article is concerned with the Hölder continuity of the IDS for particular random Schrödinger operators on manifolds. The continuity of the IDS is a matter of interest both for physicists (e.g. [Weg81]) and geometers (e.g. [DLM$^+$03]). It has been intensely studied in the theory of localization for random Schrödinger operators, see e.g. the accounts in [CFKS87, CL90, PF92, Sto01, Ves06]. In a subsequent paper we will study the Hölder continuity of the IDS for quantum graphs with randomly perturbed lengths of edges (see [HP06] as well for results on localization for certain quantum graphs).

Localization is the phenomenon, that certain quantum Hamiltonians, describing disordered solid systems, exhibit pure point spectrum almost surely. Other ergodic operators exhibit purely continuous spectrum, while it is conjectured, that for a large class of operators pure point and continuous spectra should coexist, with a (or several) sharp energy value separating them. This energy is called *mobility edge*.

Although the misconception that the IDS has a singularity of some kind at the mobility edge was discarded by Wegner in [Weg81], there is still a strong relation

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between properties of the IDS and localized states (corresponding to p.p. spectrum). Namely, the proof of localization with the so far most widely applicable method, the *multi scale analysis* introduced by Fröhlich and Spencer [FS83], uses as a key ingredient an upper bound on the *density of states*. This function is the derivative of the IDS and its existence (for certain models) may be proved by using an estimate going back to Wegner [Weg81].

For periodic operators in Euclidean geometry — the most regular form of ergodic Schrödinger operators — the continuity of the IDS is established under mild conditions on the potential, see e.g. [She02] and the references therein.

A substantial body of literature is devoted to randomly perturbed periodic operators, where the perturbation is of alloy type. Under certain conditions it is known that these random operators have also an continuous IDS, i.e., that the random perturbation conserves the continuity of the IDS. From the physical point of view it is actually expected that the IDS of the random operators should be even more regular than the one of periodic ones. However, only for certain discrete models, better regularity of the IDS than continuity has been proven, see for instance [ST85, CFS84].

For more general geometries than $\mathbb{R}^d$ the situation is somewhat different. In this situation even the periodic Laplace-Beltrami operator (without any potential) on an abelian covering may have L^2-eigenfunctions, as was already indicated in [Sun88], referring to an example in [KOS89]. This is equivalent to a discontinuity of the IDS (cf. Proposition 3.2). Other cases with jumps in the IDS are given by quasi-crystals [KLS03, LS], periodic operators on covering graphs and percolation Hamiltonians [Ves05a, Ves05b], random necklace models [KS04] or fourth order differential operators, see e.g. [Kuc93]. However, in particular cases, the continuity of the IDS of periodic Schrödinger operators on an abelian covering manifold can be established using a criterion of Sunada (cf. [Sun90]).

Our main results are Wegner estimates for particular random perturbations of periodic operators on manifolds (cf. Theorems 2.11 and 2.14). The perturbation is assumed to be of alloy-type and may enter the operators via the *potential* or the *metric*, defined in the models RAP and RAM (see Definitions 2.9 and 2.13). These estimates imply the continuity of the IDS, even if the unperturbed, periodic operator had a discontinuous IDS. Thus, while alloy type perturbations *preserve* the continuity of the IDS in the Euclidean case, they are even IDS-continuity *improving* for certain operators on manifolds.

The paper is organized as follows: In the following section we introduce our models RAP and RAM and state the main results. Section 3 is devoted to periodic operators with abelian covering group. In Sections 4 and 5 we prove Wegner estimates for both models RAP and RAM, respectively. For this aim, we need a (super) trace class estimate of an effective perturbation in each model (see Propositions 4.2 and 5.2). The proof for this trace class estimate is given in Sections 6–8. In the appendix we provide necessary uniform results on Sobolev spaces on families of manifolds which are used throughout this article.

2. Model and results

Throughout the paper we will consider the following *geometric situation*:

Let (X, g_0) be a Riemannian manifold with a smooth metric g_0 and Γ an group acting freely, cocompactly and properly discontinuously by isometries on (X, g_0) such that the quotient $M = X/\Gamma$ is a compact Riemannian manifold of the same dimension as X. The stated assumptions imply that Γ is a finitely generated group. Typically, X will be non-compact and thus Γ infinite.

Let $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ be a probability space on which Γ acts ergodically by measure preserving transformations $\gamma \colon \Omega \to \Omega, \gamma \in \Gamma$, i.e., any Γ-invariant set $B \in \mathcal{B}_\Omega$ ($\gamma B = B$ for all $\gamma \in \Gamma$) has probability 0 or 1. The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$.

We will be given two types of random objects over $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$. The first is a family of random potentials on X, the second is a family of random metrics. Put together, they will give rise to a family of random operators whose study is our primary concern here. Note that this includes the case that Ω contains only one element and thus the operator family consists of a single periodic operator.

As for the random geometry, the manifold X is equipped with a family of metrics $\{g_\omega\}_{\omega\in\Omega}$ with corresponding volume forms vol_ω. With respect to a fixed periodic metric g_0, we define a smooth section A_ω in the bundle $L(TX) \cong T^*X \otimes TX$ via

$$g_\omega(x)(v, v) = g_0(x)(A_\omega(x)v, v)$$

for all $x \in X$, $v \in T_xX$ and $\omega \in \Omega$. In the sequel, we will often suppress the dependence on $x \in X$. We denote by Δ_ω the non-positive Laplace operator on the mannifold (X, g_ω).

We need the following definition:

Definition 2.1. We say that a family $\{g_\omega\}_\omega$ of metrics on X is *relatively bounded w.r.t.* the metric g_0 on X if for each $k \in \mathbb{N}$ there are constants $C_{\mathrm{rel},k} > 0$ such that

$$C_{\mathrm{rel},0}^{-1}\, g_0(v, v) \le g_\omega(v, v) = g_0(A_\omega v, v) \le C_{\mathrm{rel},0}\, g_0(v, v) \tag{1}$$

for all $v \in TX$ and

$$|\nabla_0^k A_\omega(x)|_0 \le C_{\mathrm{rel},k} \tag{2}$$

for all $x \in X$ and all $\omega \in \Omega$. Here ∇_0^k denotes the iterated covariant derivative w.r.t g_0 and $|\cdot|_0$ is the (pointwise) norm w.r.t g_0 in the appropriate tensor bundle of T^*X and TX.

Since the periodic manifold (X, g_0) is of bounded geometry, the relative boundedness of the family $\{g_\omega\}_\omega$ implies that (X, g_ω) is also of bounded geometry with constants (r_0, C_k) *independent* of ω, as shown in Lemma A.2.

Note that the lower bound in (1) implies that we have in analogy to (2) also a uniform bound on the derivatives of A_ω^{-1}, more precisely $|\nabla_0^k A_\omega^{-1}(x)|_0 \le \widetilde{C}_{\mathrm{rel},k}$.

The functions $x \mapsto (\det(A_\omega(x)))^{1/2}$ are positive, smooth functions and satisfy

$$\int_X f(x)\big(\det(A_\omega(x))\big)^{1/2}\, \mathrm{dvol}_0(x) = \int_X f(x)\, \mathrm{dvol}_\omega(x),$$

i.e., they are densities of the measures dvol_ω with respect to the unperturbed measure dvol_0. Consequently, for any measurable subset $\Lambda \subset X$ and any pair $\omega^1, \omega^2 \in \Omega$ the operators

$$S_{\omega^1,\omega^2} \colon L^2(\Lambda, g_{\omega^1}) \to L^2(\Lambda, g_{\omega^2}), \quad S_{\omega^1,\omega^2}(f) = \big(\det(A_{\omega^1} A_{\omega^2}^{-1})\big)^{1/2} f \tag{3}$$

are unitary, see also [LPV04]. These operators will be used in Section 5 to transform different Laplace-Beltrami operators into the same Hilbert space.

The following conditions will be assumed throughout this paper:

Assumption 2.2. We assume that the family $\{g_\omega\}_\omega$ is jointly measurable, i.e., that $(\omega, v) \to g_\omega(x)(v, v)$ is measurable on $\Omega \times TX$. In addition, we suppose that $\{g_\omega\}_\omega$ is relatively bounded in the sense of Definition 2.1 with respect to a fixed periodic metric g_0. Furthermore, we assume that the metrics are compatible in the sense that the covering transformations

$$\gamma \colon (X, g_\omega) \to (X, g_{\gamma\omega}), \quad \gamma \colon x \mapsto \gamma x \tag{4}$$

are isometries. Hence, the induced maps

$$U_{(\omega,\gamma)} \colon L^2(X, g_{\gamma^{-1}\omega}) \to L^2(X, g_\omega), \qquad (U_{(\omega,\gamma)} f)(x) = f(\gamma^{-1}x)$$

are unitary operators between L^2-spaces over the manifolds $\{(X, g_\omega)\}_{\omega\in\Omega}$.

As for the random potentials, let $V \colon \Omega \times X \longrightarrow [0, \infty)$ be jointly measurable and such that $V_\omega := V(\omega, \cdot)$ is for all $\omega \in \Omega$ relatively Δ_ω-bounded with relative bound strictly less than one. Assume furthermore that $V(\gamma\omega, x) = V(\omega, \gamma^{-1}x)$ for arbitrary $x \in X$ and $\omega \in \Omega$.

Given Assumption 2.2, we can now introduce the corresponding random Schrödinger operator as

$$H_\omega := -\Delta_\omega + V_\omega \geq 0. \tag{5}$$

In fact, these operators are defined by means of quadratic forms. For more details we refer the reader to [LPV04]. The operators (5) satisfy the *equivariance condition*

$$H_\omega = U_{(\omega,\gamma)} H_{\gamma^{-1}\omega} U^*_{(\omega,\gamma)}, \tag{6}$$

for all $\gamma \in \Gamma$ and $\omega \in \Omega$. Moreover, they form a measurable family of operators in the sense of the next definition as has been shown in Theorem 1 of [LPV04].

Definition 2.3. A family of selfadjoint operators $\{H_\omega\}_\omega$, where the domain of H_ω is a dense subspace of $L^2(D, g_\omega)$, is called *measurable family of operators* if

$$\omega \mapsto \langle f_\omega(\cdot), F(H_\omega) f_\omega(\cdot)\rangle_\omega \tag{7}$$

is measurable for all $F \colon \mathbb{R} \to \mathbb{C}$ bounded and measurable and all $f \colon \Omega \times X \to \mathbb{R}$ measurable with $f_\omega(\cdot) \in L^2(X, g_\omega)$ for every $\omega \in \Omega$.

This notion of measurability is consistent with the works of Kirsch and Martinelli [KM82a, KM82b] as discussed in [LPV04].

A key object in our study is the *integrated density of states* (IDS). It will be defined next. Let $\mathcal{F} \subset X$ be a fundamental domain of Γ. We will need restrictions of operators to agglomerates of translates of $\mathcal{F}$. For a finite set

$I \subset \Gamma$ define the agglomerate $\Lambda_0(I)$ of fundamental domains associated with I by

$$\Lambda_0(I) := \bigcup_{\gamma \in I} \gamma \bar{\mathcal{F}} \subset X. \tag{8}$$

For technical reasons (e.g., the Sobolev extension Theorem A.9), it is easier to work with a "smoothed" version $\Lambda(I)$ of the agglomerate $\Lambda_0(I)$, satisfying $\Lambda(\gamma I) = \gamma \Lambda(I)$, and for some fixed radius $r > 0$ the relation

$$\Lambda_0(I) \subset \Lambda(I) \subset B_r(\Lambda_0(I)), \tag{9}$$

where $B_r(A)$ denotes the open r-neighborhood of the set $A \subset X$ with respect to the metric g_0. The construction of $\Lambda(I)$ is given in [Bro81, pp. 593]. We show in Lemma A.3 that $(\Lambda(I), g_0)$, and also $(\Lambda(I), g_\omega)$, are of bounded geometry in the sense of Definition A.1, *uniformly in I and ω.*

The restriction of H_ω to $\Lambda(I)$ with Dirichlet boundary conditions is denoted by $H_\omega^I = H_\omega^{\Lambda(I)}$. The corresponding spectral projections are denoted by P_ω^I, i.e.

$$P_\omega^I\big(]-\infty, E[\big) := \chi_{]-\infty,E[}(H_\omega^I)$$

and similarly $P_\omega\big(]-\infty, E[\big) := \chi_{]-\infty,E[}(H_\omega)$. We define the distribution function N_ω^I on $\mathbb{R}$ for H_ω^I by

$$N_\omega^I(E) := \frac{1}{\mathrm{vol}_\omega \Lambda(I)} \operatorname{Tr} P_\omega^I\big(]-\infty, E[\big).$$

The integrated density of states of the random operator $\{H_\omega\}_\omega$ is defined as the distribution function

$$N\colon \mathbb{R} \to [0,\infty[, \quad N(E) := \frac{1}{\mathbb{E}[\mathrm{vol}_\bullet \mathcal{F}]} \mathbb{E}\Big[\operatorname{Tr}\big(\chi_{\mathcal{F}} \cdot P_\omega\big(]-\infty, E[\big) \cdot \chi_{\mathcal{F}}\big)\Big], \tag{10}$$

where Tr is the trace in $L^2(\mathcal{F})$.

If the group Γ is *amenable* there exists a *tempered Følner sequence*, i.e., an increasing sequence of finite, non-empty subsets $I_l \subset \Gamma$, $l \in \mathbb{N}$, with "small boundary" cf. [Ada93, Lin01, LPV04] for details.

Theorem 4 in [LPV04] can be phrased as follows:

Theorem 2.4. *Suppose that the transformation group Γ of the covering $X \to M$ is amenable. Then at all continuity points E of N and for almost every ω the following convergence holds*

$$\lim_{l\to\infty} N_\omega^{I_l}(E) = \lim_{l\to\infty} \mathbb{E}\big[N_\bullet^{I_l}(E)\big] = N(E).$$

Remark 2.5. Note that in [LPV04] we proved the above theorem for the *non*-smoothed domains $\Lambda_0(I)$, but the statement is still true for the smoothed domains $\Lambda(I)$. More precisely, if

$$N_{0,\omega}^I(E) := \frac{1}{\mathrm{vol}_\omega \Lambda_0(I)} \operatorname{Tr} \chi_{]-\infty,E[}(H_\omega^{\Lambda_0(I)})$$

denotes the distribution function with respect to the domain $\Lambda_0(I)$, then domain monotonicity implies that

$$N_{0,\omega}^{I_l}(E) \le \left(1 + \frac{\mathrm{vol}_\omega\big(\Lambda(I_l) \setminus \Lambda_0(I_l)\big)}{\mathrm{vol}_\omega \Lambda_0(I_l)}\right) N_\omega^{I_l}(E),$$

where $H_\omega^{\Lambda_0(I_l)}$ denotes the Dirichlet operator on the (unsmoothed) agglomerate $\Lambda_0(I)$. The Følner property and (1) now immediately imply that $\lim_{l\to\infty} N_{0,\omega}^{I_l}(E) \le \liminf_{l\to\infty} N_\omega^{I_l}(E)$.

