# CONVERGENCE OF SECTORIAL OPERATORS ON VARYING HILBERT SPACES

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Abstract. Convergence of operators acting on a given Hilbert space is an old and well studied topic in operator theory. The idea of introducing a related notion for operators acting on varying spaces is natural. However, it seems that the first results in this direction have been obtained only recently, to the best of our knowledge. Here we consider sectorial operators on scales of Hilbert spaces. We define a notion of convergence that generalises convergence of the resolvents in operator norm to the case when the operators act on different spaces and show that this kind of convergence is compatible with the functional calculus of the operator and moreover implies convergence of the spectrum. Finally, we present examples for which this convergence can be checked, including convergence of coefficients of parabolic problems. Convergence of a manifold (roughly speaking consisting of thin tubes) towards the manifold's skeleton graph plays a prominent role, being our main application.

## 1. Introduction

Convergence of operators in the resolvent sense is a classical issue in operator theory. Early results go back, at least implicitly, to Rayleigh and Schrödinger. The first systematic investigations are due to Trotter, Rellich and Kato.

If the operators under consideration arise from sesquilinear forms on a Hilbert space, there are powerful methods available to study convergence of the operators, in particular in the self-adjoint case. In Kato's classical monograph [Kat95, Chapters 8–9] one finds a detailed study of various kinds of convergence with focus on strong and uniform convergence in the resolvent sense and the consequences of the respective convergence for the behaviour of the spectrum. Moreover, Kato gives criteria in terms of the forms that allow to check easily in many situations that a sequence of operators arising from uniformly sectorial forms converge either strongly or uniformly. Those criteria are particularly easy to verify if the forms satisfy some monotonicity assumptions, i.e., they converge from above or from below.

A similar, very successful approach has been developed by Mosco [Mos94] in the context of symmetric Dirichlet forms, i.e., forms associated with sub-Markovian self-adjoint $C_0$-semigroups. His notion of form convergence, which is sometimes known as *$M$-convergence*, modifies (and sometimes improves) the earlier *$\Gamma$-convergence* introduced in [DGF75]. In particular he studies compactness and closure properties with respect to the topologies associated to these notions of convergence. By this, he succeeds in obtaining *strong* resolvent convergence, spectral convergence and convergence of the generated semigroups from simple conditions on the forms, and in fact resolvent convergence can be easily characterised via the forms.

In this article, on the other hand, we are interested in convergence properties *in operator norm* of operators associated with forms that act on varying Hilbert spaces, for example differential operators on varying domains. Such situations have

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been widely studied for elliptic equations on varying domains with respect to several boundary conditions on several spaces, and we refer to Daners' survey article [Dan08] for results for these instances of problems in the spirit of form methods and further references.

In the late 1990s further convergence results in the context of homogenisation problems have been obtained by the post-Soviet school, see [Zhi00, Mel00, Pas04] and references therein. These results mostly rely upon a notion of convergence of sequences living in a family of Hilbert spaces, i.e., of $L^2$-spaces on varying domains or with respect to varying measures. While their results only provide strong convergence to the solution of a concrete elliptic or parabolic problem, an interesting feature is that their methods can also be extended to certain nonlinear settings, cf. [Mel08].

On the other hand, a natural approach to infinite dimensional problems is based on approximation via finite dimensional spaces. This is common in numerical analysis and an abstract theory of such kind of approximations has been recently developed, cf. [IK02]. If in particular one considers diffusion-like processes, form methods are a mighty tool. Convergence schemes for Dirichlet forms on scales of finite dimensional spaces have been considered by Mosco and others, particularly in the context of stochastic diffusion equations and diffusion on fractals, see e.g. [Kol06, FL08, Hin09, AVR09, MV09]. There are similar convergence results for manifolds, metric measure spaces, Hilbert spaces, quadratic forms on different Hilbert spaces in [KS03, Kas02, Kas06]. Though, in these works only the *strong* convergence of the associated operators is considered.

In this article we investigate convergence of m-sectorial operators $A_\varepsilon$, which are allowed to act on different Hilbert spaces $H_\varepsilon$, towards an m-sectorial operator $A_0$ acting on a Hilbert space $H_0$ by form methods. Our notion of form convergence resembles a sufficient condition for convergence of the resolvent in operator norm due to Kato and is designed in a way that allows to check the conditions easily in many applications. The notation is introduced in Section 2. Our main abstract results are contained in Section 3. More precisely, in Section 3.1 we show that if $A_\varepsilon$ converges to $A_0$, then also $\varphi(A_\varepsilon)$ converges to $\varphi(A_0)$ in norm if $\varphi$ is in a suitable class of bounded holomorphic functions (Theorem 3.7). We prove in Section 3.2 that the spectra of $A_\varepsilon$ converge to the spectrum of $A_0$ (Corollary 3.14 and Theorem 3.17). Similar results for self-adjoint operators can be found in [Pos06]. In [Pos09], also convergence of certain non-self-adjoint operators in a specific situation is considered. In Section 3.3 we consider invariance of subsets of the Hilbert spaces and extrapolated semigroups. In particular, if we assume that the Hilbert spaces $H_\varepsilon$ are $L^2$-spaces and the semigroups $(\mathrm{e}^{tA_\varepsilon})_{t\geq 0}$ generated by the $A_\varepsilon$ are bounded on the corresponding $L^\infty$-spaces, then we can prove that under suitable assumptions on the convergence scheme the semigroups $\mathrm{e}^{tA_\varepsilon}$ converge to $\mathrm{e}^{tA_0}$ also as operators on $L^p$ for $p \in [2,\infty)$ (Theorem 3.23).

Section 4 describes several situations to which our results can be applied without much effort. In Section 4.1 we put the Fourier series expansion with respect to eigenvectors into our framework to exhibit the ideas at an elementary example. In Section 4.2 we apply our results in a situation where $A_\varepsilon$ are elliptic operators on a domain whose coefficients converge to the coefficients of an elliptic operator $A_0$. More precisely, we consider generalised Wentzell-Robin boundary conditions, which are a natural candidate for our framework because the natural choice of inner products on the underlying Hilbert space depends on the coefficients even if the Hilbert spaces coincide as sets. In this setting we generalise results of Coclite et al. [CFG$^+$08] and complement those of [CGG08] (Theorem 4.4). In Section 4.3 we adopt a variational approach to elliptic operators whose coefficient may vanish at the boundary, as in [AC10] (Theorem 4.5). Observe that in this situation the limiting Hilbert space differs from the approximating ones – not only with respect to the inner product, but even as a set –, so that Kato's classical results cannot be applied directly.

Our main example, however, is the convergence of tube-like manifolds to the skeleton graph, which we investigate in Section 5. More precisely, we let $H_\varepsilon = L^2(X_\varepsilon)$ for $\varepsilon > 0$, where $X_\varepsilon$ is a manifold consisting of $(m+1)$-dimensional objects resembling tubes (*edge neighbourhoods*) that are connected in $(m+1)$-dimensional junction regions (*vertex neighbourhoods*). If these tubes have a uniform thickness $\varepsilon$, then it is natural to expect that the behaviour of physical processes on $X_\varepsilon$ which are described by an elliptic operator is close to the behaviour of an analogous process on the *skeleton graph* $X_0$, which is a 1-dimensional manifold with singularities at the vertices. We show that under some uniformity assumptions we indeed have resolvent convergence and convergence of finite parts of the spectrum (Theorem 5.9). Note that the convergence results for manifolds and metric measure spaces of Kasue et al. [KS03, Kas02, Kas06] do not apply here, since our families of manifolds $(X_\varepsilon)_\varepsilon$ do not satisfy the necessary curvature bounds (see e.g. [Kas02, p. 1224]).

Robin boundary conditions are closely related to Neumann boundary conditions from the perspective of the quadratic (or, more generally, sesquilinear) form approach. In fact, the form domain is the same, while the forms differs only by a (possibly non-symmetric) boundary term. This allows us to rely on the results in [Pos06] for treating the principal term, so that we only have to handle the boundary term.