As for the converse inequality, we define a sequence $J_l \subset \Gamma$ as $J_l := I_l A$ with

$$A := \{\gamma \in \Gamma \mid d_0(\gamma\overline{\mathcal{F}}, \overline{\mathcal{F}}) \le r\}.$$

Note that A is a finite set and that J_l form also a tempered Følner sequence. Furthermore we have by construction that $\Lambda(I_l) \subset B_r(\Lambda_0(I_l)) \subset \Lambda_0(J_l)$, see [PV02, Proof of Lem. 2.4]. As before, we derive $\limsup_{l\to\infty} N_\omega^{I_l}(E) \le \lim_{l\to\infty} N_{0,\omega}^{J_l}(E)$. The latter limit equals $N(E)$ since Theorem 4 in [LPV] states that for *any* tempered Følner sequence $(J_l)_l$ the finite volume approximations $N_{0,\omega}^{J_l}$ converge to N.

Now we impose *specific assumptions* to describe various situations where we can say something about the continuity of the IDS. We first study the case that

$$H \equiv H_\omega \text{ is a single } \Gamma\text{-periodic operator.} \tag{11}$$

This fits in the general framework of ergodic operators if Ω contains a single element ω.

The following is a basic result in the spectral analysis of Γ-periodic operators. Recall that a measure μ is called a *spectral measure* for the selfadjoint operator H if, for a Borel-measurable subset B of $\mathbb{R}$, the spectral projection $\chi_B(H) = 0$ if and only if $\mu(B) = 0$.

From [LPV] or [LPV04] we infer

Proposition 2.6. *Assume that H is Γ-periodic. Then the IDS of H defined in* (10) *is the distribution function of a spectral measure for H. In particular, the IDS is continuous at E if and only if $E \in \mathbb{R}$ is not an eigenvalue of H.*

If one additionally assumes that the group Γ is *abelian*, much more is known. In fact, for abelian groups Γ, strong regularity properties of the IDS are established in results of Sunada [Sun90] and Gruber [Gru02]. Sunada proves that under a certain additional assumption the spectrum has no point component. Gruber shows that the spectrum has no singularly continuous component. Putting this together one obtains the following result.

Theorem 2.7. *Assume that H is Γ-periodic and let X be the maximal abelian covering of a closed Riemannian manifold M. If the potential V is smooth and M admits a nontrivial S^1-action whose generating vector field is parallel, then H has purely absolutely continuous spectrum and, consequently, the IDS is absolutely continuous.*

Example 2.8. In [Sun90] Sunada considers, as a particular example of a manifold M which satisfies the assumptions of Theorem 2.7, a Riemannian product of a flat torus and a closed manifold.

A more detailed discussion of Γ-periodic operators for abelian Γ can be found in Section 3. After this discussion of the periodic case, we will now deal with instances of random operators. The key result in our analysis of continuity properties of the IDS will be the Wegner estimates discussed below. The first type of random operators is given in the following

Definition 2.9. A family of operators $\{H_\omega\}_\omega$ as in (5) is called ***r**andom Schrödinger operator with **a**lloy type **p**otential* and abbreviated by RAP if it satisfies the following conditions:

(P1) Let $q_\gamma\colon \Omega \to [0,\infty[$, $\gamma \in \Gamma$ be a collection of i.i.d. random variables, whose distribution measure μ has a compactly supported bounded density f with $\operatorname{supp} f \subset [q_-, q_+] \subset [0,\infty[$.

(P2) Let the function, $v\colon X \to \mathbb{R}$, called *single site potential*, satisfy

$$v \geq \lambda \chi_{\mathcal{F}}, \quad v \in L_c^{p(d)}(X, g_0), \tag{12}$$

where λ is some positive real,

$$p(d) \geq 2 \quad \text{if} \quad d \leq 3 \qquad \text{and} \qquad p(d) > d/2 \quad \text{if} \quad d \geq 4.$$

(P3) Define the family of potentials by

$$V_\omega(x) = V_{\mathrm{per}}(x) + \sum_{\gamma\in\Gamma} q_\gamma(\omega) v(\gamma^{-1}x)$$

with $V_{\mathrm{per}} \geq 0$ a bounded periodic potential.

(P4) Let $\{g_\omega\}_\omega$ be a random metric, relatively bounded with respect to g_0, which is independent of the random variables $\{q_\gamma\}_\gamma$.

Here, the random variable q_γ is called *coupling constant* and $\omega \in \Omega$ a *random configuration*.

Note that random Schrödinger operators with alloy type potentials satisfy the conditions (6) and (7).

Remark 2.10. Due to the assumptions in Definition 2.9, the potential V_ω is *uniformly* infinitesimally Δ_ω-bounded in $\omega \in \Omega$, i.e., for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ *independent* of ω such that

$$\|V_\omega u\| \leq \varepsilon \|\Delta_\omega u\| + C_\varepsilon \|u\|$$

where $\|\cdot\|$ denotes the norm on $L^2(X, g_0)$ with respect to the periodic metric g_0.

All our results hold if we replace the condition $V_{\mathrm{per}} \geq 0$ by $V_{\mathrm{per}} \geq c$ for some negative number c. However, it is crucial that the single site potential v does not change sign.

The corresponding Wegner estimate proved in Section 4 reads as follows.

Theorem 2.11 (Wegner estimate for RAP)**.** *Let $\{H_\omega\}_\omega$ be as in Definition 2.9. Then, for all $p > 1$, and $E \in \mathbb{R}$, there exists $C_{E,p} > 0$ such that for all $\varepsilon \in [0, 1/2[$, and $I \subset \Gamma$ finite,*

$$\mathbb{E}\big[\mathrm{Tr}\big(P_\bullet^I([E-\varepsilon, E+\varepsilon])\big)\big] \leq C_{E,p}\, \varepsilon^{1/p}\, (\#I^+), \tag{13}$$

where $I^+ = I^+(v)$ abbreviates

$$I^+(v) := \{\gamma \in \Gamma \mid (\operatorname{supp} v \circ \gamma^{-1}) \cap \Lambda(I) \neq \emptyset\}. \tag{14}$$

The constant $C_{E,p}$ depends only on E, p, the manifold (X, g_0), the constants $C_{\mathrm{rel},k}, k \in \mathbb{N}$ of Definition 2.1, the group Γ, the fundamental domain $\mathcal{F}$, the single site potential v, the supremum of the density $\|f\|_\infty$ and its support $\operatorname{supp} f$ *and $\|V_{\mathrm{per}}\|_\infty$.*

The proof of Theorem 2.11 shows that the same result holds true if we replace the condition (12) by the following weaker assumption

$$v \geq 0, \quad v \in L^{p(d)}_{\mathrm{c}}(X, g_0) \qquad \text{and} \qquad \sum_{\gamma\in\Gamma} v \circ \gamma^{-1} \geq \lambda \quad \text{on } X. \tag{15}$$

Theorems 2.4 and 2.11 immediately imply:

Corollary 2.12 (Hölder continuity of the IDS for RAP)**.** *Under the assumptions of Theorem 2.11 and the amenability of* Γ *the integrated density of states is locally Hölder continuous on* $\mathbb{R}$*, for any Hölder exponent* $1/p$ *strictly smaller than* 1*.*

Now, we consider Laplacians with random metrics as defined in

Definition 2.13. A family of operators $\{H_\omega\}_\omega$ as in (5) is called ***r**andom Laplace operator with **a**lloy type **m**etric* and abbreviated by (RAM) if it satisfies the following conditions:

(M1) Let $r_\gamma \colon \Omega \to \mathbb{R}$, $\gamma \in \Gamma$ be a collection of i.i.d. random variables, whose distribution measure ν has a compactly supported density h of bounded variation.
(M2) Let $u \in C^\infty_{\mathrm{c}}(X)$ with $u \geq \kappa\chi_{\mathcal{F}}$ and $\kappa > 0$ be given.
(M3) Define the family of conformally perturbed Riemannian metrics on X for $\omega \in \Omega$:
$$g_\omega(x) := a_\omega(x) g_0(x) := \Big(\sum_{\gamma\in\Gamma} \mathrm{e}^{r_\gamma(\omega)}\, u(\gamma^{-1}x)\Big) g_0(x). \tag{16}$$
(M4) Let V_ω be identically zero, i.e. $H_\omega = -\Delta_\omega$ for all $\omega \in \Omega$.

In this situation, the random variable r_γ is called *coupling constant*, $\omega \in \Omega$ a *random configuration* and u the *single site deformation*.

Note that the family $\{g_\omega\}_\omega$ is relatively bounded with respect to g_0 and that the constants $C_{\mathrm{rel},k}, k \in \mathbb{N}$ depend only on u, its derivatives and $\operatorname{supp} h$.

To state our Wegner estimate for alloy type metrics, whose proof is given in Section 5, we need one more piece of notation. Namely, for $a \geq 1$, set $J_a = [1/a, a]$.

Theorem 2.14 (Wegner estimate for RAM)**.** *Let* $\{H_\omega\}_\omega$ *with alloy-type metric be given. Then, for every* $p > 1$*,* $a \geq 1$ *there exists* $C_{a,p} > 0$ *such that for every finite* $I \subset \Gamma$

$$\mathbb{E}\big[\mathrm{Tr}\big(P^I_\bullet([E-\varepsilon, E+\varepsilon])\big)\big] \leq C_{a,p}\, \varepsilon^{1/p}\, (\# I^+) \tag{17}$$

whenever $\varepsilon < 1/2$ *and* $[E-\varepsilon, E+\varepsilon] \subset J_a$*. Here* $I^+ = I^+(u)$ *as in* (14)*. The constant* $C_{a,p}$ *depends on* a*,* p*, the manifold* (X, g_0)*, the group* Γ*, the fundamental domain* $\mathcal{F}$*, the single site deformation* u *and the density* h*.*

Note that this Wegner estimate in contrast to Theorem 2.11 does not apply to a neighbourhood of the energy zero. An intuitive explanation for this phenomenon is given in Example 3.3. Similarly as before, Theorems 2.4 and 2.14 imply:

Corollary 2.15 (Hölder continuity of the IDS for RAM)**.** *Under the assumptions of Theorem 2.14 and the amenability of* Γ *the integrated density of states is locally Hölder continuous on* $\mathbb{R}\setminus\{0\}$*, for any Hölder exponent strictly smaller than* 1*.*

In Example 3.3 below, we mention a class of abelian, non-compact covering manifolds (X, g_0), where the corresponding Laplace operator has an L^2-eigenfunction with positive eigenvalue. (Note that this phenomenon does not occur for periodic Schrödinger operators on the Euclidean space.) Consequently, the IDS of the periodic Laplace operator has a discontinuity away from 0, whereas the IDS of the random family RAM is continuous away from 0. In this example, the introducion of randomness improves the regularity of the IDS.

3. Periodic operators on manifolds

In this section, we consider a covering manifold X with *abelian* covering group Γ and Γ-periodic metric g. In this case, all irreducible, unitary representations are one-dimensional. Therefore, the set of their equivalence classes forms a group, called the *dual group* of Γ and denoted by $\hat{\Gamma}$. In particular, if $\Gamma = \mathbb{Z}^r$, we have $\hat{\Gamma} = \mathbb{T}^r$, the r-dimensional torus together with its Haar measure denoted by $\mathrm{d}\theta$.

A periodic Schrödinger operator H admits a direct integral decomposition

$$UHU^* = \int_{\hat{\Gamma}}^{\oplus} H^\theta \,\mathrm{d}\theta$$

where U is a unitary map from $L^2(X) \cong \ell^2(\Gamma) \otimes L^2(\mathcal{F})$ onto $\int_{\hat{\Gamma}}^{\oplus} L^2(\mathcal{F})\,\mathrm{d}\theta \cong L^2(\hat{\Gamma}) \otimes L^2(\mathcal{F})$ acting as a *partial Fourier transformation* on the group part. The operators H^θ, $\theta \in \hat{\Gamma}$ are defined on the set of θ-periodic functions, i.e., functions ψ such that $\psi(\gamma x) = \theta(\gamma)\psi(x)$ for all $x \in X$ and $\gamma \in \Gamma$. It suffices to consider such functions on a fundamental domain $\mathcal{F}$. Since H^θ can be considered as an elliptic operator on a complex line bundle of a compact manifold (cf. the notion of a "twisted" Laplacian in [Sun88]), it has purely discrete spectrum for all $\theta \in \hat{\Gamma}$. We denote by $E_1(\theta) \le E_2(\theta) \le \dots$ the eigenvalues of H^θ in non-decreasing order and including multiplicities.

Let us start with the following proposition giving a formula to calculate the IDS:

Proposition 3.1. *Let* Γ *be an abelian group and* H *be* Γ*-periodic and*

$$N(E) := \frac{1}{\mathrm{vol}\mathcal{F}} \operatorname{Tr}\big(\chi_{\mathcal{F}} \cdot \chi_{]-\infty,E[}(H) \cdot \chi_{\mathcal{F}}\big),$$

where Tr *is the trace in* $L^2(\mathcal{F})$*. Then*

$$N(E) = \frac{1}{\mathrm{vol}\mathcal{F}} \int\limits_{\hat{\Gamma}} \operatorname{Tr}\chi_{]-\infty,E[}(H^\theta)\,\mathrm{d}\theta = \frac{1}{\mathrm{vol}\mathcal{F}} \sum_n \mathrm{meas}\{\theta \in \hat{\Gamma} \mid E_n(\theta) < E\}.$$

Proof. Let $\{\varphi_n\}$ be an orthonormal basis of $L^2(\mathcal{F})$ and $\widetilde{\varphi}_n$ the trivial extension of φ_n in $L^2(X)$. Then $\widetilde{\varphi}_n = \delta_e \otimes \varphi_n$ via the identification $L^2(X) \cong \ell^2(\Gamma) \otimes L^2(\mathcal{F})$.