One of our motivations for this example is given by recent articles of Grieser [Gri08] and Cacciapuoti–Finco [CF08]. Grieser considered general boundary conditions (Dirichlet, Robin or Neumann) on a manifold (if embedded, the embedding is "straight") shrinking to a metric graph. He showed that the limit behaviour depends on the scattering matrix at the threshold of the essential spectrum, so that, generically, the limit operator is decoupling. Cacciapuoti and Finco use a simple wave-guide model (in our terminology, a flat manifold converging to a graph consisting of two (half-infinite) edges and one vertex only). Using curved embedded edges with different scalings of the transversal and longitudinal curvature, they obtain non-trivial couplings starting with Robin boundary conditions. However, their notion of convergence differs significantly from ours since one can use separation of variables due to the simple product topology of the space. For the convergence of unitary groups in a similar setting we refer to the recent work of Teufel and Wachsmuth [TW09].

Grieser and Cacciapuoti-Finco use scale-invariant Robin boundary conditions of the form $\frac{\partial u}{\partial \nu} = \beta_\varepsilon u$ with $\beta_\varepsilon = \beta/\varepsilon$. This scaling leads to transversal eigenvalues of the order $\varepsilon^{-2}$. In particular, a rescaling of the limit operator is necessary in order to expect convergence, see Remark 5.2. Using Robin boundary conditions with coupling of order $\beta_\varepsilon = \mathrm{O}(1)$ near the vertices and $\beta_\varepsilon = \mathrm{O}(\varepsilon^{3/2})$ along the edge neighbourhoods, we are able to construct a family of manifolds with boundary, such that, in the limit, the corresponding Laplacians converge to a Laplacian on the underlying metric graph with generalised, possibly non-local $\delta$-interactions in the vertices. Using the same idea as in [EP09], we can further approximate other couplings like the $\delta$'-interaction.

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## 2. Notation

We consider m-sectorial operators (in the sense of Kato) on Hilbert spaces. For our approach, it is convenient to work with such an operator in terms of its associated form. We briefly sketch the correspondence of m-sectorial operators and sesquilinear forms. For these results and much more information we refer to [Kat95, Chapter VI].

We point out that there is a one-to-one correspondence between bounded, $H$-elliptic forms and m-sectorial operators, so there is no loss of generality in working with an m-sectorial operator only in terms of its form.

Let $H$ be a Hilbert space and let $V$ be a dense subspace of $H$ that is a Hilbert space in its own right, and which is continuously embedded into $H$. We say that a sesquilinear form $a\colon V \times V \to \mathbb{C}$ is *bounded* if there exists $M \geq 0$ such that

$$|a(u,v)| \leq M\|u\|_V\|v\|_V \qquad \text{for all } u,v \in V, \tag{2.1}$$

and we call $a$ *$H$-elliptic* or simply *elliptic* if there exist $\omega \in \mathbb{R}$ and $\alpha > 0$ such that

$$\operatorname{Re} a(u,u) + \omega\|u\|_H^2 \geq \alpha\|u\|_V^2 \qquad \text{for all } u \in V. \tag{2.2}$$

In this case

$$\|u\|_a := \sqrt{\operatorname{Re} a(u,u) + \omega\|u\|_H^2}$$

defines an equivalent norm on $V$. More precisely, since $V$ is continuously embedded into $H$, there exists $c \geq 0$ such that

$$\|u\|_H \leq c_V\|u\|_V \qquad \text{for all } u \in V. \tag{2.3}$$

For any such constant $c_V$, we obtain

$$\alpha\|u\|_V^2 \leq \|u\|_a^2 \leq \bigl(M + c_V^2\omega\bigr)\|u\|_V^2 \qquad \text{for all } u \in V. \tag{2.4}$$

We define the *associated operator* $A$ of $a$ by

$$u \in D(A) \text{ and } Au = f \quad :\Longleftrightarrow \quad u \in V \text{ and } a(u,v) = \langle f|v\rangle_H \ \forall v \in V,$$

and we emphasise that since the form $a$ is not assumed to be symmetric, the associated operator $A$ is in general not self-adjoint.

Consider for a moment the form $b\colon V \times V \to \mathbb{C}$ given by

$$b(u,v) := a(u,v) + \omega\langle u|v\rangle_H,$$

which is associated with the operator $A + \omega$. Then by (2.1) and (2.4)

$$|\operatorname{Im} b(u,u)| = |\operatorname{Im} a(u,u)| \leq |a(u,u)| \leq M\|u\|_V^2 \leq \frac{M}{\alpha}\|u\|_a^2 = \frac{M}{\alpha} b(u,u).$$

The proof of [Ouh05, Theorem 1.53] now shows that $\sigma(A+\omega) \subset \overline{\Sigma}_{\arctan\frac{M}{\alpha}}$, where

$$\Sigma_\theta := \bigl\{\, z \in \mathbb{C} \setminus \{0\} \;:\; |\arg(z)| < \theta \,\bigr\}. \tag{2.5}$$

Moreover, denoting here and in the following

$$R(z,A) := (z - A)^{-1},$$

for every $\theta \in (\arctan\frac{M}{\alpha}, \pi]$ we have

$$\|zR(z, A+\omega)\|_{\mathscr{L}(H)} \leq D_\theta \quad \text{for all } z \notin \Sigma_\theta,$$

i.e., $\sigma(A) \subset -\omega + \Sigma_{\arctan\frac{M}{\alpha}}$ and

$$\|R(z,A)\|_{\mathscr{L}(H)} \leq \frac{D_\theta}{|z+\omega|} \quad \text{for all } z \notin \Sigma_\theta - \omega \tag{2.6}$$

with

$$D_\theta := \frac{1}{\sin(\theta - \arctan\frac{M}{\alpha})}.$$

Operators satisfying such a condition are frequently called *m-sectorial* (in the sense of Kato).

**Definition 2.1.** Let $(H_\varepsilon)_{\varepsilon\geq 0}$ be a family of Hilbert spaces. We say that $(a_\varepsilon)_{\varepsilon\geq 0}$ is an *equi-sectorial family of sesquilinear forms with form domains* $(V_\varepsilon)_{\varepsilon\geq 0}$, if there exist $M$, $\omega$, $\alpha$ and $c_V$ not depending on $\varepsilon$ such that (2.1), (2.2) and (2.3) are satisfied for all $\varepsilon \geq 0$, i.e., all the constants are uniform with respect to $\varepsilon$.

*Remark* 2.2. If $(a_\varepsilon)_{\varepsilon\ge 0}$ is an equi-sectorial family of sesquilinear forms, then by (2.4) the norms $\|\cdot\|_{V_\varepsilon}$ and $\|\cdot\|_{a_\varepsilon}$ are equivalent with a uniform constant. This allows us to use either of these two norm interchangeably in the following. For the theoretical part, the form norm is more convenient. But for applications, we usually prefer to equip $V_\varepsilon$ with other norms that are easier to handle.

Now let $(a_\varepsilon)_{\varepsilon\ge 0}$ be a family of equi-sectorial family of sesquilinear forms on Hilbert spaces $(H_\varepsilon)_{\varepsilon\ge 0}$. We want to "measure" the distance between the associated operators $(A_\varepsilon)_{\varepsilon>0}$ and $A_0$. For this, we introduce *identification operators* $J^{\uparrow\varepsilon}\colon H_0 \longrightarrow H_\varepsilon$ and $J^{\downarrow\varepsilon}\colon H_\varepsilon \longrightarrow H_0$, $\varepsilon>0$, which are considered to be "almost unitary", i.e., unitary up to some error. For technical reasons, it is also convenient to introduce identification operators $J_1^{\uparrow\varepsilon}\colon V_0 \longrightarrow V_\varepsilon$ and $J_1^{\downarrow\varepsilon}\colon V_\varepsilon \longrightarrow V_0$ for the form domains, which are considered to be "almost the restrictions" of $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$ to $V_0$ and $V_\varepsilon$, respectively.

We make this more explicit and use the following terminology, inspired by the technique developed in [Pos06, Appendix A] and [Pos09] (see also [EP09]).

**Definition 2.3.** Let $\varepsilon>0$, and let $a_0$ and $a_\varepsilon$ be bounded, elliptic, sesquilinear forms on Hilbert spaces $H_0$ and $H_\varepsilon$ with form domains $V_0$ and $H_0$. Denote the associated operators by $A_0$ and $A_\varepsilon$, respectively. For parameters $\delta_\varepsilon>0$ and $\kappa\ge 1$ we say that $A_0$ and $A_\varepsilon$ are *$\delta_\varepsilon$-$\kappa$-quasi-unitarily equivalent* if there exist bounded operators $J^{\uparrow\varepsilon}\in\mathscr{L}(H_0,H_\varepsilon)$, $J^{\downarrow\varepsilon}\in\mathscr{L}(H_\varepsilon,H_0)$, $J_1^{\uparrow\varepsilon}\in\mathscr{L}(V_0,V_\varepsilon)$ and $J_1^{\downarrow\varepsilon}\in\mathscr{L}(V_\varepsilon,V_0)$ that satisfy the following conditions.

$$\|J^{\uparrow\varepsilon}-J_1^{\uparrow\varepsilon}\|_{\mathscr{L}(V_0,H_\varepsilon)}\le\delta_\varepsilon \quad\text{and}\quad \|J^{\downarrow\varepsilon}-J_1^{\downarrow\varepsilon}\|_{\mathscr{L}(V_\varepsilon,H_0)}\le\delta_\varepsilon; \tag{2.7a}$$

$$\|J^{\downarrow\varepsilon}-(J^{\uparrow\varepsilon})^*\|_{\mathscr{L}(H_\varepsilon,H_0)}\le\delta_\varepsilon \quad\text{and}\quad \|J^{\uparrow\varepsilon}-(J^{\downarrow\varepsilon})^*\|_{\mathscr{L}(H_0,H_\varepsilon)}\le\delta_\varepsilon; \tag{2.7b}$$

$$\|\mathrm{id}-J^{\downarrow\varepsilon}J^{\uparrow\varepsilon}\|_{\mathscr{L}(V_0,H_0)}\le\delta_\varepsilon \quad\text{and}\quad \|\mathrm{id}-J^{\uparrow\varepsilon}J^{\downarrow\varepsilon}\|_{\mathscr{L}(V_\varepsilon,H_\varepsilon)}\le\delta_\varepsilon; \tag{2.7c}$$

$$\|J^{\uparrow\varepsilon}\|_{\mathscr{L}(H_0,H_\varepsilon)}\le\kappa \quad\text{and}\quad \|J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon,H_0)}\le\kappa; \tag{2.7d}$$

$$\bigl|a_0(f,J_1^{\downarrow\varepsilon}u)-a_\varepsilon(J_1^{\uparrow\varepsilon}f,u)\bigr|\le\delta_\varepsilon\|f\|_{V_0}\|u\|_{V_\varepsilon}\quad\text{for all } f\in V_0 \text{ and } u\in V_\varepsilon. \tag{2.7e}$$

If $(a_\varepsilon)_{\varepsilon\in\ge 0}$ is an equi-sectorial family of sesquilinear forms and if there exists $\kappa\ge 1$ and a family $(\delta_\varepsilon)_{\varepsilon>0}$ of positive real numbers with $\lim_{\varepsilon\to 0}\delta_\varepsilon\to 0$ such that $A_\varepsilon$ is $\delta_\varepsilon$-$\kappa$-quasi-unitarily equivalent to $A_0$, then we say that the family $(A_\varepsilon)_{\varepsilon>0}$ *converges to* $A_0$ as $\varepsilon\to 0$.

*Remark* 2.4.

(i) For $\delta_\varepsilon=0$ the operators $A_0$ and $A_\varepsilon$ are unitarily equivalent. In fact, if $\delta_\varepsilon=0$, conditions (2.7b) and (2.7c) states that $J^{\uparrow\varepsilon}$ is unitary with inverse $J^{\downarrow\varepsilon}$. Since by (2.7a) the operators $J_1^{\uparrow\varepsilon}$ and $J_1^{\downarrow\varepsilon}$ are the restrictions of $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$, condition (2.7e) states that $J^{\uparrow\varepsilon}$ realises the unitary equivalence of $A_0$ and $A_\varepsilon$.

(ii) In applications, it typically is easy to check that $J_1^{\uparrow\varepsilon}\colon V_0\longrightarrow V_\varepsilon$ and $J_1^{\downarrow\varepsilon}\colon V_\varepsilon\longrightarrow V_0$ are bounded: if $J_1^{\uparrow\varepsilon}$ is bounded as an operator into $H_\varepsilon$ and takes values in $V_\varepsilon$, then it is bounded as an operator into $V_\varepsilon$ by the closed graph theorem, and an analogous argument applies to $J_1^{\downarrow\varepsilon}$.

(iii) The two conditions in (2.7b) are equivalent to each other. In fact, they can be rephrased as

$$|\langle J^{\uparrow\varepsilon}f|u\rangle_{H_\varepsilon}-\langle f|J^{\downarrow\varepsilon}u\rangle_{H_0}|\le\delta_\varepsilon\|f\|_{H_0}\|u\|_{H_\varepsilon}\ \text{for all } f\in H_0 \text{ and } u\in H_\varepsilon. \tag{2.7b'}$$

(iv) Condition (2.7c) does *not* imply that $J^{\downarrow\varepsilon}J^{\uparrow\varepsilon}$ or $J^{\uparrow\varepsilon}J^{\downarrow\varepsilon}$ are invertible operators. In fact, in most of our examples one of the two operators will have a large kernel, whereas the other has a small range.

(v) Only (2.7e) depends on the evolution processes acting on the scale $H_\varepsilon$, while the first four conditions are solely related to the function spaces. So if we have verified (2.7a)–(2.7d) in one situation, those conditions are satisfied for a large class of examples. To be more specific, we can reuse the results obtained in [Pos06] for Neumann boundary conditions and do not have to check these four conditions once again for the discussion of the Laplace operator with Robin boundary conditions in Section 5.

**Example 2.5.** Let $a_0$ and $a_\varepsilon$ be forms on a single Hilbert space $H$ with equal form domain $V$ and let $J^{\uparrow\varepsilon}$, $J_1^{\uparrow\varepsilon}$, $J^{\downarrow\varepsilon}$ and $J_1^{\downarrow\varepsilon}$ be the identity on $H$ resp. $V$. Then the conditions of Definition 2.3 are satisfied if and only if

$$|a_0(u,v) - a_\varepsilon(u,v)| \le \delta_\varepsilon \|u\|_V \|v\|_V \tag{2.8}$$

for all $u, v \in V$.

In the setting of Example 2.5, if (2.8) is satisfied for a family $(\delta_\varepsilon)_{\varepsilon>0}$ satisfying $\lim_{\varepsilon\to 0} \delta_\varepsilon = 0$, then the resolvent of $A_\varepsilon$ converges to the resolvent of $A_0$ in operator norm uniformly on compact subsets of $\varrho(A_0)$. In fact, it would suffice if (2.8) is satisfied whenever $u = v$, see [Kat95, Theorem VI.3.6]. In this sense, our results are a generalisation of this classical result to the setting of varying spaces. We can also deduce similar consequences like in the classical situation, e.g. convergence of the spectra.

## 3. Abstract results

For the whole section, let $(a_\varepsilon)_{\varepsilon\ge 0}$ be an equi-sectorial family of sesquilinear forms for constants $M$, $\omega$, $\alpha$ and $c_V$ as in (2.1), (2.2) and (2.3), and let $(A_\varepsilon)_{\varepsilon\ge 0}$ denote the associated operators. We always let the operators $J^{\uparrow\varepsilon}$, $J^{\downarrow\varepsilon}$, $J_1^{\uparrow\varepsilon}$ and $J_1^{\downarrow\varepsilon}$ and the constant $\kappa$ be as in Definition 2.3.

3.1. **Functional calculus.** In our situation, each operator $A_\varepsilon + \omega$ is invertible by the Lax-Milgram theorem due to (2.2). It is known that in this situation the operators $A_\varepsilon + \omega$ have bounded $H^\infty$-calculus, see [KW04, §11], [Are04, §5.2] or [Haa06, §7.3]. We are going to show that under the conditions of Section 2, for an admissible holomorphic function $\varphi$ the operators $\varphi(A_\varepsilon)$ converge to $\varphi(A_0)$ as $\varepsilon \to 0$.