We have

$$
\begin{aligned}
(\mathrm{vol}\mathcal{F})\, N(E) &= \mathrm{Tr}\big(\chi_{\mathcal{F}} \cdot \chi_{]-\infty,E[}(H) \cdot \chi_{\mathcal{F}}\big) \\
&= \sum_n \langle \widetilde{\varphi}_n, \chi_{]-\infty,E[}(H) \widetilde{\varphi}_n \rangle_{L^2(X)} \\
&= \sum_n \langle U\widetilde{\varphi}_n, U\chi_{]-\infty,E[}(H) U^* U\widetilde{\varphi}_n \rangle_{L^2(\hat{\Gamma}) \otimes L^2(\mathcal{F})}.
\end{aligned}
$$

Here, $U\widetilde{\varphi}_n = U(\delta_e \otimes \varphi_n) = \mathbb{1} \otimes \varphi_n$ where $\mathbb{1}$ is the constant function on $\hat{\Gamma}$ and

$$
U\chi_{]-\infty,E[}(H)U^* = \int_{\hat{\Gamma}}^{\oplus} \chi_{]-\infty,E[}(H^\theta)\, \mathrm{d}\theta.
$$

Therefore, $(\mathrm{vol}\mathcal{F})N(E)$ equals

$$
\begin{aligned}
\sum_n \int_{\hat{\Gamma}} \langle \varphi_n, \chi_{]-\infty,E[}(H^\theta) \varphi_n \rangle_{L_2(\mathcal{F})}\, \mathrm{d}\theta &= \int_{\hat{\Gamma}} \mathrm{Tr}\, \chi_{]-\infty,E[}(H^\theta)\, \mathrm{d}\theta \\
&= \sum_n \int_{\hat{\Gamma}} \langle \varphi_n^\theta, \chi_{]-\infty,E[}(H^\theta) \varphi_n^\theta \rangle_{L_2(\mathcal{F})}\, \mathrm{d}\theta,
\end{aligned}
$$

by Fubini, where $\{\varphi_n^\theta\}_n$ is an orthogonal basis of eigenfunctions of H^θ associated to $E_n(\theta)$ in each fiber. But the latter integral equals the measure of $\{\theta \in \hat{\Gamma} \mid E_n(\theta) < E\}$ and the result follows. □

Proposition 3.2. *Let Γ be an abelian group and H be Γ-periodic. Then the following assertions are equivalent:*

(a) *$E \in \mathbb{R}$ is an eigenvalue of H.*
(b) *$N(\cdot)$ is not continuous at E.*
(c) *There is an $n \in \mathbb{N}$ such that* $\mathrm{meas}\{\theta \in \hat{\Gamma} \mid E_n(\theta) = E\} > 0$.

Proof. The equivalence of (a) and (b) follows from Proposition 2.6. The equivalence of (b) and (c) is an immediate consequence of Proposition 3.1. □

Example 3.3. In [KOS89, Prop. 4] a class of examples of periodic Laplacians on an infinite covering manifold with an L^2-eigenfunction is constructed using Atiyah's L^2-index theorem. This class includes in particular periodic Laplace operators on abelian covering manifolds, e.g., the principal spin-bundle of a connected sum of a $K3$-surface and a 4-dimensional torus.

Note that the corresponding eigenvalue is strictly positive. This can be seen as follows: Brooks' Theorem [Bro81] implies that the bottom of the spectrum is strictly positive for non-amenable groups. For the case of amenable groups the bottom of the spectrum equals 0, but it cannot be an eigenvalue, which follows from [Sar82] and [Sul87], cf. [Sun88, Prop. 3].

4. Wegner estimate for alloy-type potentials

Now we are in the position to prove Theorem 2.11 following Wegner's original idea [Weg81] and using adaptations from [Kir96, Sto00, CHN01].

We will apply the Hellman-Feynman theorem, i.e., first order perturbation theory, cf. [Kat66] or [IZ88], to the purely discrete spectrum of H^I_ω. By assumption (12) in Definition 2.9 the derivatives of the eigenvalues $E_n(\omega) := E^I_n(\omega)$ of H^I_ω obey

$$\sum_{\gamma\in I^+} \frac{\partial}{\partial q_\gamma} E_n(\omega) = \sum_{\gamma\in I^+} \big\langle \psi_n, (v\circ\gamma^{-1})\,\psi_n\big\rangle \geq \lambda. \tag{18}$$

Here ψ_n denotes the eigenfunction corresponding to $E_n(\omega)$ and $I^+ = I^+(v)$ as in (14).

For $0<\varepsilon<1/2$, let $\rho := \rho_{E,\varepsilon}\colon \mathbb{R}\to[-1,0]$ be a smooth, monotone switch function, i.e., ρ satisfies $\rho\equiv -1$ on $]-\infty, E-\varepsilon]$, $\rho\equiv 0$ on $[E+\varepsilon,\infty[$ and $\|\rho'\|_\infty \leq 1/\varepsilon$. Then we have $\chi_{[E-\varepsilon,E+\varepsilon]}(x) \leq \int_{-2\varepsilon}^{2\varepsilon} \rho'(x+t)\,\mathrm{d}t$ and thus by the spectral theorem

$$P^I_\omega([E-\varepsilon,E+\varepsilon]) \leq \int_{-2\varepsilon}^{2\varepsilon} \rho'(H^I_\omega+t)\,\mathrm{d}t.$$

The chain rule implies

$$\sum_{\gamma\in I^+} \frac{\partial}{\partial q_\gamma}\rho(E_n(\omega)+t) = \rho'(E_n(\omega)+t)\sum_{\gamma\in I^+}\frac{\partial}{\partial q_\gamma}E_n(\omega)$$

which is by (18) bounded from below by $\lambda\,\rho'(E_n(\omega)+t)$. Thus we can divide by $\lambda>0$ and obtain $\rho'(E_n(\omega)+t) \leq \frac{1}{\lambda}\sum_{\gamma\in I^+}\frac{\partial}{\partial q_\gamma}\rho(E_n(\omega)+t)$ and consequently

$$\operatorname{Tr}\Big(P^I_\omega([E-\varepsilon,E+\varepsilon])\Big) \leq \frac{1}{\lambda}\int_{-2\varepsilon}^{2\varepsilon}\sum_{n\in\mathbb{N}}\sum_{\gamma\in I^+}\frac{\partial}{\partial q_\gamma}\rho(E_n(\omega)+t)\,\mathrm{d}t.$$

By our independence assumption on the various random ingredients of the model, the expectation value corresponds to an integration with respect to a product measure. The averaging effect we need is produced by integration over a single coupling constant. Afterwards we take expectation over all the remaining randomness.

$$\mathbb{E}\big[\operatorname{Tr} P^I_\bullet\big([E-\varepsilon,E+\varepsilon]\big)\big] \leq \\ \frac{1}{\lambda}\int_{-2\varepsilon}^{2\varepsilon}\mathrm{d}t\sum_{\gamma\in I^+}\mathbb{E}\Big[\int_{q_-}^{q_+} f(q_\gamma)\sum_{n\in\mathbb{N}}\frac{\partial}{\partial q_\gamma}\rho(E_n(\bullet)+t)\,\mathrm{d}q_\gamma\Big].$$

Now, ρ is increasing and E_n is increasing in q_γ. Thus, $\frac{\partial}{\partial q_\gamma}\rho(E_n(\omega)+t)\geq 0$ and the modulus of the $\mathrm{d}q_\gamma$-integral in the square brackets is bounded by

$$\|f\|_\infty\int_{q_-}^{q_+}\sum_{n\in\mathbb{N}}\frac{\partial\rho}{\partial q_\gamma}(E_n(\omega)+t)\,\mathrm{d}q_\gamma = \|f\|_\infty\int_{q_-}^{q_+}\operatorname{Tr}\Big(\frac{\partial\rho}{\partial q_\gamma}(H^I_\omega+t)\Big)\,\mathrm{d}q_\gamma.$$

Here, the integral on the right hand side is equal to

$$\operatorname{Tr}\big(\rho(H_2+t)-\rho(H_1+t)\big) \tag{19}$$

where $H_1 := H^I_\omega + (q_- - q_\gamma(\omega))\cdot(v\circ\gamma^{-1})$, $H_2 := H_1 + (q_+ - q_-)\cdot(v\circ\gamma^{-1})$ and q_-, q_+ denote the two extremal values which the random variable q_γ may take.

Krein's trace identity, see e.g. [BY93], now tells us that (19) equals

$$\int \rho'(\widetilde{E})\, \xi(\widetilde{E}, H_2 + t, H_1 + t)\, \mathrm{d}\widetilde{E} \tag{20}$$

where ξ is the spectral shift function.

In the following definition we introduce a technical piece of notation which plays an important role in the sequel:

Definition 4.1. Let p be the inverse Hölder exponent chosen in Theorem 2.11. Let $0 < \alpha < 1$ be given by $1/p + \alpha = 1$ and $q \in 2\mathbb{N}$ be the smallest even integer satisfying $q \geq \max\{6, d/2 + 2\}$. Finally, k denotes the smallest integer such that $k/q \geq 1/\alpha$ and $g(x) := (x+1)^{-k}$.

Since $H_\omega \geq 0$ for all ω the operator $g(H_\omega)$ is well defined. As discussed in Section 6, $g(H_2) - g(H_1)$ is trace class and even belongs to $\mathcal{J}_\alpha$. Here, $\mathcal{J}_\alpha$ denotes the (super) trace class ideal of compact operators whose singular values are summable to the power α. This class of operators is discussed in more detail at the beginning of Section 6. Note that since $\alpha < 1$ the ideal $\mathcal{J}_\alpha$ is a subset of the trace class ideal.

The invariance principle, see e.g. [BY93], tells us that the modulus of the expression (20) equals

$$\Big| \int \rho'(\widetilde{E}) \xi\big(g(\widetilde{E} - t), g(H_2), g(H_1)\big)\, \mathrm{d}\widetilde{E} \Big|$$

The Hölder inequality for $1/p + \alpha = 1$ gives an upper bound

$$\Big(\int (\rho'(\widetilde{E}))^p \, \mathrm{d}\widetilde{E} \Big)^{1/p} \Big(\int_{\operatorname{supp} \rho'} \big| \xi(g(\widetilde{E} - t), g(H_2), g(H_1)) \big|^{1/\alpha} \, \mathrm{d}\widetilde{E} \Big)^{\alpha}.$$

The first factor can be estimated by

$$\Big(\|\rho'\|_\infty^{p-1} \int \rho'(\widetilde{E})\, \mathrm{d}\widetilde{E} \Big)^{1/p} \leq \varepsilon^{-1+1/p}$$

and the second obeys the upper bound,

$$\Big(\frac{1}{k} (E+2)^{k+1} \int_{\mathbb{R}} \big| \xi(E', g(H_2), g(H_1)) \big|^{1/\alpha} \, \mathrm{d}E' \Big)^{\alpha}.$$

By a result of [CHN01],

$$\Big(\int_{\mathbb{R}} \big| \xi(E', g(H_2), g(H_1)) \big|^{1/\alpha} \, \mathrm{d}E' \Big)^{\alpha} \leq \| g(H_2) - g(H_1) \|_{\mathcal{J}_\alpha}^{\alpha}. \tag{21}$$

The operator $g(H_2) - g(H_1)$ appearing on the right side of (21) is a kind of *effective perturbation*. To estimate its $\|\cdot\|_{\mathcal{J}_\alpha}$-norm we use the following immediate consequence of Theorem 6.1.

Proposition 4.2. *For given $p > 1$ let $\alpha = 1 - 1/p$, k and g be as in Definition 4.1. Furthermore, let $H_1 := H^I_\omega + (q_- - q_\gamma(\omega))\, v \circ \gamma^{-1}$, $H_2 := H_1 + (q_+ - q_-) \cdot (v \circ \gamma^{-1})$ be as defined earlier in this section. Then there exists a constant C_α, which does not depend on ω, I and γ, such that*

$$\| g(H_2) - g(H_1) \|_{\mathcal{J}_\alpha} \leq C_\alpha. \tag{22}$$

Collecting the estimate of this section we obtain the desired result:

$$\mathbb{E}\big[\mathrm{Tr}\big(P^I_\bullet([E-\varepsilon,E+\varepsilon])\big)\big] \le \frac{4\,(C_\alpha)^\alpha \|f\|_\infty}{k^\alpha \lambda}\,(E+2)^{\alpha(k+1)}\,(\#I^+)\,\varepsilon^{1/p}. \qquad (23)$$

5. Wegner estimate for alloy-type metrics

This section is devoted to the proof and discussion of Theorem 2.14. Without loss of generality we may assume $\sum_\gamma u(\gamma^{-1}x) \equiv 1$ on X by replacing simultaneously the single site deformation $u(x)$ by $u(x)/\sum_\gamma u(\gamma^{-1}x)$ and $g_0(x)$ by $g_0(x)\sum_\gamma u(\gamma^{-1}x)$. In the sequel we will tacitly identify Ω and $\times_{\gamma\in\Gamma}\mathbb{R}$ via $\omega = \{r_\gamma(\omega)\}_\gamma$.

The following lemma describes how eigenvalues are moved by a special change of parameters in the random Hamiltonian. It is an analogue of estimate (18).

Lemma 5.1. *Denote by $E_n(\omega) = E^I_n(\omega)$ the eigenvalues of $-\Delta^I_\omega$. Then*

$$\sum_{\gamma\in I^+} \frac{\partial E_n(\omega)}{\partial r_\gamma} = -E_n(\omega)$$

for all $n\in\mathbb{N}$, $\omega\in\Omega$, and $I\subset\Gamma$. Here $I^+ = I^+(u)$ as in (14).

Proof. Since $g_{\omega+t(1,\dots,1)}\restriction_{\Lambda(I)} = \mathrm{e}^t g_\omega\restriction_{\Lambda(I)}$ for $(1,\dots,1)\in\mathbb{R}^{I^+}$, the operator $\Delta^I_{\omega+t(1,\dots,1)}$ is a conformal perturbation of Δ^I_ω with perfactor e^{-t}. Hence

$$E_n(\omega+t(1,\dots,1)) = \mathrm{e}^{-t}E_n(\omega).$$

This gives

$$\sum_{\gamma\in I^+}\frac{\partial E_n(\omega)}{\partial r_\gamma} = \frac{\partial}{\partial t}\Big|_{t=0} E_n(\omega+t(1,\dots,1)) = -E_n(\omega),$$

and the proof is finished. □

If we consider an eigenvalue which is bounded away from zero, the lemma tells us that the absolute value of its derivative has a positive lower bound. Thus it is ensured, that this eigenvalue is moved by the chosen change in the coupling constants. This approach is analogous to the vector field method of [Klo95] and related to Wegner estimates for multiplicative perturbations.

We have to analyze the change of elements $\omega\in\Omega$ at a single coordinate. To do so, we define $\theta^s_\gamma(\omega)$ for $\gamma\in\Gamma$ and $s\in\mathbb{R}$ by

$$\big(\theta^s_\gamma(\omega)\big)_\beta := \begin{cases}\omega_\beta, & \text{for } \beta\neq\gamma,\\ s, & \text{for } \beta=\gamma,\end{cases}$$

i.e., the sequence $\theta^s_\gamma(\omega)$ coincides with ω up to position γ, where its value is s.