We start with some auxiliary estimates. For brevity, in the proofs we write

$$R_\varepsilon(z) := R(z, A_\varepsilon) = (z - A_\varepsilon)^{-1}.$$

**Lemma 3.1.** *Let $\theta \in (\arctan \frac{M}{\alpha}, \pi]$. There exists $C_\theta \ge 0$ such that for all $\varepsilon \ge 0$ and $z \notin \Sigma_\theta - \omega$*

$$\|R(z, A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon, V_\varepsilon)} \le \frac{C_\theta}{\sqrt{|z+\omega|}} \quad \text{and} \quad \|R(z, A_\varepsilon)^*\|_{\mathscr{L}(H_\varepsilon, V_\varepsilon)} \le \frac{C_\theta}{\sqrt{|z+\omega|}}.$$

*Proof.* Let $u \in H_\varepsilon$ be fixed. Then by (2.4)

$$\begin{aligned}
\alpha \|R_\varepsilon(z)u\|_{V_\varepsilon}^2 &\le \|R_\varepsilon(z)u\|_{a_\varepsilon}^2 = \operatorname{Re} a_\varepsilon(R_\varepsilon(z)u, R_\varepsilon(z)u) + \omega \|R_\varepsilon(z)u\|_{H_\varepsilon}^2 \\
&= \operatorname{Re}\langle(\omega + A_\varepsilon)R_\varepsilon(z)u | R_\varepsilon(z)u\rangle_{H_\varepsilon} \\
&= \operatorname{Re}\langle(\omega + z)R_\varepsilon(z)u | R_\varepsilon(z)u\rangle_{H_\varepsilon} - \operatorname{Re}\langle u | R_\varepsilon(z)u\rangle_{H_\varepsilon} \\
&\le \big(|\omega + z| \cdot \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} + 1\big) \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} \|u\|_{H_\varepsilon}^2
\end{aligned}$$

Now (2.6) implies the first estimate for

$$C_\theta := \sqrt{\frac{(1 + D_\theta)D_\theta}{\alpha}}.$$

The second estimate can be proved like the first. In fact, $R(z, A_\varepsilon)^* = R(\overline{z}, A_\varepsilon^*)$ and $A_\varepsilon^*$ is associated with the form $a_\varepsilon^*$ given by

$$a_\varepsilon^*(u,v) := \overline{a(v,u)}.$$

Thus it suffices to realise that $a_\varepsilon^*$ satisfies (2.1) and (2.2) for the same constants as $a_\varepsilon$. ☐

**Lemma 3.2.** *Let $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$ be as in Definition 2.3, and let $B_\varepsilon \in \mathscr{L}(H_\varepsilon, V_\varepsilon)$ and $B_0 \in \mathscr{L}(H_0, V_0)$. Then*

$$\|B_\varepsilon - J^{\uparrow\varepsilon} B_0 J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} \leq \kappa \|J^{\downarrow\varepsilon} B_\varepsilon - B_0 J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon, H_0)} + \delta_\varepsilon \|B_\varepsilon\|_{\mathscr{L}(H_\varepsilon, V_\varepsilon)}.$$

*Proof.* By (2.7c) and (2.7d) we have

$$\begin{aligned}\|B_\varepsilon - J^{\uparrow\varepsilon} B_0 J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} &\leq \|B_\varepsilon - J^{\uparrow\varepsilon} J^{\downarrow\varepsilon} B_\varepsilon\|_{\mathscr{L}(H_\varepsilon)} + \|J^{\uparrow\varepsilon} J^{\downarrow\varepsilon} B_\varepsilon - J^{\uparrow\varepsilon} B_0 J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} \\ &\leq \delta_\varepsilon \|B_\varepsilon\|_{\mathscr{L}(H_\varepsilon, V_\varepsilon)} + \kappa \|J^{\downarrow\varepsilon} B_\varepsilon - B_0 J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon, H_0)}.\end{aligned}$$ ☐

**Lemma 3.3.** *Let $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$ be as in Definition 2.3, and let $f \in V_0$ and $u \in H_\varepsilon$. Then*

$$\|J^{\downarrow\varepsilon} u - f\|_{H_0} \leq \kappa \|u - J^{\uparrow\varepsilon} f\|_{H_\varepsilon} + \delta_\varepsilon \|f\|_{V_0}.$$

*Proof.* Let $g \in H_0$. By (2.7c) and (2.7d)

$$\begin{aligned}\left|\langle J^{\downarrow\varepsilon} u - f | g\rangle_{H_0}\right| &\leq \left|\langle J^{\downarrow\varepsilon}(u - J^{\uparrow\varepsilon} f) | g\rangle_{H_\varepsilon}\right| + \left|\langle J^{\downarrow\varepsilon} J^{\uparrow\varepsilon} f - f | g\rangle_{H_\varepsilon}\right| \\ &\leq \kappa \|u - J^{\uparrow\varepsilon} f\|_{H_\varepsilon} \|g\|_{H_0} + \delta_\varepsilon \|f\|_{V_0} \|g\|_{H_0}.\end{aligned}$$

Since $g$ is arbitrary, this proves the claim. ☐

Now we prove the key estimate of this section.

**Proposition 3.4.** *Let $A_\varepsilon$ and $A_0$ be $\delta_\varepsilon$-$\kappa$-quasi-unitarily equivalent, m-sectorial operators with vertex $\omega$ and semi-angle $\theta \in (\arctan \frac{M}{\alpha}, \pi]$, and let $r > 0$. Then there exist constants $C_{\theta,r,1} > 0$ and $C_{\theta,r,2} > 0$ such that*

$$\|R(z, A_\varepsilon) J^{\uparrow\varepsilon} - J^{\uparrow\varepsilon} R(z, A_0)\|_{\mathscr{L}(H_0, H_\varepsilon)} \leq \frac{\delta_\varepsilon C_{\theta,r,1}}{\sqrt{|z+\omega|}} \tag{3.1}$$

*and*

$$\|R(z, A_\varepsilon) - J^{\uparrow\varepsilon} R(z, A_0) J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} \leq \frac{\delta_\varepsilon C_{\theta,r,2}}{\sqrt{|z+\omega|}} \tag{3.2}$$

*for all $z \notin \Sigma_\theta - \omega$ satisfying $|z + \omega| \geq r$.*

*Proof.* Let $D_\theta$ be as in (2.6) and let $C_\theta$ be as in Lemma 3.1. Let $f \in H_0$ and $u \in H_\varepsilon$ be arbitrary, and fix $z \notin \Sigma_\theta - \omega$. Then by (2.7b) (see also (2.7b')), (2.6), (2.7a) and Lemma 3.1

$$\begin{aligned}&\left|\left\langle \left(R_\varepsilon(z) J^{\uparrow\varepsilon} - J^{\uparrow\varepsilon} R_0(z)\right) f \middle| u \right\rangle_{H_\varepsilon}\right| \\ &\quad\leq \left|\langle f | J^{\downarrow\varepsilon} R_\varepsilon(z)^* u\rangle_{H_0} - \langle J^{\uparrow\varepsilon} R_0(z) f | u\rangle_{H_\varepsilon}\right| + \frac{\delta_\varepsilon D_\theta}{|z+\omega|} \|u\|_{H_\varepsilon} \|f\|_{H_0} \\ &\quad\leq \left|\langle (z - A_0) R_0(z) f | J_1^{\downarrow\varepsilon} R_\varepsilon(z)^* u\rangle_{H_0} - \langle J_1^{\uparrow\varepsilon} R_0(z) f | (z - A_\varepsilon)^* R_\varepsilon(z)^* u\rangle_{H_\varepsilon}\right| \\ &\qquad + \left(\frac{2\delta_\varepsilon C_\theta}{|z+\omega|^{1/2}} + \frac{\delta_\varepsilon D_\theta}{|z+\omega|}\right) \|u\|_{H_\varepsilon} \|f\|_{H_0} \\ &\quad\leq \left|a_0(R_0(z) f, J_1^{\downarrow\varepsilon} R_\varepsilon(z)^* u) - a_\varepsilon(J_1^{\uparrow\varepsilon} R_0(z) f, R_\varepsilon(z)^* u)\right| \\ &\qquad + |z| \left|\langle R_0(z) f | J_1^{\downarrow\varepsilon} R_\varepsilon(z)^* u\rangle_{H_0} - \langle J_1^{\uparrow\varepsilon} R_0(z) f | R_\varepsilon(z)^* u\rangle_{H_\varepsilon}\right| \\ &\qquad + \left(\frac{2\delta_\varepsilon C_\theta}{|z+\omega|^{1/2}} + \frac{\delta_\varepsilon D_\theta}{|z+\omega|}\right) \|u\|_{H_\varepsilon} \|f\|_{H_0}.\end{aligned}$$