Let ρ be as in Section 4. Using Lemma 5.1, the chain rule, and the arguments from Section 4, we obtain for $E_n(\omega)\ge 1/a$

$$0 \le \rho'(E_n(\omega)+t) \le a\Big(-\sum_{\gamma\in I^+}\frac{\partial\rho}{\partial r_\gamma}(E_n(\omega)+t)\Big)$$

and thus the bound

$$\mathbb{E}\big[\operatorname{Tr}\big(P^I_\bullet([E-\varepsilon,E+\varepsilon])\big)\big] \le a\int_{-2\varepsilon}^{2\varepsilon}\sum_{\gamma\in I^+}\mathbb{E}(T_\bullet(\gamma))\,\mathrm{d}t$$

with

$$T_\omega(\gamma) := -\int h(r_\gamma)\sum_{n\in\mathbb{N}}\frac{\partial}{\partial r_\gamma}\rho(E_n(\omega)+t)\,\mathrm{d}r_\gamma.$$

Let $\omega^1 := \theta^0_\gamma(\omega)$. Since ω^1 does not depend on r_γ, we can replace $\frac{\partial}{\partial r_\gamma}\rho(E_n(\omega)+t)$ by $\frac{\partial}{\partial r_\gamma}\rho(E_n(\omega)+t) - \frac{\partial}{\partial r_\gamma}\rho(E_n(\omega^1)+t)$. Such a normalisation is also used in [HK02]. Thus

$$\begin{aligned} T_\omega(\gamma) &= -\int h(r_\gamma)\frac{\partial}{\partial r_\gamma}\Big(\sum_{n\in\mathbb{N}}\rho(E_n(\omega)+t) - \sum_{n\in\mathbb{N}}\rho(E_n(\omega^1)+t)\Big)\,\mathrm{d}r_\gamma \\ &= -\int h(r_\gamma)\frac{\partial}{\partial r_\gamma}\Big(\operatorname{Tr}_\omega\rho(H^I_\omega+t) - \operatorname{Tr}_1\rho(H_1+t)\Big)\,\mathrm{d}r_\gamma. \end{aligned}$$

Here, $H_1 := H^I_{\omega^1}$, Tr_1 denotes the trace in the space $L^2(\Lambda(I), g_{\omega^1})$ and Tr_ω denotes the trace in the space $L^2(\Lambda(I), g_\omega)$. By partial integration for functions of bounded variation, this can be bounded in modulus by

$$\|h\|_{\mathrm{BV}}\max_{s\in\operatorname{supp}h}\Big|\operatorname{Tr}_\omega\rho(H^I_{\theta^s_\gamma(\omega)}+t) - \operatorname{Tr}_1\rho(H_1+t)\Big|. \tag{24}$$

That bounded variation regularity of the density function is sufficient in such a situation was laready noted in [KV06]. Choosing $\widetilde{s}$ such that the maximum is attained and setting $H_2 := H^I_{\omega^2}$, $\omega^2 := \theta^{\widetilde{s}}_\gamma(\omega)$, we can finally bound $|T_\omega(\gamma)|$ by

$$\|h\|_{\mathrm{BV}}\Big|\operatorname{Tr}_2\rho(H_2+t) - \operatorname{Tr}_1\rho(H_1+t)\Big|.$$

Here, Tr_2 denotes the trace in the space $L^2(\Lambda(I), g_{\omega^2})$. To be able to apply the theory of the spectral shift function, we want to transform the two operators H_1 and H_2 into the same Hilbert space. To do so, we use the operator $S = S_{\omega^1,\omega^2}$ defined in (3). It is a multiplication operator given by

$$S\colon L^2(\Lambda(I), g_{\omega^1}) \to L^2(\Lambda(I), g_{\omega^2}), \quad S\varphi(x) = \Big(\frac{a_{\omega^1}}{a_{\omega^2}}\Big)^{d/4}\varphi(x), \tag{25}$$

with a_ω defined in (16). Now both operators $\widetilde{H}_1 := SH_1S^*$ and H_2 act on the same Hibert space $L^2(\Lambda(I), g_{\omega^2})$. Since S is unitary, we have $\operatorname{Tr}_1\rho(H_1+t) = \operatorname{Tr}_2\rho(\widetilde{H}_1+t)$.

Similarly as in Section 4 we can bound $|\operatorname{Tr}_2[\rho(H_2+t) - \rho(\widetilde{H}_1+t)]|$ by

$$\varepsilon^{-1+1/p}\Big(\frac{1}{k}(E+2)^{k+1}\Big)^\alpha\|g(H_2) - g(\widetilde{H}_1)\|^\alpha_{\mathcal{J}_\alpha}.$$

Again we are left to estimate the $\|\cdot\|_{\mathcal{J}_\alpha}$-norm of the effective perturbation $g(H_2) - g(\widetilde{H}_1)$. This is provided by the following direct consequence of Theorem 6.2.

Proposition 5.2. *For given $p > 1$ let $\alpha = 1 - 1/p$, k and g be as in Definition 4.1. Furthermore, let $\widetilde{H}_1 := SH^I_{\omega^1}S^*$ with $\omega^1 := \theta^0_\gamma(\omega)$ and $H_2 := H^I_{\omega^2}$ with $\omega^2 := \theta^{\widetilde{s}}_\gamma(\omega)$ be as defined above. Then there exists a constant $\hat{C}_\alpha$, which does not depend on ω, I and γ, such that*

$$\|g(H_2) - g(\widetilde{H}_1)\|_{\mathcal{J}_\alpha} \leq \hat{C}_\alpha. \tag{26}$$

Thus we obtain, for alloy type metrics, the Wegner estimate

$$\mathbb{E}\big[\mathrm{Tr}\big(P^I_\bullet([E-\varepsilon, E+\varepsilon])\big)\big] \leq \frac{4a\,(\hat{C}_\alpha)^\alpha\,\|h\|_{\mathrm{BV}}}{k^\alpha}(E+2)^{\alpha(k+1)}\,(\#I^+)\,\varepsilon^{1/p}. \tag{27}$$

Example 5.3. While our Wegner estimate *for alloy type potentials* is valid for all bounded energy intervals, in the case of an *alloy type metric* we are only able to prove it for energy intervals away from zero. Let us indicate the reason why our proof does not apply to low energies in the random metric case.

For this we use a simplified example, where the probability space is the one-dimensional interval $[1, 2]$, and the random operator the multiple of the Laplace operator on Euclidean space $H_s = -s\Delta$ for $s \in [1, 2]$. The Fourier transformation of H_s is $f(s, p) := sp^2$, and its derivative with respect to s is p^2. This derivative is positive, except for the value $p = 0$. This shows that moving the perturbation parameter s smears out the spectrum of H_s on any spectral subspace corresponding to energies away from zero. However, the effect of the perturbation parameter on a spectral subspace corresponding to energies around zero can be arbitrarily small. A similar phenomenon occurs in Lemma 5.1 and thus in Theorem 2.14.

6. Trace class bounds on the effective perturbations

In this section we estimate certain effective perturbation operators, which played a crucial role in Sections 4 and 5. More precisely, we want to show that the effective perturbations are in some (super) trace class spaces $(\mathcal{J}_\alpha, \|\cdot\|_{\mathcal{J}_\alpha})$, and need to bound these operators in the $\|\cdot\|_{\mathcal{J}_\alpha}$-topology. More informations on (super) trace class spaces can be found, e.g., in [BS77, Sim79, CHN01].

Olaf: So in Ordnung?

Let us start by shortly introducing the $\|\cdot\|_{\mathcal{J}_\alpha}$-topology: For $\alpha > 0$, $\mathcal{J}_\alpha = \mathcal{J}_\alpha(\mathcal{H})$ is a subspace of the compact operators on a Hilbert space $\mathcal{H}$. For $A \in \mathcal{J}_\alpha$ we define

$$\|A\|_{\mathcal{J}_\alpha} := \Big(\sum_{n\in\mathbb{N}} \mu_n(A)^\alpha\Big)^{1/\alpha}$$

to be the ℓ^α-quasi-norm of the singular values of A. Here we denote by $\mu_n(A)$ the singular values of the operator A. It has the following properties:

- *Quasi-norm property:* We have, for $c \in \mathbb{C}$ and $A, B \in \mathcal{J}_\alpha$:

  $$\|cA\|_{\mathcal{J}_\alpha} = |c|\,\|A\|_{\mathcal{J}_\alpha}$$

  and

  $$\|A+B\|^\alpha_{\mathcal{J}_\alpha} \leq \|A\|^\alpha_{\mathcal{J}_\alpha} + \|B\|^\alpha_{\mathcal{J}_\alpha} \quad \text{for } \alpha \leq 1,$$
  $$\|A+B\|_{\mathcal{J}_\alpha} \leq \|A\|_{\mathcal{J}_\alpha} + \|B\|_{\mathcal{J}_\alpha} \quad \text{for } \alpha \geq 1.$$

These inequalities imply that there are constants $C(\alpha, m) > 0$ such that

$$\Big\|\sum_{j=1}^{m} A_j\Big\|_{\mathcal{J}_\alpha} \le C(\alpha, m)\sum_{j=1}^{m}\|A_j\|_{\mathcal{J}_\alpha}.$$

For $\alpha \ge 1$ one can choose $C(\alpha, m) = 1$.

- *Hölder inequality:* Let $1/\alpha + 1/\beta = 1/\gamma$ for *any* $\alpha, \beta, \gamma > 0$ and $A \in \mathcal{J}_\alpha, B \in \mathcal{J}_\beta$. Then $AB \in \mathcal{J}_\gamma$ and
$$\|AB\|_{\mathcal{J}_\gamma} \le \|A\|_{\mathcal{J}_\alpha}\,\|B\|_{\mathcal{J}_\beta}.$$
- *Ideal property:* Let $A \in \mathcal{J}_\alpha$ and B be a bounded operator on the Hilbert space $\mathcal{H}$. Then we have AB, $BA \in \mathcal{J}_\alpha$ and
$$\|AB\|_{\mathcal{J}_\alpha} \le \|A\|_{\mathcal{J}_\alpha}\,\|B\|, \quad \|BA\|_{\mathcal{J}_\alpha} \le \|B\|\,\|A\|_{\mathcal{J}_\alpha},$$
where $\|\cdot\|$ denotes the usual norm of bounded operators on $\mathcal{H}$.
- *Monotonicity:* For $\alpha \le \beta$ and $A \in \mathcal{J}_\alpha$, we have $A \in \mathcal{J}_\beta$ and $\|A\|_{J_\beta} \le \|A\|_{J_\alpha}$.

For $p > 1$ let $\alpha = 1 - 1/p$, q be even, $k \in \mathbb{N}$ and $g(x) = (1 + x)^{-k}$ be given as in Definition 4.1. With this choice of parameters, the following results hold:

Theorem 6.1. *Let H_ω be a family satisfying RAP. There exists a constant $C_\alpha > 0$, such that*
$$\|g(H^I_{\omega^2}) - g(H^I_{\omega^1})\|_{\mathcal{J}_\alpha} \le C_\alpha,$$
for all subsets $I \subset \Gamma$ and all ω^1,$\omega^2 \in \Omega$ differing in only one coordinate.

Note that $\mathcal{J}_\alpha = \mathcal{J}_\alpha(L^2(X, g_\omega))$ and that $g_\omega = g_{\omega^1} = g_{\omega^2}$ since ω^1 and ω^2 differ only in the coupling constant of the potential.

Theorem 6.2. *Let H_ω be a family satisfying RAM. There exists a constant $\hat{C}_\alpha > 0$, such that*
$$\|g(H^I_{\omega^2}) - g(SH^I_{\omega^1}S^*)\|_{\mathcal{J}_\alpha} \le \hat{C}_\alpha$$
for all subsets $I \subset \Gamma$ and all ω^1, $\omega^2 \in \Omega$ differing in only one coordinate and $S := S_{\omega^1,\omega^2}$, defined in (3).

Note here, that $\mathcal{J}_\alpha = \mathcal{J}_\alpha(L^2(X, g_{\omega^2}))$. The proofs of the two theorems are similar. We only present the proof of Theorem 6.2, since it concerns the more complicated case. Assume that ω^1 and ω^2 differ only in the coordinate $\gamma \in \Gamma$. For simplicity, set

$$H_1 := H^I_{\omega^1}, \quad \widetilde{H}_1 := SH^I_{\omega^2}S^* \quad \text{and} \quad H_2 := H^I_{\omega^2}.$$

Since the single site deformation u is compactly supported, there exists a radius R, such that

$$\widetilde{H}_1\varphi = H_2\varphi \qquad \text{for all} \quad \varphi \in C^\infty_{\rm c}(\Lambda(I) \setminus B_R(\gamma\mathcal{F})),$$

where $B_R(\gamma\mathcal{F})$ denotes the open R-neighborhood of $\gamma\mathcal{F}$ with respect to the metric g_0. Choose $f_0, F_0 \in C^\infty_{\rm c}(X)$ such that

$$f_0\,|_{B_R(\mathcal{F})} \equiv 1, \quad \operatorname{supp} f_0 \subset B_{2R}(\mathcal{F}) \qquad \text{and} \qquad F_0|_{\operatorname{supp} f_0} \equiv 1, \tag{28}$$

Let $f = f_0 \circ \gamma^{-1}$, $F =_0 \circ\gamma^{-1}$ be their γ-translates. Then we have

$$\begin{aligned}
D_{\text{eff}} &:= g(H_2) - g(\widetilde{H}_1) \\
&= - \sum_{m=0}^{k-1} (H_2+1)^{-(k-m)} \, (H_2 - \widetilde{H}_1) \, (\widetilde{H}_1 + 1)^{-(m+1)} \\
&= - \sum_{m=0}^{k-1} (H_2+1)^{-(k-m)} f \, (H_2 - \widetilde{H}_1) \, f (\widetilde{H}_1 + 1)^{-(m+1)} \\
&= - \sum_{m=0}^{k-1} \big[f \, (H_2+1)^{-(k-m)} \big]^* \, (H_2 - \widetilde{H}_1) \, \big[f \, (\widetilde{H}_1 + 1)^{-(m+1)} \big].
\end{aligned}$$

Note that all operators in the previous calculation are defined in the same Hilbert space $L^2(\Lambda(I), g_{\omega^2})$.

By monotonicity, the quasi-norm property and the Hölder inequality, we obtain

$$\begin{aligned}
&\|D_{\text{eff}}\|_{\mathcal{J}_\alpha} \le \|D_{\text{eff}}\|_{\mathcal{J}_{q/k}} \\
&\le C(q/k,k) \sum_{m=0}^{k-1} \|f \, (H_2{+}1)^{-(k-m)}\|_{\mathcal{J}_{q/(k-m)}} \, \|(H_2{-}\widetilde{H}_1) \, f \, (\widetilde{H}_1{+}1)^{-(m+1)}\|_{\mathcal{J}_{q/m}}.
\end{aligned}$$

It remains to estimate each of the terms at the right side, independently of ω, I and γ. We explain this for the most difficult term $\|H_2 f (\widetilde{H}_1 + 1)^{-(m+1)}\|_{\mathcal{J}_{q/m}}$. The term $\|\widetilde{H}_1 f (\widetilde{H}_1 + 1)^{-(m+1)}\|_{\mathcal{J}_{q/m}}$ can be treated similarly, and the term $\|f(H_2 + 1)^{-(k-m)}\|_{\mathcal{J}_{q/(k-m)}}$ is even simpler. In each case we use the following fact, which is in the Euclidean situation essentially due to Nakamura [Nak01]:

Proposition 6.3. *Let $\omega \in \Omega$ and $I \subset \Gamma$ be arbitrary. Let $f, F \in C_c^\infty(X)$ with $F = 1$ on $\operatorname{supp} f$, $R = (H_\omega^I + 1)^{-1}$, and $\nu \in \mathbb{N}$ be fixed. Then we have*

$$f R^\nu = \sum_{i=1}^{N_\nu} \prod_{j=1}^{\nu} f_{ij} R B_{ij} = \sum_{i=1}^{N_\nu} (f_{i1} R B_{i1}) \cdots (f_{i\nu} R B_{i\nu}),$$

where $f_{ij} = F$ for $j < \nu$, the functions $f_{i\nu}$ agree with certain ω-dependent derivatives of f, and the B_{ij} are bounded operators.

There exist a constant $\hat{C}_1(\nu)$, which is independent of ω and I, such that $\|f_{ij}\|_\infty \le \hat{C}_1(\nu)$. The bound $\hat{C}_1(\nu)$ does not change when replacing f, F by any translate $f \circ \gamma^{-1}, F \circ \gamma^{-1}$ with $\gamma \in \Gamma$.

Moreover, there exists a constant $\hat{C}_2$, which is independent of f, F, ν, ω and I, such that $\|B_{ij}\| \le \hat{C}_2$.

We will prove this proposition in full detail in Section 7 and describe f_{ij} and B_{ij} explicitely.

Note that all considerations are carried out in the Hilbert space $L^2(\Lambda(I), g_{\omega^2})$, unless stated otherwise. However, $\|\cdot\|_{\mathcal{J}_\alpha,\omega^1}$ denotes the $\mathcal{J}_\alpha$-norm with respect to the Hilbert space $L^2(\Lambda(I), g_{\omega^1})$ and $\|\cdot\|_{\omega^1}$ is the corresponding operator norm. Furthermore, we introduce $R_1 := (H_1 + 1)^{-1}$.

Let us now return to the study of the term $H_2 f(\widetilde{H}_1+1)^{-(m+1)}$. The spectral theorem and Proposition 6.3 yield

$$\begin{aligned} H_2 f(\widetilde{H}_1+1)^{-(m+1)} &= H_2 S(fR_1^{m+1})S^* \\ &= \sum_{i=1}^{N_{m+1}} H_2 S\Big((f_{i1}R_1B_{i1})\prod_{j=2}^{m+1}(f_{ij}R_1B_{ij})\Big)S^* \\ &= \sum_{i=1}^{N_{m+1}} (H_2 f_{i1}SR_1S^*)\Big(SB_{i1}\Big(\prod_{j=2}^{m+1} f_{ij}R_1B_{ij}\Big)S^*\Big). \end{aligned}$$

Using the quasi-norm property, the ideal property and the Hölder inequality, we obtain

$$\begin{aligned} &\|H_2 f(\widetilde{H}_1+1)^{-(m+1)}\|_{\mathcal{J}_{q/m}} \\ \le\ & C(q/m, N_{m+1}) \sum_{i=1}^{N_{m+1}} \|H_2 f_{i1}SR_1S^*\| \cdot \|SB_{i1}\Big(\prod_{j=2}^{m+1} f_{ij}R_1B_{ij}\Big)S^*\|_{\mathcal{J}_{q/m}} \\ =\ & C(q/m, N_{m+1}) \sum_{i=1}^{N_{m+1}} \|H_2 f_{i1}(\widetilde{H}_1+1)^{-1}\| \cdot \|B_{i1}\prod_{j=2}^{m+1}(f_{ij}R_1B_{ij})\|_{\mathcal{J}_{q/m},\omega^1} \\ \le\ & C(q/m, N_{m+1}) \sum_{i=1}^{N_{m+1}} \|H_2 f_{i1}(\widetilde{H}_1+1)^{-1}\| \cdot \|B_{i1}\|_{\omega^1} \cdot \prod_{j=2}^{m+1} \|f_{ij}R_1B_{ij}\|_{\mathcal{J}_q,\omega^1}. \end{aligned}$$

Note that, by the ideal property of the spaces $\mathcal{J}_q$, we have

$$\|f_{ij}R_1\|_{\mathcal{J}_q,\omega^1} \le \hat{C}_1(\nu)\|FR_1\|_{\mathcal{J}_q,\omega^1}$$

due to Proposition 6.3, since the support of any derivative of f is contained in the support of f. As this proposition also gives $\|B_{ij}\|_{\omega^1} \le \hat{C}_2$, we continue our estimate as follows:

$$\begin{aligned} &\|H_2 f(\widetilde{H}_1+1)^{-(m+1)}\|_{\mathcal{J}_{q/m}} \\ &\le C(q/m, N_{m+1})(\hat{C}_2)^{m+1} \sum_{i=1}^{N_{m+1}} \|H_2 f_{i1}(\widetilde{H}_1+1)^{-1}\| \cdot \prod_{j=2}^{m+1} \|f_{ij}R_1\|_{\mathcal{J}_q,\omega^1} \\ &\le C(q/m, N_{m+1})(\hat{C}_2)^{m+1}\hat{C}_1(m+1)^m \sum_{i=1}^{N_{m+1}} \|H_2 f_{i1}(\widetilde{H}_1+1)^{-1}\|\|FR_1\|^m_{\mathcal{J}_q,\omega^1}. \end{aligned} \tag{29}$$

Note that f_{i1} in the above formula agrees with f or F. Thus the left hand factor in the last sum above can be estimated by the following lemma:

Lemma 6.4. *There exists a constant $C_0 > 0$, independent of ω^1, ω^2, I, and γ such that*

$$\|H_2 f(\widetilde{H}_1+1)^{-1}\|, \|H_2 F(\widetilde{H}_1+1)^{-1}\| \le C_0.$$

where $\|\cdot\|$ is the norm of bounded operators in $L^2(\Lambda(I), g_{\omega^2})$.

Note that γ enters into the definition of f and F, see (28).

Proof. We use the notation of the appendix. Due to the relative boundedness of the family $\{g_\omega\}_\omega$ with respect to the periodic metric g_0, the Sobolev spaces $W^k(\Lambda(I), g_\omega)$ and $W^k(\Lambda(I), g_0)$ are equivalent with constants *independent* of ω. We do not mention these identifications in the rest of the proof.

By Lemma A.8, the operators, given by the multiplication with the smooth functions $f = f_0 \circ \gamma^{-1}$ and $F = F_0 \circ \gamma^{-1}$, are bounded in $W^2(\Lambda(I), g_0)$ with constants obviously independent of I. The independence of γ for $(\Lambda(I), g_0)$ follows by periodicity of the metric g_0.

Moreover, using Theorem A.7 and the uniform infinitesimal boundedness of the potential (see Remark 2.10), the identification operators

$$\mathrm{Id}_1 \colon W^2(\Lambda(I), H_1) \to W^2(\Lambda(I), g_{\omega^1}) \quad \text{and } \mathrm{Id}_2 \colon W^2(\Lambda(I), g_{\omega^2}) \to W^2(\Lambda(I), H_2)$$

are also bounded uniformly in I and ω. Recall the definition of the multiplication operator $S = S_{\omega^1,\omega^2}$ in (3) or (25). It follows from Lemma A.8, that S acting on $W^2(\Lambda(I), g_0)$ (resp. S^* acting on $L^2(\Lambda(I), g_0)$) is uniformly bounded in ω (and in I) by the relatively boundedness of $\{g_\omega\}_\omega$.

Finally, $R_1 \colon L^2(\Lambda(I), g_{\omega^1}) \to W^2(\Lambda(I), H_1)$ is an isometry and

$$H_2 \colon W^2(\Lambda(I), H_2) \to L^2(\Lambda(I), g_{\omega^2})$$

is bounded in norm by 1. The statement of the lemma follows now by writing the two operators as the compositions $H_2 \,\mathrm{Id}_2\, f S \,\mathrm{Id}_1 R_1 S^*$ and $H_2 \,\mathrm{Id}_2\, F S \,\mathrm{Id}_1 R_1 S^*$ of uniformly bounded operators (and the hidden identification of the spaces depending on g_{ω^1}, g_{ω^2}, and g_0). $\square$

For the remaining terms in (29) we use the following lemma:

Lemma 6.5. *There is a constant $C_1 > 0$, independent of ω, I, γ, such that*

$$\|FR_1\|_{\mathcal{J}_q,\omega^1} \le C_1.$$

The proof of Lemma 6.5 is somewhat involved and is presented in Section 8. By Lemmas 6.4 and 6.5, we finally obtain the estimate

$$\|H_2 f(\widetilde{H}_1 + 1)^{-(m+1)}\|_{\mathcal{J}_{q/m}} \le N_{m+1} C(q/m, N_{m+1}) C_0 \hat{C}_2 (\hat{C}_1 (m+1)(C_1)\hat{C}_2))^m.$$

Note that all constants are independent of ω, I and γ. This completes the proof of the uniform boundedness of $\|g(H_2) - g(\widetilde{H}_1)\|_{\mathcal{J}_\alpha}$ up to the proofs of Proposition 6.3 and Lemma 6.5. These proofs are given in the next two sections.

7. Commutator relations and estimates

This section is devoted to the proof of Theorem 7.4 below. It implies Proposition 6.3 and, moreover, provides an explicit description of the operators f_{ij} and B_{ij}. Roughly, we want to rewrite fR^ν as as product of ν factors of the type $f_{ij} R B_{ij}$. The key idea is to use a certain commutator relation iteratively, similarly as in [Nak01].

To clarify some formulae in this section we will occasionally use the notation M_f for the multiplication operator by f. Let $\omega \in \Omega$ and $I \subset \Gamma$ be arbitrary. For simplicity, we drop the dependency on ω, I in this section and write grad, Δ, div, V for grad_ω, Δ_ω, div_ω, V_ω and H for H^I_ω. Only for the metric we keep the notation g_ω to distinguish it from the periodic metric g_0. Recall that we use the convention $\Delta = \operatorname{div} \operatorname{grad} \le 0$. Moreover, let $R := (H+1)^{-1}$.

Lemma 7.1 (Commutator lemma)**.** *For any function $h \in C_c^\infty(X)$ we have*

$$hR = Rh - Rh^{\{1\}}R - R \operatorname{div} h^{\{2\}} R, \tag{30}$$

where $h^{\{1\}} = \Delta h, h^{\{2\}} = -2 \operatorname{grad} h$.

Proof. We first prove

$$[-\Delta, M_h] = M_{h^{\{1\}}} + \operatorname{div} M_{h^{\{2\}}}. \tag{31}$$

This follows from

$$\begin{aligned}
[-\Delta, M_h]\varphi &= -\Delta(h\varphi) + h\Delta\varphi \\
&= -\operatorname{div}(\varphi \operatorname{grad} h + h \operatorname{grad} \varphi) + \operatorname{div}(h \operatorname{grad} \varphi) - g_\omega(\operatorname{grad} h, \operatorname{grad} \varphi) \\
&= -\operatorname{div}(\varphi \operatorname{grad} h) - g_\omega(\operatorname{grad} h, \operatorname{grad} \varphi) \\
&= -2\operatorname{div}(\varphi \operatorname{grad} h) + \varphi \operatorname{div}(\operatorname{grad} h) \\
&= \operatorname{div}(M_{h^{\{2\}}}\varphi) + M_{h^{\{1\}}}\varphi.
\end{aligned}$$

From the resolvent equation we obtain

$$[M_h, R] = R(-\Delta M_h + M_h \Delta)R = R[-\Delta, M_h]R,$$

and, using (31), we conclude that

$$[M_h, R] = R(M_{h^{\{1\}}} + \operatorname{div} M_{h^{\{2\}}})R = RM_{h^{\{1\}}}R + R \operatorname{div} M_{h^{\{2\}}} R,$$

which proves the lemma. ☐

A key idea is to apply the above lemma, a second time, to the expression $h^{\{2\}}R$ in (30). However, $h^{\{2\}}$ is a vector field. We solve this problem by introducing the operators $\operatorname{div}_{i,\beta}$, acting on functions, in the following way: Let $(\psi_\beta)_{\beta \in B}$ be a finite partition of unity on the compact manifold M, i.e.,

$$\sum_{\beta=1}^{n} \psi_\beta = 1.$$

Moreover, for all β, let $X_{1,\beta}, \dots, X_{d,\beta}$ be vector fields which are a local orthonormal frame on the subset $\operatorname{supp} \psi_\beta \subset M$ with respect to the metric g_0.

Let $\pi \colon X \to M$ be the canonical projection and let us denote the periodic lifts $\psi_\beta \circ \pi$ and $X_{i,\beta} \circ D\pi$ on X, again, by ψ_β and $X_{i,\beta}$, for simplicity. Note that every vector field $Z \in C^\infty(TX)$ can be written as

$$Z = \sum_{i,\beta} \psi_\beta \, g_0(Z, X_{i,\beta}) X_{i,\beta},$$

We define the operator $\operatorname{div}_{i,\beta}$ by

$$\operatorname{div}_{i,\beta}(h) := \operatorname{div}(\psi_\beta h X_{i,\beta}) = g_\omega(\psi_\beta X_{i,\beta}, \operatorname{grad} h) + h \operatorname{div}(\psi_\beta X_{i,\beta})$$

and obtain

$$\operatorname{div} Z = \sum_{i,\beta} \operatorname{div}_{i,\beta}(g_0(Z, X_{i,\beta})). \tag{32}$$

Note that the operator div and therefore also $\operatorname{div}_{i,\beta}$ is ω-dependent, since div is defined via the metric g_ω.

Using the differential operators $\mathrm{div}_{i,\beta}$, we can reformulate the above lemma in the following way:

Corollary 7.2. *For any function $h \in C_c^\infty(X)$ we have*

$$hR = Rh + Rh^{\{1\}}R + \sum_{i,\beta} R \,\mathrm{div}_{i,\beta}\, h^{2,i,\beta} R,$$

where $h^{\{1\}} = \Delta h, h^{2,i,\beta} = -2\, g_0(\operatorname{grad} h, X_{i,\beta})$.

Now, we can apply Corollary 7.2 twice and obtain the following result, which is of central importance. In formula (33) below we use the convention that expressions of the form (Dh) denote multiplication operators by the function Dh.

Proposition 7.3. *For any function $h \in C_c^\infty(X)$ we have*

$$\begin{aligned} hR = Rh + R(D^{(1)}h)R + R\,\mathrm{div}_{i,\beta} R(D^{(2,i,\beta)}h) \\ + R\,\mathrm{div}_{i,\beta} R(D^{(3,i,\beta)}h)R + R\,\mathrm{div}_{i,\beta} R\,\mathrm{div}_{j,\mu}(D^{(4,i,\beta,j,\mu)}h)R, \end{aligned} \quad (33)$$

where $D^{(1)}h = \Delta h$, $D^{(2,i,\beta)}h = -2\,g_0(\operatorname{grad} h, X_{i,\beta})$, $D^{(3,i,\beta)}h = D^{(1)}D^{(2,i,\beta)}h$ and $D^{(4,i,\beta,j,\mu)}h = D^{(2,j,\mu)}D^{(2,i,\beta)}h$ are compactly supported function with support contained in $\operatorname{supp} h$.