Using (2.7e) and once again (2.7b) we can further estimate

$$\begin{aligned}
&\bigl|\langle \bigl(R_\varepsilon(z)J^{\uparrow\varepsilon} - J^{\uparrow\varepsilon}R_0(z)\bigr)f|u\rangle_{H_\varepsilon}\bigr| \\
&\quad\le \delta_\varepsilon \|R_0(z)f\|_{V_0}\|R_\varepsilon(z)^*u\|_{V_\varepsilon} \\
&\qquad + |z|\bigl|\langle R_0(z)f|J^{\downarrow\varepsilon}R_\varepsilon(z)^*u\rangle_{H_0} - \langle J^{\uparrow\varepsilon}R_0(z)f|R_\varepsilon(z)^*u\rangle_{H_\varepsilon}\bigr| \\
&\qquad + \Bigl(\frac{\delta_\varepsilon|z|C_\theta D_\theta}{|z+\omega|^{3/2}} + \frac{2\delta_\varepsilon C_\theta}{|z+\omega|^{1/2}} + \frac{\delta_\varepsilon D_\theta}{|z+\omega|}\Bigr)\|u\|_{H_\varepsilon}\|f\|_{H_0} \\
&\quad\le \Bigl(\frac{\delta_\varepsilon(D_\theta + C_\theta^2)}{|z+\omega|} + \frac{\delta_\varepsilon|z|D_\theta^2}{|z+\omega|^2} + \frac{\delta_\varepsilon|z|C_\theta D_\theta}{|z+\omega|^{3/2}} + \frac{2\delta_\varepsilon C_\theta}{|z+\omega|^{1/2}}\Bigr)\|u\|_{H_\varepsilon}\|f\|_{H_0}.
\end{aligned}$$

For $|z+\omega| \ge r$, this implies (3.1) with

$$C_{\theta,r,1} := (C_\theta D_\theta + 2C_\theta) + \frac{D_\theta + C_\theta^2 + D_\theta^2}{r^{1/2}} + \frac{|\omega|C_\theta D_\theta}{r} + \frac{|\omega|D_\theta^2}{r^{3/2}}.$$

Estimate (3.2) is a consequence of (3.1) since by Lemma 3.2 and Lemma 3.1 we have

$$\|R_\varepsilon(z)u - J^{\uparrow\varepsilon}R_0(z)J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} \le \kappa\|J^{\downarrow\varepsilon}R_\varepsilon(z) - R_0(z)J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon,H_0)} + \frac{\delta_\varepsilon C_\theta}{\sqrt{|z+\omega|}},$$

so we can choose $C_{\theta,r,2} := \kappa C_{\theta,r,1} + C_\theta$. □

*Remark* 3.5. Estimate (3.2) tells us that we can find a good approximation of the operator $A_\varepsilon$ in terms of the (often simpler) operators $A_0$, $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$, at least for small $\varepsilon$. This is interesting by itself. In fact, we even have a rather explicit error estimate; in the proof we have given concrete (though certainly not optimal) constants. However, since these expressions are quite cumbersome, we prefer to work with the general constants $C_{\theta,r,1}$ and $C_{\theta,r,2}$.

Define

$$H^\infty(\Sigma_\theta - \omega) := \bigl\{\, \varphi\colon \Sigma_\theta - \omega \to \mathbb{C} \,:\, \varphi \text{ is holomorphic and bounded} \,\bigr\}$$

and

$$H^\infty_{00}(\Sigma_\theta - \omega) := \bigl\{\, \psi \in H^\infty(\Sigma_\theta - \omega) \,:\, \exists \mu > \tfrac12 \text{ such that } \ \psi(z) \in \mathrm{O}(|z|^{-\mu}) \ (z \to \infty) \,\bigr\}$$

and equip these spaces with the supremum norm. Let $\theta \in (\arctan \frac{M}{\alpha}, \pi]$. We define the primary functional calculus of $A_\varepsilon$ for $\psi \in H^\infty_{00}(\Sigma_\theta - \omega)$ by

$$\psi(A_\varepsilon) := \frac{1}{2\pi \mathrm{i}} \int_{\partial(\Sigma_\sigma - \omega)} \psi(z) R(z, A_\varepsilon)\,\mathrm{d}z, \tag{3.3}$$

where $\sigma \in (\arctan \frac{M}{\alpha}, \theta)$. By Cauchy's integral theorem, this definition is independent of the choice of $\sigma$ and agrees with the usual definition of the functional calculus, compare also [Haa06, §2.5.1].

*Remark* 3.6. In our setting, the natural space for the primary functional calculus would be the larger space

$$H^\infty_0(\Sigma_\theta) := \bigl\{\, \psi \in H^\infty(\Sigma_\theta) \,:\, \exists\, \mu > 0 \text{ such that } \psi(z) \in \mathrm{O}(|z|^{-\mu}) \ (z \to \infty) \,\bigr\}$$

since in fact (3.3) is defined even for $\psi \in H^\infty_0(\Sigma_\theta - \omega)$. However, using estimate (3.2) we can show convergence of $\psi(A_\varepsilon)$ to $\psi(A_0)$ only for $\psi \in H^\infty_{00}(\Sigma_\theta - \omega)$.

Since the operators $A_\varepsilon$ are m-sectorial in the sense of Kato, this functional calculus has a natural extension to $\varphi \in H^\infty(\Sigma_\theta - \omega)$, and the operator $\varphi(A_\varepsilon)$ is bounded with norm

$$\|\varphi(A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon)} \le \Bigl(2 + \frac{2}{\sqrt{3}}\Bigr)\|\varphi\|_\infty, \tag{3.4}$$

cf. [Haa06, Corollary 7.1.17]. It is important for us to have a bound on the norm of $\varphi(A_\varepsilon)$ that is uniform with respect to $\varepsilon$.

We are now able to show that $\varphi(A_\varepsilon)$ converges to $\varphi(A_0)$ is the following sense if $(A_\varepsilon)_{\varepsilon>0}$ converges to $A_0$ in the sense of Definition 2.3, which is our main result in the context of the functional calculus.

**Theorem 3.7.** *Let $A_0$ and $A_\varepsilon$ be as in Section 2 and assume that $A_\varepsilon$ be $\delta_\varepsilon$-$\kappa$-quasi-unitarily equivalent to $A_0$. Let $\theta \in (\arctan \frac{M}{\alpha}, \pi]$. Then for all $\psi \in H^\infty_{00}(\Sigma_\theta - \omega)$ there exists $C_\psi \geq 0$ such that*

$$\|J^{\uparrow\varepsilon}\psi(A_0)J^{\downarrow\varepsilon} - \psi(A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon)} \leq C_\psi \delta_\varepsilon. \tag{3.5}$$

*Moreover, for all $\psi \in H^\infty(\Sigma_\theta - \omega)$ there exists $C_\psi \geq 0$ such that*

$$\|J^{\uparrow\varepsilon}\varphi(A_0)J^{\downarrow\varepsilon}u - \varphi(A_\varepsilon)u\|_{H_\varepsilon} \leq C_\varphi \delta_\varepsilon \|(\omega + 1 + A_\varepsilon)u\|_{H_\varepsilon} \tag{3.6}$$

*for all $u \in D(A_\varepsilon)$.*

*Proof.* Fix $\psi \in H^\infty_{00}(\Sigma_\theta - \omega)$ and $\sigma \in (\arctan \frac{M}{\alpha}, \theta)$. Let $\nu \in (0, \alpha c_V^{-2})$ be such that

$$\theta' := \arctan\Big(\frac{M}{\alpha - \nu c_V^2}\Big) < \sigma,$$

where $\alpha$, $M$ and $c_V$ are as in (2.1), (2.2) and (2.3). Since

$$a_\varepsilon(u,u) + (\omega - \nu)\|u\|^2_{H_\varepsilon} \geq \alpha\|u\|^2_{V_\varepsilon} - \nu\|u\|^2_{H_\varepsilon} \geq (\alpha - \nu c_V^2)\|u\|^2_{V_\varepsilon},$$

the operator $A_\varepsilon$ is m-sectorial with vertex $-\omega + \nu$ and semi-angle $\theta'$, and the same is true for $A_0$ on $H_0$. Hence by Proposition 3.4

$$\|R(z, A_\varepsilon) - J^{\uparrow\varepsilon}R(z, A_0)J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} \leq \frac{\delta_\varepsilon C_{\theta',\nu/2,2}}{\sqrt{|z + \omega - \nu|}}$$

for all $z \notin \Sigma_{\theta'} - \omega + \nu$ such that $|z + \omega - \nu| \geq \frac{\nu}{2}$. If $r > 0$ is sufficiently small, then $B(-\omega, r) \cap (\partial\Sigma_\sigma - \omega)$ has distance at least $\frac{\nu}{2}$ to $\Sigma_{\theta'} - \omega + \nu$, and hence

$$\|R(z, A_\varepsilon) - J^{\uparrow\varepsilon}R(z, A_0)J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} \leq \delta_\varepsilon c_{\sigma,\nu} \tag{3.7}$$

for all $z \in \partial(\Sigma_\sigma - \omega)$ satisfying $|z + \omega| \leq r$, where $c_{\sigma,\nu}$ and $r$ are constants depending on $\sigma$ and $\nu$, so in principle only on $M$, $\alpha$ and $c_V$.

There exist $\mu > \frac{1}{2}$ and $K \geq 0$ such that

$$|\psi(z)| \leq \frac{K}{|z + \omega|^\mu} \quad \text{and} \quad |\psi(z)| \leq K$$

for all $z \in \Sigma_\theta - \omega$. Thus, by (3.3), Proposition 3.4 and (3.7)

$$\begin{aligned}
&\|J^{\uparrow\varepsilon}\psi(A_0)J^{\downarrow\varepsilon} - \psi(A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon)} \\
&\quad \leq \frac{1}{2\pi}\int_{\partial(\Sigma_\sigma - \omega)} |\psi(z)|\, \|J^{\uparrow\varepsilon}R_0(z)J^{\downarrow\varepsilon} - R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)}\, \mathrm{d}z \\
&\quad \leq \frac{1}{2\pi}\int_{\partial(\Sigma_\sigma - \omega)\setminus B(-\omega,r)} \frac{\delta_\varepsilon C_{\sigma,r,2}K}{|z + \omega|^{\mu + \frac{1}{2}}}\, \mathrm{d}z + \frac{1}{2\pi}\int_{\partial(\Sigma_\sigma - \omega)\cap B(-\omega,r)} \delta_\varepsilon c_{\sigma,\nu}K\, \mathrm{d}z.
\end{aligned}$$

Therefore, we have shown (3.5) with

$$C_\psi := \frac{C_{\sigma,r,2}K}{(\mu - \frac{1}{2})r^{\mu - \frac{1}{2}}\pi} + \frac{c_{\sigma,\nu}rK}{\pi}.$$

In particular, there exists a constant $C_{\psi^*}$ belonging to the function $\psi^* \in H^\infty_{00}(\Sigma_\theta)$ defined by

$$\psi^*(z) := \frac{1}{\omega + 1 + z}$$

such that

$$\|J^{\uparrow\varepsilon}\psi^*(A_0)J^{\downarrow\varepsilon} - \psi^*(A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon)} \leq C_{\psi^*}\delta_\varepsilon,$$

i.e.,

$$\|J^{\uparrow\varepsilon}(\omega+1+A_0)^{-1}J^{\downarrow\varepsilon}-(\omega+1+A_\varepsilon)^{-1}\|_{\mathscr{L}(H_\varepsilon)}\le C_{\psi^*}\delta_\varepsilon. \tag{3.8}$$

Now let $\varphi\in H^\infty(\Sigma_\theta-\omega)$ be fixed and define $\psi\in H^\infty_{00}(\Sigma_\theta-\omega)$ by

$$\psi(z):=\frac{\varphi(z)}{\omega+1+z}$$

Then $\varphi(A_\varepsilon)(\omega+1+A_\varepsilon)^{-1}=\psi(A_\varepsilon)$ by the construction of the functional calculus [Haa06, §2.3.2]. Hence for all $u\in D(A_\varepsilon)$ we have

$$\begin{aligned}&\|J^{\uparrow\varepsilon}\varphi(A_0)(\omega+1+A_0)^{-1}J^{\downarrow\varepsilon}(\omega+1+A_\varepsilon)u-\varphi(A_\varepsilon)u\|_{H_\varepsilon}\\&\quad=\|J^{\uparrow\varepsilon}\psi(A_0)J^{\downarrow\varepsilon}(\omega+1+A_\varepsilon)u-\psi(A_\varepsilon)(\omega+1+A_\varepsilon)u\|_{H_\varepsilon}\\&\quad\le C_\psi\delta_\varepsilon\|(\omega+1+A_\varepsilon)u\|_{H_\varepsilon}.\end{aligned}$$

Moreover, from (2.7d) and (3.4) we obtain that

$$\begin{aligned}&\|J^{\uparrow\varepsilon}\varphi(A_0)J^{\downarrow\varepsilon}u-J^{\uparrow\varepsilon}\varphi(A_0)(\omega+1+A_0)^{-1}J^{\downarrow\varepsilon}(\omega+1+A_\varepsilon)u\|_{H_\varepsilon}\\&\quad\le\kappa\big(2+\frac{2}{\sqrt{3}}\big)\|\varphi\|_\infty\|J^{\downarrow\varepsilon}u-(\omega+1+A_0)^{-1}J^{\downarrow\varepsilon}(\omega+1+A_\varepsilon)u\|_{H_0}.\end{aligned}$$

Finally, by Lemma 3.3, (3.8), Lemma 3.1 (for $z=-\omega-1$) and (2.7d)

$$\begin{aligned}&\|J^{\downarrow\varepsilon}u-(\omega+1+A_0)^{-1}J^{\downarrow\varepsilon}(\omega+1+A_\varepsilon)u\|_{H_0}\\&\quad\le\kappa\|u-J^{\uparrow\varepsilon}(\omega+1+A_0)^{-1}J^{\downarrow\varepsilon}(\omega+1+A_\varepsilon)u\|_{H_\varepsilon}\\&\qquad+\delta_\varepsilon\|(\omega+1+A_0)^{-1}J^{\downarrow\varepsilon}(1+\omega+A_\varepsilon)u\|_{V_0}\\&\quad\le\kappa C_{\psi^*}\delta_\varepsilon\|(\omega+1+A_\varepsilon)u\|_{H_\varepsilon}+\delta_\varepsilon C_\theta\kappa\|(\omega+1+A_\varepsilon)u\|_{H_0}.\end{aligned}$$

Combining the previous three estimates, we have proved (3.6) with

$$C_\varphi:=C_\psi+\kappa^2\Big(2+\frac{2}{\sqrt{3}}\Big)\|\varphi\|_\infty\big(C_{\psi^*}+C_\theta\big).\qquad\square$$

**Corollary 3.8.** *If $(A_\varepsilon)_{\varepsilon>0}$ converges to $A_0$ in the sense of Definition 2.3, then the family $(J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon})_{\varepsilon>0}$ converges in operator norm to $\varphi(A_0)$ (regarded as operators on $H_0$) for every $\varphi\in H^\infty_{00}(\Sigma_\theta-\omega)$, $\theta\in(\arctan\frac{M}{\alpha},\pi]$. If merely $\varphi\in H^\infty(\Sigma_\theta-\omega)$, then we have at least convergence in the strong operator topology.*