Note that we used Einstein notation and omitted sum signs, for simplicity.

Proof. A first application of Corollary 7.2 gives

$$hR = Rh + R(D^{(1)}h)R + R\,\mathrm{div}_{i,\beta}\big((D^{(2,i,\beta)}h)R\big).$$

We now apply Corollary 7.2 again to the term $(D^{(2,i,\beta)}h)R$ and obtain the desired statement. □

Now, we can formulate a more detailed version of Proposition 6.3:

Theorem 7.4. *Let $f, F \in C_c^\infty(X)$ with $F = 1$ on $\operatorname{supp} f$ and $\nu \in \mathbb{N}$ be fixed. Then we have*

$$fR^\nu = \sum_{i=1}^{N_\nu} \prod_{j=1}^{\nu} f_{ij} R B_{ij} = \sum_{i=1}^{N_\nu} (f_{i1} R B_{i1}) \cdots (f_{i\nu} R B_{i\nu}). \quad (34)$$

Here, $f_{ij} = F$ for $j < \nu$, and the functions $f_{i\nu}$ are of the form Df, where D is a composition of $\nu-1$ operators of the set $\mathcal{D} := \{\mathrm{Id}, D^{(1)}, D^{(2,i,\beta)}, D^{(3,i,\beta)}, D^{(4,i,\beta,j,\mu)}\}$. Morover, the operators B_{ij} are bounded and of the form BR^l with $B \in \mathcal{B} := \{\mathrm{Id}, R, \mathrm{div}_{i,\beta} R, \mathrm{div}_{i,\beta} R\,\mathrm{div}_{j,\mu}\}$ and $0 \le l \le \nu - 1$.

There is a constant $\hat{C}_1(\nu)$, which does not depend on $\omega \in \Omega$ and $I \subset \Gamma$ such that

$$\|f_{ij}\|_\infty \le \hat{C}_1(\nu). \quad (35)$$

The bound $\hat{C}_1(\nu)$ does not change when replacing f, F by any translate $f \circ \gamma^{-1}, F \circ \gamma^{-1}$ with $\gamma \in \Gamma$.

Finally, there is a constant $\hat{C}_2$, which does not depend on $\nu \in \mathbb{N}$, $\omega \in \Omega$, $I \subset \Gamma$, and $f, F \in C_c^\infty(X)$ such that

$$\|B_{ij}\| \le \hat{C}_2. \quad (36)$$

The proof of this theorem needs some preparation and will be given at the end of this section.

Lemma 7.5. *Let $f \in C_{\mathrm{c}}^{\infty}(X)$ and $\nu \in \mathbb{N}$ be fixed. Then there exists a constant $C_1(\nu) > 0$, independent of ω such that*

$$\|D(f \circ \gamma^{-1})\|_\infty \le C_1(\nu),$$

for all $\gamma \in \Gamma$ and every composition D of $2\nu - 2$ operators of the set $\{\mathrm{Id}, D^{(1)}, D^{(2,i,\beta)}\}$, where $D^{(1)} f = \Delta_\omega f$ and $D^{(2,i,\beta)} f = -2\, g_0(\mathrm{grad}_\omega f, X_{i,\beta})$.

Proof. The dependence of γ can easily be eliminated by the observation, that all operators D satisfy the equivariance condition (6). For example we have $D_\omega^{(2,i,\beta)}(f \circ \gamma^{-1}) = (D_{\gamma^{-1}\omega}^{(2,i,\beta)} f) \circ \gamma^{-1}$ because of

$$\begin{aligned} D_\omega^{(2,i,\beta)} U_{\omega,\gamma} f &= -2 g_0(\mathrm{grad}_\omega(f \circ \gamma^{-1}), X_{i,\beta}) = -2(A_\omega^{-1} X_{i,\beta})(f \circ \gamma^{-1}) \\ &= -2((A_{\gamma^{-1}\omega}^{-1} X_{i,\beta}) f) \circ \gamma^{-1} = U_{\omega,\gamma} D_{\gamma^{-1}\omega}^{(2,i,\beta)} f, \end{aligned}$$

where we used the Γ-periodicity of $X_{i,\beta}$. Therefore, the dependence of γ can be moved into a dependence of ω. The supremum norm estimates follow easily from the observation, that the operators D depend only on g_ω, its derivatives and on $X_{i,\beta}$, which are bounded in a suitable atlas (see (v') of Definition A.1). $\square$

Lemma 7.6. *The operators R, $R \,\mathrm{div}$, $\mathrm{grad}\, R$ and $\mathrm{grad}\, R \,\mathrm{div}$ on $L^2(\Lambda(I), g_\omega)$ are bounded operators with norm ≤ 1.*

Proof. By the spectral theorem, both R and $R^{1/2}$ are bounded by one. Next we prove boundedness of $R^{1/2} \,\mathrm{div}$. For the proof we use the differential form calculus. $\|R^{1/2} \,\mathrm{div}\| \le 1$ translates then into the condition

$$\langle R d^* \eta, d^* \eta \rangle \le \|\eta\|^2$$

for all one-forms $\eta \in \Omega_{\mathrm{c}}^1(\Lambda(I))$ with compact support. Since inversion is a monotone operator function and $-\Delta \le H = -\Delta + V$, we conclude that $R = (H+1)^{-1} \le (-\Delta + 1)^{-1}$, so it remains to prove

$$\langle d(-\Delta + 1)^{-1} d^* \eta, \eta \rangle \le \|\eta\|^2 \qquad \text{for all } \eta \in \Omega_{\mathrm{c}}^1(\Lambda(I)).$$

Adding non-negative terms and using $-\Delta = (d + d^*)^2$, it suffices to prove that

$$\langle (d + d^*)(-\Delta + 1)^{-1}(d + d^*)\eta, \eta \rangle \le \|\eta\|^2 \qquad \text{for all } \eta \in \Omega_{\mathrm{c}}^1(\Lambda(I)),$$

which follows from the spectral theorem applied to the elliptic operator $d + d^* \colon \Omega_{\mathrm{c}}(\Lambda(I)) \to \Omega_{\mathrm{c}}(\Lambda(I))$.

The formal adjoint of $R^{1/2} \,\mathrm{div}$ is $-\,\mathrm{grad}\, R^{1/2}$, so we conclude $\|\,\mathrm{grad}\, R^{1/2}\| \le 1$, and finally $\|\,\mathrm{grad}\, R \,\mathrm{div}\,\| \le 1$, by composition. $\square$

Lemma 7.7. *There is a constant $\hat{C}_2 > 0$, which does not depend on $\omega \in \Omega$ and $I \subset \Gamma$, such that*

$$\|R\|, \ \|\,\mathrm{div}_{i,\beta} R\|, \ \|\,\mathrm{div}_{i,\beta} R \,\mathrm{div}_{j,\mu}\,\| \le \hat{C}_2.$$

Proof. Since by definition

$$\mathrm{div}_{i,\beta} R\varphi = (R\varphi)\,\mathrm{div}(\psi_\beta X_{i,\beta}) + g_\omega(\mathrm{grad}\, R\varphi, \psi_\beta X_{i,\beta}),$$

we conclude with Lemma 7.6 that

$$\begin{aligned}\|\operatorname{div}_{i,\beta} R\varphi\| &\leq \|\operatorname{div}(\psi_\beta X_{i,\beta})\|_\infty \cdot \|R\varphi\| + \|\psi_\beta X_{i,\beta}\|_{\infty,g_\omega} \cdot \|\operatorname{grad} R\varphi\| \\ &\leq \big(\|\operatorname{div}(\psi_\beta X_{i,\beta})\|_\infty + C_{\mathrm{rel},0}^{1/2}\|\psi_\beta X_{i,\beta}\|_{\infty,g_0}\big)\|\varphi\|,\end{aligned}$$

where $C_{\mathrm{rel},0}$ is the uniform quasi-isometry constant in (1) comparing the metrics g_0 and g_ω. Note that $X_{i,\beta}, \psi_\beta$ are periodic and independent of the choices ω, I and that the term $\|\operatorname{div}_\omega(\psi_\beta X_{i,\beta})\|_\infty$ can be uniformly bounded for all $\omega \in \Omega$ by the relative boundedness assumptions on the metrics g_ω.

Similarly,

$$\begin{aligned}&\operatorname{div}_{i,\beta} R \operatorname{div}_{j,\mu} \varphi \\ =\ &\operatorname{div}\big((R \operatorname{div} M_{\psi_\mu X_{j,\mu}}\varphi)\psi_\beta X_{i,\beta}\big) \\ =\ &g_\omega(\psi_\beta X_{i,\beta}, \operatorname{grad} R \operatorname{div} M_{\psi_\mu X_{j,\mu}}\varphi) + M_{\operatorname{div}(\psi_\beta X_{i,\beta})} R \operatorname{div} M_{\psi_\mu X_{j,\mu}}\varphi\end{aligned}$$

implies that

$$\begin{aligned}&\|\operatorname{div}_{i,\beta} R \operatorname{div}_{j,\mu} \varphi\| \\ &\quad\leq \big(C_{\mathrm{rel},0}^{1/2}\|\psi_\beta X_{i,\beta}\|_{\infty,g_0}\, \|\operatorname{grad} R \operatorname{div}\| + \|\operatorname{div}(\psi_\beta X_{i,\beta})\|_\infty\, \|R\operatorname{div}\|\big)\cdot \\ &\qquad\qquad \cdot C_{\mathrm{rel},0}^{1/2}\, \|\psi_\mu X_{j,\mu}\|_{\infty,g_0}\, \|\varphi\| \\ &\quad\leq \big(C_{\mathrm{rel},0}^{1/2}\|\psi_\beta X_{i,\beta}\|_{\infty,g_0} + \|\operatorname{div}(\psi_\beta X_{i,\beta})\|_\infty\big) C_{\mathrm{rel},0}^{1/2}\, \|\psi_\mu X_{j,\mu}\|_{\infty,g_0}\, \|\varphi\|.\end{aligned}$$

□

Proof of Theorem 7.4. We first prove the commutator relation (34) by induction. The equation is obviously satisfied in the case $\nu = 1$ with $N_1 = 1$, $f_{11} = f$, $B_{11} = \mathrm{Id}$. Assume that the equation is true for $\nu - 1$. Using Proposition 7.3 we obtain

$$\begin{aligned}fR^\nu &= F(fR)R^{\nu-1} \\ &= (FR)(fR^{\nu-1}) + (FR)((D^{(1)}f)R^{\nu-1})R \\ &\quad + (FR(\operatorname{div}_{i,\beta} R))((D^{(2,i,\beta)}f)R^{\nu-1}) \\ &\quad + (FR(\operatorname{div}_{i,\beta} R))((D^{(3,i,\beta)}f)R^{\nu-1})R \\ &\quad + (FR(\operatorname{div}_{i,\beta} R \operatorname{div}_{j,\mu}))((D^{(4,i,\beta,j,\mu)}f)R^{\nu-1})R.\end{aligned}$$

Note that each term involved is of the form $(FRB)((\widetilde{D}f)R^{\nu-1})R^s$ with $B \in \mathcal{B}$, $s \in \{0,1\}$ and $\widetilde{D} \in \mathcal{D}$. Using the induction hypothesis we conclude that

$$(\widetilde{D}f)R^{\nu-1} = \sum_{i=1}^{N_{\nu-1}} \prod_{j=1}^{\nu-1} g_{ij} R B_{ij},$$

where g_{ij} is of the form $D\widetilde{D}f$ and D is a composition of $\nu - 2$ operators in $\mathcal{D}$, and the operators B_{ij} are of the form BR^l with $B \in \mathcal{B}$ and $0 \leq l \leq \nu - 2$. This finishes the induction step.

The norm estimates (35) and (36) are easy consequences of Lemmas 7.5 and 7.7. □

8. A trace class estimate of the resolvent

In this final section we prove the following proposition:

Proposition 8.1. *Let $F_0 \in C_c^\infty(X)$ be a fixed smooth function with compact support. For $I \subset \Gamma$ and $\omega \in \Omega$, let $R_\omega^I := (H_\omega^I + 1)^{-1}$. Then there is a constant $C > 0$, independent of ω and I such that*

$$\|F_0\, R_\omega^I\|_{\mathcal{J}_q,\omega} \le C.$$

Recall that Lemma 6.5 claims $\|(F_0 \circ \gamma) R_\omega^I\|_{\mathcal{J}_q,\omega} \le C$, independently of the choice of ω, I and $\gamma \in \Gamma$. This, however, is an immediate consequence of Proposition 8.1 and the equivariance property (6) of the operators H_ω: Using the unitary map $U_{(\gamma\omega,\gamma)} \colon L^2(\Lambda(\gamma I), g_{\gamma\omega}) \to L^2(\Lambda(I), g_\omega)$ for a given $\gamma \in \Gamma$, we conclude that

$$U_{(\gamma\omega,\gamma)}\,(F_0 \circ \gamma)\, R_\omega^I\, U_{(\gamma\omega,\gamma)}^* = F_0\, R_{\gamma\omega}^{\gamma I},$$

and therefore

$$\|(F_0 \circ \gamma) R_\omega^I\|_{\mathcal{J}_q,\omega} = \|F_0 R_{\gamma\omega}^{\gamma I}\|_{\mathcal{J}_q,\gamma\omega}.$$

Hence, Proposition 8.1 implies Lemma 6.5 and we are left with the proof of the proposition.

Proof. The proof of Proposition 8.1 is carried out in three steps:

First Step: Removal of the potential and the $\Lambda(I)$-restriction: Let $R_{0,\omega}^I := (-\Delta_\omega^I + 1)^{-1}$. Using the ideal property, we obtain

$$\|F_0 R_\omega^I\|_{\mathcal{J}_q,\omega} = \|F_0 R_{0,\omega}^I\|_{\mathcal{J}_q,\omega}\, \|(-\Delta_\omega^I + 1) R_\omega^I\|_{L^2(\Lambda(I),g_\omega)} \le C_0 \|F_0 R_{0,\omega}^I\|_{\mathcal{J}_q,\omega}.$$

Note that the constant $C_0 > 0$ is independent of ω and I, because of the uniform infinitesimal Δ_ω-boundedness of the potential V_ω (see Remark 2.10). Therefore if suffices to estimate the potential free case.

From the appendix, we infer that the constants of bounded geometry of the manifolds (X, g_ω) and $(\Lambda(I), g_\omega)$ can be chosen independently of ω and I (see Lemmas A.2 and A.3). Set $R_{0,\omega} = (-\Delta_\omega + 1)^{-1}$. Using the ideal property, we obtain

$$\begin{aligned}\|F_0 R_{0,\omega}^I\|_{\mathcal{J}_q,\omega} &\le \|F_0(-\Delta_\omega + 1)^{-1}\|_{\mathcal{J}_q,\omega}\, \|(-\Delta_\omega + 1)\mathcal{E}(-\Delta_\omega^I + 1)^{-1}\| \\ &= \|F_0 R_{0,\omega}\|_{\mathcal{J}_q,\omega}\, \|(-\Delta_\omega + 1)\mathcal{E}(-\Delta_\omega^I + 1)^{-1}\|,\end{aligned}$$

where the first norm at the right side is a (super-)trace norm of $L^2(X, g_\omega)$ and the second is the operator norm. Here, $\mathcal{E}$ is the extension operator from $W^2(\Lambda(I), \mathcal{A})$ into $W^2(X, \widetilde{\mathcal{A}})$ as given in Theorem A.9.