*Proof.* Since all conditions in Definition 2.3 are symmetric with respect to $A_0$ and $A_\varepsilon$, interchanging the roles of the two operators we obtain as in Theorem 3.7 that

$$\|\psi(A_0)-J^{\downarrow\varepsilon}\psi(A_\varepsilon)J^{\uparrow\varepsilon}\|_{\mathscr{L}(H_0)}\le C_\psi\delta_\varepsilon$$

for $\psi\in H^\infty_{00}(\Sigma_\theta-\omega)$, which proves the first claim. Similarly,

$$\|\varphi(A_0)f-J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{\mathscr{L}(H_0)}\le C_\varphi\delta_\varepsilon\|(\omega+1+A_0)f\|_{H_0}$$

for all $f\in D(A_0)$ if $\varphi\in H^\infty(\Sigma_\theta-\omega)$, implying that $(J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon})_{\varepsilon>0}$ converges to $\varphi(A_0)$ on a dense subspace of $H_0$. Since the operators are uniformly bounded by (2.7d) and (3.4), this implies strong convergence. $\square$

**Example 3.9.** For every $t\in\Sigma_\sigma$, $\sigma\in(0,\frac{\pi}{2})$, the function $z\mapsto\mathrm{e}^{-tz}$ is in $H^\infty_{00}(\Sigma_\theta-\omega)$ for $\theta\in(0,\frac{\pi}{2}-\sigma)$. Hence the semigroups $(\mathrm{e}^{-tA_\varepsilon})_{\varepsilon>0}$ converge to $\mathrm{e}^{-tA_0}$ "in operator norm" (in the sense of Theorem 3.7 and Corollary 3.8) for $t$ in the common sector of holomorphy of the semigroups, i.e., for every fixed $t\in\Sigma_\sigma$, where $\sigma:=\frac{\pi}{2}-\arctan\frac{M}{\alpha}$.

Note, however, that we cannot expect this for $t=0$ since typically $J^{\uparrow\varepsilon}J^{\downarrow\varepsilon}$ does not tend to the identity in operator norm even if $H_\varepsilon=H_0$ for all $\varepsilon\ge0$. Thus we cannot expect uniform convergence near $t=0$. However, the explicit constant $C_\psi$ in the proof of Theorem 3.7 shows that the convergence is uniform on compact subsets of $\Sigma_\sigma$.

3.2. **Spectral convergence.** It has been proven in [Pos06, §A.5] that if a family of self-adjoint operators $(A_\varepsilon)_{\varepsilon>0}$ converges to $A_0$ in the sense of Definition 2.3, then their spectra $\sigma(A_\varepsilon)$ converge to $\sigma(A_0)$. In [Pos09], a similar result for certain non-self-adjoint operators arising in the treatment of resonances via complex scaling are considered. Here we prove that a similar result is true in the general (m-sectorial) case, where the spectra need not be real. This is a part of the justification why we regard our notion of convergence as a sensible generalisation of the classical resolvent convergence, compare also Example 2.5.

We consider the following notion of spectral convergence, which is quite natural. It is often called "upper semi-continuity" of the spectrum. This type of convergence is precisely what we obtain if in a fixed Hilbert space we have a family of operators whose resolvents converge in operator norm, see [Kat95, Theorem IV.3.1].

**Definition 3.10.** We say that *the spectra $\sigma(A_\varepsilon)$ of the family $(A_\varepsilon)_{\varepsilon>0}$ converge to the spectrum $\sigma(A_0)$ of $A_0$ as $\varepsilon \to 0$* if for each compact set $K \subset \varrho(A_0)$ there exists $\varepsilon_1 > 0$ such that $K \subset \varrho(A_\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_1)$.

Ideally, we could hope that the spectra $\sigma(A_\varepsilon)$ converge to $\sigma(A_0)$ if $(A_\varepsilon)_{\varepsilon>0}$ converges to $A_0$. In fact, this is true if in addition $\varrho(A_0)$ is connected, see Corollary 3.14.

We start with an auxiliary lemma, allowing us to estimate the resolvent of $A_\varepsilon$ if we have a priori information about the resolvent of $A_0$. For the whole section, the operators $(A_\varepsilon)_{\varepsilon>0}$ are assumed to satisfy the conditions in Section 2.

**Lemma 3.11.** *For every $\ell > 0$ and $r > 0$ there exist $\delta_0 = \delta_0(\ell, r, \omega) > 0$ and $L = L(\ell, r, \omega) > 0$ with the following property: if $A_\varepsilon$ is $\delta_\varepsilon$-$\kappa$-quasi-unitarily equivalent to $A_0$ for some $\delta_\varepsilon \in (0, \delta_0]$, if $z \in \varrho(A_0) \cap \varrho(A_\varepsilon) \cap B(0, r)$, and if $\|R(z, A_0)\|_{\mathscr{L}(H_0)} \le \ell$, then $\|R(z, A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon)} \le L$.*

*Proof.* For $z \in \varrho(A_0) \cap \varrho(A_\varepsilon)$ we define
$$V(z) := J^{\downarrow\varepsilon} R_\varepsilon(z) - R_0(z) J^{\downarrow\varepsilon}.$$
Let $z$ and $z_0$ be in $\varrho(A_0) \cap \varrho(A_\varepsilon)$. Then by the resolvent identity we have
$$\big(R_0(z_0) - R_0(z)\big) J^{\downarrow\varepsilon} R_\varepsilon(z) R_\varepsilon(z_0) = R_0(z) R_0(z_0) J^{\downarrow\varepsilon} \big(R_\varepsilon(z_0) - R_\varepsilon(z)\big)$$
and thus
$$R_0(z_0) V(z) R_\varepsilon(z_0) = R_0(z) V(z_0) R_\varepsilon(z).$$
Hence
$$\begin{aligned} V(z) &= (z_0 - A_0) R_0(z) V(z_0) R_\varepsilon(z)(z_0 - A_\varepsilon) \\ &= (\operatorname{id} + (z_0 - z) R_0(z)) V(z_0) (\operatorname{id} + (z_0 - z) R_\varepsilon(z)) \end{aligned}$$
on $D(A_\varepsilon)$ and thus on $H_\varepsilon$ by density. Setting $z_0 := -\omega - 1$ and using the dual version of (3.1), which follows by exchanging the roles of $A_\varepsilon$ and $A_0$ in Proposition 3.4, to estimate $V(z_0)$ we deduce that

$$\|V(z)\|_{\mathscr{L}(H_\varepsilon, H_0)} \le \delta_\varepsilon C_{\theta,1,1} \big(1 + \ell|\omega + 1 + z|\big)\big(1 + |\omega + 1 + z|\, \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)}\big). \tag{3.9}$$

Next, we note that for all $u \in H_\varepsilon$
$$\begin{aligned} \|R_\varepsilon(z) u\|_{a_\varepsilon}^2 &= \langle (\omega + A_\varepsilon) R_\varepsilon(z) u | R_\varepsilon(z) u \rangle_{H_\varepsilon} \\ &\le \big(\|u\|_{H_\varepsilon} + |\omega + z|\, \|R_\varepsilon(z) u\|_{H_\varepsilon}\big) \|R_\varepsilon(z) u\|_{H_\varepsilon}, \end{aligned}$$
proving by (2.4) that
$$\begin{aligned} \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon, V_\varepsilon)}^2 &\le \frac{1}{\alpha} \big(1 + |\omega + z|\, \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)}\big) \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} \\ &\le \frac{1}{\alpha} \big(1 + \beta \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)}\big)^2 \end{aligned} \tag{3.10}$$
with $\beta := \max\{1, |\omega + z|\}$.

Now write

$$R_\varepsilon(z) = \bigl(\mathrm{id} - J^{\uparrow\varepsilon} J^{\downarrow\varepsilon}\bigr) R_\varepsilon(z) + J^{\uparrow\varepsilon}\bigl(J^{\downarrow\varepsilon} R_\varepsilon(z) - R_0(z) J^{\downarrow\varepsilon}\bigr) + J^{\uparrow\varepsilon} R_0(z) J^{\downarrow\varepsilon}.$$

This representation, combined with (3.9) and (3.10), shows that

$$\begin{aligned}
\|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} &\le \delta_\varepsilon \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon, V_\varepsilon)} + \kappa \|V(z)\|_{\mathscr{L}(H_\varepsilon)} + \kappa^2 \ell \\
&\le \Bigl(\frac{\delta_\varepsilon}{\sqrt{\alpha}} + \kappa \delta_\varepsilon C_{\theta,1,1}\bigl(1 + \ell|\omega + 1 + z|\bigr) + \kappa^2 \ell\Bigr) \\
&\qquad + \delta_\varepsilon \Bigl(\frac{\beta}{\sqrt{\alpha}} + C_{\theta,1,1}\bigl(1 + \ell|\omega + 1 + z|\bigr)\,|\omega + 1 + z|\Bigr) \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} \\
&=: \ell_1 + \delta_\varepsilon c \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)}.
\end{aligned}$$

Thus, if $\delta_\varepsilon \in (0, \delta_0]$ with $\delta_0 := \frac{1}{2c}$, then

$$\frac{1}{2}\|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} \le (1 - \delta_\varepsilon c)\|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} \le \ell_1,$$

i.e., we have proved the claim with $L := 2\ell_1$. □

Now we come to our main theorem regarding convergence of the spectrum.