From the equivalence of the Sobolev norms (see Lemma A.6 and Theorem A.7) and the Sobolev extension Theorem A.9, we conclude that there is another constant $C_1 > 0$ (independent of I and ω) such that

$$\|F_0 R_{0,\omega}^I\|_{\mathcal{J}_q,\omega} \le C_1 \|F_0 R_{0,\omega}\|_{\mathcal{J}_q,\omega}.$$

It remains to prove $F_0 R_{0,\omega} \in \mathcal{J}_q(L^2(X, g_\omega))$ and to derive a uniform estimate for the (super-)trace class norm.

Second Step: Hilbert-Schmidt norm estimate for $F_0(R_{0,\omega})^{q/2}$: Note that $K := \operatorname{supp} F_0 \subset X$ is a compact set. We first convince ourselves that

$$F_0(-\Delta_\omega + 1)^{-q/2} \colon L^2(X, g_\omega) \to W^q(X, -\Delta_\omega)$$

is bounded: $(-\Delta_\omega + 1)^{-q/2} \colon L^2(X, g_\omega) \to W^q(X, -\Delta_\omega)$ is by definition norm-preserving. By Lemma A.8, the multiplication with F_0 is a bounded operator in $W^q(X, g_\omega)$ with norm bounded by a constant C_2 depending only on q and d, and pointwise bounds on $|\nabla^i_\omega F_0|_\omega$, $i = 0, \dots, q$. But the latter can be estimated by ω-independent constants using the constants $C_{\mathrm{rel},i}$ of Definition 2.1 and bounds on $|\nabla^i_0 F_0|_0$.

By the Sobolev embedding Theorem A.10, the identity map $W^q(X, -\Delta_\omega) \to C_{\mathrm{b}}(X)$ is bounded and its norm can be estimated by geometric constants which hold uniformly for all manifolds (X, g_ω); note that $q/2 \geq d/4 + 1$ by Definition 4.1. Consequently,

$$F_0(R_{0,\omega})^{q/2} \colon L^2(X, g_\omega) \to C_{\mathrm{b}}(K)$$

is a bounded operator with norm bounded by a constant $C_3 > 0$, depending only on F_0 and uniform ω-independent geometric constants. Now we can apply Theorem A.11 and obtain that $F_0(R_{0,\omega})^{q/2}$ is Hilbert-Schmidt with norm bounded by $C_3\,(\mathrm{vol}_\omega K)^{1/2} \leq C_3'\,(\mathrm{vol}_0 K)^{1/2}$.

Final Step: Trace class estimate for $F_0 R_{0,\omega}$: Using Lemma 2 of [Bra01] (where J equals the multiplication operator by F_0, $r = 0$, $t = 1$, $u = q/2$, $p = q$, $\alpha = 1$ and $G_\alpha = R_{0,\omega}$), we conclude from the second step that $F_0 R_{0,\omega} \in \mathcal{J}_q((L^2(X, g_\omega))$ and

$$\|F_0 R_{0,\omega}\|^q_{\mathcal{J}_q,\omega} \leq \|F_0\|^{q-2}_\infty \, \|F_0(R_{0,\omega})^{q/2}\|^2_{\mathcal{J}_2,\omega} \leq (C_3')^2 \, \mathrm{vol}_0(\operatorname{supp} F_0) \, \|F_0\|^{q-2}_\infty$$

and were are done. □

Appendix A

In this appendix, we first define several Sobolev spaces and show that they are equivalent under certain geometric assumptions. Most of the material is standard (see e.g. [Eic88, Sch01]). Afterwards we prove an extension theorem and a Sobolev embedding theorem. Note, that it is crucial for our applications, that the involved constants are *independent* of the random parameter ω in the random metric family $\{g_\omega\}_\omega$ and the choice of $I \subset \Gamma$ in the agglomerates $\Lambda(I)$. Finally, we recall a Hilbert-Schmidt norm estimate for operators with continuous kernels.

A.1. **Sobolev spaces on manifolds.**

Suppose that M is a manifold (possibly with boundary). Suppose, in addition, that $\{\varphi_\alpha\}_\alpha$ is an atlas of M with charts $\varphi_\alpha \colon V_\alpha \to U_\alpha$, where U_α is an open cover of M and $V_\alpha \subset [0, \infty[\, \times \mathbb{R}^{d-1}$. Let $\{\chi_\alpha\}_\alpha$ be a subordinated family of smooth functions satisfying $\sum_\alpha \chi^2_\alpha = 1$. Note that $\{\chi^2_\alpha\}_\alpha$ forms a *partition of unity*. We refer to the pair of families $\mathcal{A} := \{\varphi_\alpha, \chi_\alpha\}_\alpha$ as an *atlas*.

Now, we will define three different types of Sobolev spaces. The *local Sobolev space* $W^k(M, \mathcal{A})$ of order k with respect to the atlas $\mathcal{A}$ is given as the space of function with finite norm

$$\|u\|^2_{W^k(M,\mathcal{A})} := \sum_{\alpha \in A} \|\chi_\alpha u_\alpha\|^2_{W^k(V_\alpha)} \tag{37}$$

where $u_\alpha := u \circ \varphi_\alpha$ and the norm on the RHS is the usual Sobolev norm in $\mathbb{R}^d$.

Associated with a Riemannian metric g on M, we define the *global Sobolev space* $W^k(M, g)$ as the space of function with finite norm

$$\|u\|^2_{W^k(M,g)} := \sum_{i=0}^{k} \||\nabla^i_g u|_g\|^2_{L^2(X,g)} \tag{38}$$

where $|\nabla^i_g u|_g$ is the pointwise norm of the ith covariant derivative (in the weak sense) with respect to the metric g.

Finally, associated with a non-negative (self-adjoint) operator H on M (usually $H = -\Delta_M$ or $H = -\Delta_M + V$) we define the *graph norm Sobolev space* with respect to the operator H as $W^k(M, H) := \operatorname{dom}(H + 1)^{k/2}$ with norm

$$\|u\|_{W^k(M,H)} := \|(H + 1)^{k/2} u\|_{L^2(X,g)}. \tag{39}$$

A.2. **Manifolds of bounded geometry.** In the following we provide the general geometric setting for which we will establish our results on Sobolev spaces. We adopt the notion of [Sch96, Sec. 3] or [Sch01]. Denote by $B_M(x, r)$ the open ball of radius r around x in (M, g).

Definition A.1. A Riemannian manifold (M, g) with boundary ∂M is of *bounded geometry* iff the following conditions are fulfilled for constants $r_0 > 0$, and $C_k > 0$ for $k \in \mathbb{N}, k \geq 0$:

(i) The collar map

$$[0, r_0[\times \partial M \to M, \quad (t, x) \mapsto \exp^M_x(t\mathrm{n}_x)$$

is a diffeomorphism onto its image where $\mathrm{n}_x \in T_x M$ is the unit normal inward vector at $x \in \partial M$. Set $\partial_\tau M := \{x \in M \mid d(x, \partial M) < \tau\}$.

(ii) The injectivity radius of ∂M as a $(d-1)$-dimensional manifold is bounded from below by r_0.

(iii) We have normal boundary coordinates at $x_0 \in \partial M$, i.e.,

$$\varphi_{x_0} \colon [0, r_0[\times B_{\partial M}(x_0, r_0) \to M, \quad (t, x) \mapsto \exp^M_x(t\mathrm{n}_x). \tag{40}$$

(iv) The injectivity radius of $M \setminus \partial_{2r_0/3} M$ is bounded from below by $r_0/3$. In particular, (inner) normal coordinates

$$\varphi_x \colon B(0, r_0/3) \to M, \quad v \to \exp^M_x(v)$$

exist where $x \in M \setminus \partial_{2r_0/3} M$.

(v) We have

$$|(\nabla^M)^k R| \leq C_k \quad \text{and} \quad |(\nabla^{\partial M})^k \ell| \leq C_k, \qquad \text{for all } k \geq 0,$$

where ∇^M and $\nabla^{\partial M}$ are the covariant derivatives in M and ∂M, resp., R the Riemann curvature tensor of M and ℓ the second fundamental form of ∂M in M.

We refer to an atlas $\{\varphi_{x_0}, \varphi_x\}$ of the above type (iii) and (iv) as a *normal atlas*.

Note that we can replace (v) by the following condition (cf. [Sch96, Prop. 3.7] and [Sch01, Thm. 2.5 (c)]):

(v') Denote by g_{ij} the metric components in (boundary) normal coordinates and by g^{ij} the components of its inverse. We assume that there exists $C_0' > 0$ such that

$$(C_0')^{-1}|v|^2 \leq \sum_{ij} g_{ij}(x) v_i v_j \leq C_0' |v|^2 \tag{41}$$

for all x in the chart, $v \in \mathbb{R}^d$. Furthermore, we assume that for each $k \in \mathbb{N}, k \geq 1$ there exists a universal constant $C_k' > 0$ such that

$$|D^\kappa g_{ij}(x)| \leq C_k' \quad \text{and} \quad |D^\kappa g^{ij}(x)| \leq C_k'$$

for all x, all multi-indices κ with $|\kappa| \leq k$ and all $k \geq 1$. Here, D^κ denotes the partial derivative with respect to the coordinates.

We now explain how the concept of bounded geometry fits into the framework of relatively bounded families of metrics introduced in Definition 2.1:

Lemma A.2. *Let (X, g_0) be a Riemannian covering manifold with compact quotient. Let $\{g_\omega\}_\omega$ be a family of Riemannian metrics, relatively bounded with respect to g_0. Then (X, g_ω) is of bounded geometry with constants (r_0, C_k)* independent *of ω.*

Proof. Let us first show that (X, g_0) is of bounded geometry. Since X has no boundary, we only have to verify (iv) and (v'): Obviously, the injectivity radius is bounded from below by $\rho_0 > 0$. Furthermore, if we introduce a so-called *periodic atlas*, namely a lift of a finite atlas on the compact quotient, it is clear, by compactness of the quotient and periodicity of the metric, that its components g_{ij} with respect to a periodic atlas fulfill the estimates in (v').

Now, the injectivity radius of (X, g_ω) is still bounded from below by $\rho_0(C_{\mathrm{rel},0})^{-1/2}$, due to (1). The estimate (41) follows similarly. Furthermore, the coordinate derivatives $D^\kappa g_{\omega,ij}$ can be expressed in terms of covariant derivatives $\nabla^l A_\omega$ for all $l \leq |\kappa|$, since $g_\omega(v, v) = g_0(A_\omega v, v)$ and

$$\partial_{i_1} \ldots \partial_{i_k} = \nabla_{\partial_{i_1}} \ldots \nabla_{\partial_{i_k}} + \sum_{|\kappa|<k} p_\kappa \partial^\kappa \tag{42}$$

on tensor fields, where p_κ is a polynomial depending only on the metric g_0 and its first $k-1$ derivatives. Here, ∇ is the covariant derivative with respect to the periodic metric g_0. Finally, the *uniform* bounded geometry of (X, g_ω) follows from (1) and (2). $\square$

Let (X, g_0) be a Riemannian covering manifold with covering group Γ and compact quotient. We fix a (relatively compact) fundamental domain $\mathcal{F}$. For any subset $I \subset \Gamma$ let $\Lambda_0(I)$ be the I-agglomerate defined in (8). Furthermore, let $\Lambda(I)$ be the smoothed version of $\Lambda_0(I)$ as constructed in [Bro81, pp. 593] and satisfying (9).

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Lemma A.3. *Let $\{g_\omega\}_\omega$ be a family of Riemannian metrics on X, relatively bounded with respect to g_0. Then $(\Lambda(I), g_\omega)$ and $(X \setminus \Lambda(I), g_\omega)$ are of bounded geometry with constants (r_0, C_k)* independent *of ω and $I \subset \Gamma$.*

Note that the constants (r_0, C_k) of the previous lemma might differ from the ones found in Lemma A.2.

Proof. After showing that $(\Lambda(I), g_0)$ and $(X \setminus \Lambda(I), g_0)$ are of bounded geometry, the general result follows as in the previous proof. Note that the construction of Brooks yields the following property of the boundaries of the smoothed agglomerates $\Lambda(I)$: There are finitely many relatively compact smooth hypersurfaces $H_1, \dots, H_n \subset X$ with boundaries, such that for every finite $I \subset \Gamma$ the boundary $\partial\Lambda(I)$ can be covered by Γ-translates of these finitely many hypersurfaces, i.e., for each I there exists $N \in \mathbb{N}$ and $\{\gamma_j\}_{1 \le j \le N}$ and a map $\sigma \colon \{1, \dots, N\} \to \{1, \dots, n\}$ such that

$$\partial\Lambda(I) = \bigcup_{j=1}^{N} \gamma_j H_{\sigma(j)}.$$

Note that n (the number of hypersurfaces) does *not* depend on I, and that only *finitely* many hypersurfaces are needed is due to the fact that, up to translates, the local shape of $\partial\Lambda(I)$ depends only on the geometry of $\mathcal{F}$ and its nearest neighbors. This finiteness, together with the periodicity of (X, g_0) ensures that all properties of Definition A.1 (with appropriate constants r_0, C_k) for $(\Lambda(I), g_0)$ and its complement are satisfied. Obviously, the constants r_0 and C_k are independent of I. □

A.3. **Equivalence of Sobolev norms.** To show the equivalence of the local Sobolev norm with the others it is important to ensure that the normal charts φ_{x_0} and φ in Definition A.1 satisfy the following conditions, which we call *admissible*:

Definition A.4. Let M be a differentiable manifold (with or without boundary). An atlas $\mathcal{A} = \{\varphi_\alpha, \chi_\alpha\}_\alpha$ of M with coordinate charts $\varphi_\alpha \colon V_\alpha \to U_\alpha \subset M$ and subordinated functions χ_α is called *admissible* with constants $N_0 \in \mathbb{N}$ and $\hat{C}_k > 0$ for $k \in \mathbb{N}, k \ge 0$, if the following properties are satisfied:

(i) The partial derivatives of all coordinate change maps $\varphi_{\alpha_1,\alpha_2} := \varphi_{\alpha_1}^{-1} \circ \varphi_{\alpha_2}$ are bounded up to all orders, i.e.,

$$|\partial^\kappa \varphi_{\alpha_1,\alpha_2}| \le \hat{C}_k$$

for all multi-indices $|\kappa| \le k$ and all k.

(ii) The multiplicity of the covering of the altas, i.e., the supremum of the number of neighbors of every fixed chart U_α is bounded from above by N_0.

(iii) We have $\sum_\alpha \chi_\alpha^2 = 1$ and

$$|\partial^\kappa \chi_\alpha| \le \hat{C}_k$$

for all multi-indices $|\kappa| \le k$ and all k.

The proof of the following lemma can be found in [Sch96, Prop. 3.8 and 3.22] or [Sch01, Props. 3.2, 3.3]:

Lemma A.5. *Suppose that M (with or without boundary) is of bounded geometry with constants C_k and r_0. Then there are constants $N_0 \in \mathbb{N}$ and $\hat{C}_k > 0$, depending only on C_k and r_0, such that an appropriate subatlas of the normal atlas in Definition A.1 is admissible with constants N_0 and $\hat{C}_k$.*

We now show that the different Sobolev spaces defined in Section A.1 have equivalent norms.

Lemma A.6. *Suppose that (M, g) is of bounded geometry. Then the local Sobolev space $W^k(M, \mathcal{A})$ defined with respect to an admissible atlas $\mathcal{A}$ (see Definition A.4) and the global Sobolev space $W^k(M, g)$ agree and have equivalent norms*

$$\frac{1}{C_{\mathrm{sob}}} \|u\|_{W^k(M,g)} \le \|u\|_{W^k(M,\mathcal{A})} \le C_{\mathrm{sob}} \|u\|_{W^k(M,g)}, \tag{43}$$

where C_{sob} depends only on the constants of bounded geometry, namely C_k and r_0.

Proof. We only sketch the proof. For $k = 0$ this follows immediately from (41). For the higher derivatives we use (42), in order to express partial derivatives recursively by covariant derivatives, as well as the properties of the atlas in Definition A.4. □

Denote by $H = -\Delta_M^{\mathrm{D}} \ge 0$ the Laplace operator on (M, g) with Dirichlet boundary conditions (if $\partial M \neq \emptyset$). Comparing the graph norm Sobolev space defined via H with the global Sobolev spaces needs elliptic estimates.

Theorem A.7. *Suppose that (M, g) is of bounded geometry. Then for $m \ge 0$ the global Sobolev space $W^{2m}(M, g)$ and the graph Sobolev space $W^{2m}(M, -\Delta_M^{\mathrm{D}})$ have equivalent norms*

$$\begin{aligned} \frac{1}{C'_{\mathrm{sob}}} \|u\|_{W^{2m}(M,g)} \le \|u\|_{W^{2m}(M,H)} = \|(-\Delta_M^{\mathrm{D}} + 1)^m u\|_{L^2(M,g)} \\ \le C'_{\mathrm{sob}} \|u\|_{W^{2m}(M,g)} \end{aligned} \tag{44}$$

for $u \in W^{2m}(M, -\Delta_M^{\mathrm{D}})$, where C'_{sob} depends only on the constants of bounded geometry, namely C_k and r_0.

Proof. The second inequality can easily be seen using the local Sobolev space, since Δ_M^{D} contains the metric and its derivative and the fact that the local and global Sobolev spaces have equivalent norms by the last lemma. The proof of the first inequality in the case $\partial M = \emptyset$ can be found e.g. in [Dod81, Thm. 1.3] (with a correction in [Sal01, Sec. 2]). The case with boundary can be found in [Sch96, Sec 4]. □

The next lemma is needed in the proof of Lemma 6.4 and a simple consequence of the product rule:

Lemma A.8. *Let $k \in \mathbb{N}$ be given. Let f be a smooth function on (M, g) such that its covariant derivatives $\nabla^i f$ are pointwise bounded by a constant $C_{f,k}$ for all $i = 0, \dots, k$. Then the multiplication operator*

$$M_f \colon W^k(M, g) \longrightarrow W^k(M, g), \qquad \psi \mapsto f\psi$$

is bounded, and its norm is a universal constant in k and $C_{f,k}$.

A.4. **Extension operators.** Our next result deals with an extension operator. Let (X, g) be a complete d-dimensional Riemannian manifold and $M \subset X$ be a submanifold of the same dimension with smooth boundary. Suppose that M and $X \setminus M$ are of of bounded geometry with constants r_0 and C_k. Then X is also of bounded geometry with the same constants.

Let $\mathcal{A}$ and $\mathcal{A}'$ be normal atlasses of M and $X \setminus M$. An associated atlas $\widetilde{\mathcal{A}}$ of X is given by the inner normal charts of $\mathcal{A}$ and $\mathcal{A}'$ and by extensions of the normal boundary charts $\varphi_{x_0} \colon [0, r_0[\times B_{\partial M}(x_0, r_0) \to M$ of $\mathcal{A}$ to collar maps $\widetilde{\varphi}_{x_0} \colon]-r_0, r_0[\times B_{\partial M}(x_0, r_0) \to X$. Clearly, by Lemma A.5, we can choose an admissible subatlas of $\widetilde{A}$ (and the corresponding subatlas of $\mathcal{A}$). We denote the subatlasses by the same symbols $\mathcal{A}$ and $\widetilde{\mathcal{A}}$.

We denote inner and boundary charts on M by $\varphi_\alpha \colon V_\alpha \to U_\alpha \subset M$ and on X by $\widetilde{\varphi}_\alpha \colon \widetilde{V}_\alpha \to \widetilde{U}_\alpha \subset X$ and similarly, we denote by χ_α and $\widetilde{\chi}_\alpha$ the associated partitions of unity. Note that now, V_α is an open subset of the half-space $\mathbb{R}^d_+ = [0, \infty[\times \mathbb{R}^{d-1}$ whereas $\widetilde{V}_\alpha$ is open in $\mathbb{R}^d$. Clearly, we can assume that also the chart domains satisfy $\widetilde{V}_\alpha \cap \mathbb{R}^d_+ = V_\alpha$.

Now we can define a Sobolev extension operator by a local procedure, using the extension operator $\mathcal{E}_0 \colon W^k(\mathbb{R}^d_+) \to W^k(\mathbb{R}^d)$ on the half-space $\mathbb{R}^d_+$ (see e.g. [GT77, Thm. 7.25]).

Theorem A.9. *Suppose that M and $X \setminus M$ have bounded geometry with constants r_0 and C_l. Suppose, in addition, that $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are atlasses of M and X as defined above. Then, for every $k \in \mathbb{N}$, there exists an extention operator*

$$\mathcal{E} \colon W^k(M, \mathcal{A}) \to W^k(X, \widetilde{\mathcal{A}}), \tag{45}$$

such that $\|\mathcal{E}\|$ only depends on k, C_l and r_0.

Proof. We set

$$\mathcal{E}u(x) := \sum_{\alpha \in \mathcal{A}} \widetilde{\chi}_\alpha(x) \cdot \big(\mathcal{E}_0(\chi_\alpha u_\alpha)\big)(\widetilde{\varphi}_\alpha^{-1} x) \tag{46}$$

for $x \in X$ and $u \in W^k(M, \mathcal{A})$. Note that (46) is well-defined: In a neighborhood of x at most N_0 terms are non-zero, so the sum is essentially finite. In addition, $\chi_\alpha u_\alpha \in W^k(\mathbb{R}^d_+)$ and $(\widetilde{\chi}_\alpha \circ \widetilde{\varphi}_\alpha^{-1}) \cdot \mathcal{E}_0(\chi_\alpha u_\alpha) \in W^k(\widetilde{V}_\alpha)$ with compact support in $\widetilde{V}_\alpha$. Finally, $\mathcal{E}u \in W^k(X)$. Clearly, (46) defines an extension operator.

For the norm estimate, we have

$$\begin{aligned}\|\mathcal{E}u\|^2_{W^k(X,\widetilde{\mathcal{A}})} &= \sum_{\alpha'\in\widetilde{\mathcal{A}}} \|\widetilde{\chi}_{\alpha'}(\mathcal{E}u)_{\alpha'}\|^2_{W^k(\widetilde{V}_{\alpha'})}\\ &= \sum_{\alpha'\in\widetilde{\mathcal{A}}} \Big\|\widetilde{\chi}_{\alpha'} \sum_{\alpha\in\mathcal{A}} \big[\chi_\alpha\cdot\mathcal{E}_0(\chi_\alpha u_\alpha)\big]\circ(\widetilde{\varphi}_\alpha^{-1}\circ\widetilde{\varphi}_{\alpha'})\Big\|^2_{W^k(\widetilde{V}_{\alpha'})}\\ &\le \sum_{\alpha',\alpha} N_0 \Big\|\widetilde{\chi}_{\alpha'}\big[\chi_\alpha\cdot\mathcal{E}_0(\chi_\alpha u_\alpha)\big]\circ(\widetilde{\varphi}_\alpha^{-1}\circ\widetilde{\varphi}_{\alpha'})\Big\|^2_{W^k(V_{\alpha'})}\end{aligned}$$

where the last sum is taken over all $\alpha \in \mathcal{A}$, $\alpha' \in \widetilde{\mathcal{A}}$ such that $\widetilde{U}_\alpha \cap \widetilde{U}_{\alpha'} \neq \emptyset$. Due to Lemma A.5, there are at most N_0 indices α' for a fixed α, and we can estimate the remaining sum (using the product and chain rule) by a constant, depending only on $\hat{C}_k$ and k, multiplied with $N_0^2 \sum_\alpha \|\mathcal{E}_0(\chi_\alpha u_\alpha)\|^2_{W^k(M,\mathcal{A})} \le N_0^2 \|\mathcal{E}_0\|^2 \, \|u\|^2_{W^k(M,\mathcal{A})}$. □

A.5. **Sobolev embedding.** In this subsection, we show that there is a continous embedding of the graph Sobolev space defined with respect to the Laplacian $H := -\Delta_X \geq 0$ into $C_{\mathrm{b}}(X)$, where $C_{\mathrm{b}}(X)$ denotes the space of *bounded* continuous functions on X.

Theorem A.10. *Suppose that (X, g) is a complete d-dimensional manifold with sectional curvature K bounded by $|K(x)| \leq K_0$ and positive injectivity radius $r_0 := \operatorname{inj\,rad} X > 0$. Suppose, in addition, that $m \geq d/4 + 1$. Then the embedding*

$$H^{2m}(X, -\Delta_X) \to C_{\mathrm{b}}(X) \tag{47}$$

is defined and bounded with norm depending only on m, $d = \dim X$, r_0 and K_0.

Proof. The theorem follows directly from

$$\begin{aligned}|\psi(x)| \le c(d) \sum_{i=0}^{m} r^{-d/2+i} \|\Delta_X^i \psi\|_{L^2(B(x,r),g)}\\ \le c'(d,m) r^{-d/2} \|(-\Delta_X + 1)^m \psi\|_{L^2(X,g)},\end{aligned}$$

for $m \geq d/4 + 1$, $x \in X$ and $r \leq \min\{K_0^{-1/2}, r_0, 1\}$ (cf. [CGT82, Prop. 1.3]), and the spectral calculus. Consequently, we obtain

$$\|\psi\|_\infty \le c'(d,m) \max\big\{K_0^{d/4}, r_0^{-d/2}, 1\big\}\, \|\psi\|_{W^{2m}(X,-\Delta_X)}.$$

□

A.6. **A Hilbert-Schmidt norm estimate.** The following standard estimate is used in Section 8. Since we could not find a reference, we present it with a proof, for the reader's convenience.

Let (X, m) be an measurable space such that $L^2(X, m)$ is separable. In addition, let Y be a topological space with finite Borel measure m'. Denote by $C_{\mathrm{b}}(Y)$ the space of bounded continuous functions on Y with supremum norm $\|\cdot\|_\infty$. We denote $J\colon C_{\mathrm{b}}(Y) \to L^2(Y, m')$ the canonical embedding.

Theorem A.11. *Assume that*

$$K\colon L^2(X,m) \to C_{\mathrm{b}}(Y)$$

is a bounded operator. Then the composition $JK\colon L^2(X,m) \to L^2(Y,m')$ *is a Hilbert-Schmidt operator with Hilbert-Schmidt norm bounded by*

$$\|JK\|_{\mathcal{J}_2} \le \|K\|(m'(Y))^{1/2}.$$

Proof. Let $\{\varphi_n\}_n$ be an orthonormal base of $L^2(X,m)$. Since for fixed $y \in Y$, the map $L^2(Y,m') \to \mathbb{C}$, $f \mapsto Kf(y)$ is a bounded functional, there exists $g_y \in L^2(X,m)$ such that $Kf(y) = \langle g_y, f\rangle$ and $\|g_y\| \le \|K\|$. Denoting $c_n(y) := \langle g_y, \varphi_n\rangle$ the Fourier coefficients of g_y, we obtain

$$\sum_n |c_n(y)|^2 = \|g_y\|^2_{L_2} = |Kg_y(y)| \le \|K\|\|g_y\|_{L_2},$$

and conclude that $\sum_n |c_n(y)|^2 \le \|K\|^2$. Moreover, we have

$$Kf(y) = \langle g_y, f\rangle = \sum_n c_n(y)\,\langle\varphi_n, f\rangle\,.$$

The function

$$k_N\colon X \times Y \to \mathbb{C}, \qquad k_N(x,y) := \sum_{n=1}^{N} c_n(y)\varphi_n(x)$$

is obviously measurable and in $L^2(X \times Y)$. Its L^2-norm can be estimated uniformly as

$$\|k_N\|^2_{L^2(X\times Y)} = \sum_{n=1}^{N} \int_Y |c_n(y)|^2 \,\mathrm{d}m'(y) \le \|K\|^2 m'(Y).$$

Similarly, it can be shown that $\{k_N\}_N$ is actually a Cauchy sequence in $L^2(X \times Y)$ with limit k. Denote the operators associated to k_N and k by K_N and $\widetilde{K}$, respectively. It remains to show that $\widetilde{K} = JK$. For $f \in L^2(X,m)$ we have

$$\begin{aligned}\|\tilde{K}f - K_N f\|^2 &= \int_Y \Big|\int_X (k(x,y) - k_N(x,y))f(x)\,\mathrm{d}m(x)\Big|^2 \,\mathrm{d}m'(y)\\ &\le \int_Y \int_X |k(x,y) - k_N(x,y)|^2 \,\mathrm{d}m(x)\|f\|^2_{L^2}\,\mathrm{d}m'(y)\\ &\qquad\qquad \le \|k - k_N\|^2_{L^2(X\times Y)}\|f\|^2_{L^2}.\end{aligned}$$

Passing to a subsequence we conclude that $\lim_{N\to\infty} K_N f(y) \to \tilde{K}f(y)$ for almost all $y \in Y$. On the other hand, we have

$$K_N f(y) = \Big\langle \sum_{n=1}^{N} c_n(y)\varphi_n, f\Big\rangle \to \langle g_y, f\rangle = (Kf)(y)$$

for all $y \in Y$ and hence, $\tilde{K}f(y) = Kf(y)$ for almost all $y \in Y$. Since Kf is continuous and bounded, we conclude $\widetilde{K}f = Kf = JKf$. $\square$

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