**Theorem 3.12.** *Let $A_0$ be an m-sectorial operator with vertex $\omega$ and semi-angle $\theta$. Let $K \subset \varrho(A_0)$ be compact and connected. Then there exist constants $\delta_0 > 0$ and $C_{\theta,K}, D_{\theta,K} \ge 0$ (that also depend on $A_0$) with the following property: if $A_\varepsilon$ is $\delta_\varepsilon$-$\kappa$-quasi-unitarily equivalent to $A_0$ for $\delta_\varepsilon \in (0, \delta_0]$, and if in addition $K \cap \varrho(A_\varepsilon) \neq \emptyset$, then $K \subset \varrho(A_\varepsilon)$,*

$$\|J^{\downarrow\varepsilon} R(z, A_\varepsilon) - R(z, A_0) J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon, H_0)} \le C_{\theta,K}\delta_\varepsilon \tag{3.11}$$

*and*

$$\|J^{\uparrow\varepsilon} R(z, A_0) J^{\downarrow\varepsilon} - R(z, A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon)} \le D_{\theta,K}\delta_\varepsilon \tag{3.12}$$

*for all $z \in K$.*

*Proof.* Since $K$ is compact, $K \subset B(0, r)$ for some $r > 0$ and

$$\ell := \sup_{z \in K} \|R_0(z)\|_{\mathscr{L}(H_\varepsilon)} < \infty.$$

Choose $\delta_0 = \delta_0(\ell, r, \omega)$ as in Lemma 3.11. Let $\delta_\varepsilon \in (0, \delta_0)$ and let $A_\varepsilon$ be $\delta_\varepsilon$-$\kappa$-quasi-unitarily equivalent to $A_0$. Let $K_0 := \varrho(A_\varepsilon) \cap K$, which is non-empty by assumption. Since $\varrho(A_\varepsilon)$ is open, the set $K_0$ is relatively open in $K$.

Let $(z_n)$ be a sequence in $K_0$ converging to $z \in K$. Then from Lemma 3.11 we know that $\|R_\varepsilon(z_n)\|_{\mathscr{L}(H_\varepsilon)}$ is bounded, hence $z \in \varrho(A_\varepsilon)$. We have shown that $K_0$ is closed in $K$. Since $K$ is connected, $K_0 = K$, i.e., $K \subset \varrho(A_\varepsilon)$.

Since by Lemma 3.11 we have $\|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} \le L$ for some $L > 0$, it follows from (3.9) that

$$\|J^{\downarrow\varepsilon} R_\varepsilon(z) - R_0(z) J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon, H_0)} \le \delta_\varepsilon C_{\theta,1,1}\bigl(1 + \ell(|\omega| + 1 + r)\bigr)\bigl(1 + L(|\omega| + 1 + r)\bigr).$$

for all $z \in K$. This is (3.11) for

$$C_{\theta,K} := C_{\theta,1,1}\bigl(1 + \ell(|\omega| + 1 + r)\bigr)\bigl(1 + L(|\omega| + 1 + r)\bigr)$$

Now (3.12) follows from Lemma 3.2 and estimate (3.10) with

$$D_{\theta,K} := \kappa C_{\theta,K} + \frac{1}{\sqrt{\alpha}}\bigl(1 + \beta L\bigr).$$

□

*Remark* 3.13. It can be difficult to check the condition $K \cap \varrho(A_\varepsilon) \neq \emptyset$ of the previous theorem. On the other hand, in the classical situation, i.e., if $H_\varepsilon = H_0$ for all $\varepsilon \geq 0$ and $R(z, A_\varepsilon)$ converges to $R(z, A_0)$ in operator norm, this is automatically fulfilled by the fact that the set of invertible operators is open in $\mathscr{L}(H_0)$:

Let $\lambda \in \varrho(A_0) \cap K$ and $\mu < -\omega$ such that $\lambda \neq \mu$. Then $R_\varepsilon(\mu)$ converges in operator norm to $R_0(\mu)$, since $\mu$ is outside the sector $\Sigma_\theta$. Moreover, $\lambda \in \varrho(A_\varepsilon)$ is equivalent with the invertibility of $\frac{1}{\mu-\lambda} - R_\varepsilon(\mu)$ by the spectral mapping theorem. For the same reason, $\frac{1}{\mu-\lambda} - R_0(\mu)$ is invertible. Since the set of invertible operators is open, $\lambda \in \varrho(A_\varepsilon)$ for sufficiently small $\varepsilon$.

If the resolvent set is connected, a given compact set $K \subset \varrho(A_0)$ can be enlarged to a connected compact set $K' \subset \varrho(A_0)$ in such a way that we can guarantee $K' \cap \varrho(A_\varepsilon) \neq \emptyset$, so that the theorem applies. We make this explicit in the following corollary. Note that in particular if the spectrum is real or discrete, the resolvent set is connected. Hence for self-adjoint operators and operators with compact resolvent we obtain spectral convergence.

**Corollary 3.14.** *Assume that $\varrho(A_0)$ is connected and that $(A_\varepsilon)_{\varepsilon>0}$ converges to $A_0$ in the sense of Definition 2.3. Then $\sigma(A_\varepsilon)$ converges to $\sigma(A_0)$ in the sense of Definition 3.10.*

*Proof.* Let $K \subset \varrho(A_0)$ be compact. Since $\varrho(A_0)$ is connected, there exists a connected compact set $K' \subset \varrho(A_0)$ such that $K \subset K'$ and $-\omega - 1 \in K'$. In fact, let $R > |\omega| + 1$ be such that $K \subset B(0, R)$, and let $(O_{\rho,\mu})_\mu$ denote the family of (open) connected components of the open set

$$O_\rho := \left\{ z \in \varrho(A_0) \cap B(0, R) \,:\, \operatorname{dist}(z, \sigma(A_0)) < \rho \right\}.$$

Then

$$K \cup \{-\omega - 1\} \subset \bigcup\nolimits_{\rho,\mu} O_{\rho,\mu},$$

and hence by compactness there exist a finite subcover $(O_{\rho_i,\mu_i})$. Let $K''$ the be the union of the compact, connected sets $\overline{O}_{\rho_i,\mu_i} \subset \varrho(A_0)$. Now $K''$ has only finitely many connected components. Since $\varrho(A_0)$ is arcwise connected, we can join these connected components by finitely many paths $(\gamma_k)$ in $\varrho(A_0)$. Then $K' := K'' \cup \bigcup_k \gamma_k$ is a connected, compact subset of $\varrho(A_0)$ that contains $K$ and $-\omega - 1$.

Since $-\omega - 1 \in \varrho(A_\varepsilon)$ for all $\varepsilon \geq 0$ we obtain from Theorem 3.12 that $K' \subset \varrho(A_\varepsilon)$ if $\delta_\varepsilon$ is sufficiently small. Hence $K \subset \varrho(A_\varepsilon)$ for small $\varepsilon$, which implies the claim. ☐

In the rest of this section, we show that the discrete spectra of $(A_\varepsilon)_{\varepsilon>0}$ converge to the discrete spectrum of $A_0$ as the operators $(A_\varepsilon)_{\varepsilon>0}$ converge to $A_0$. In fact, we show that for an eigenvalue $\lambda$ of $A_0$ of finite algebraic multiplicity $m_0(\lambda)$ and for sufficiently small $\delta_\varepsilon$, there exist exactly $m_0(\lambda)$ eigenvalues of $A_\varepsilon$ near $\lambda$, where we count the eigenvalues according to their algebraic multiplicity.

Recall that the *algebraic multiplicity* $m_0(\lambda)$ of an isolated point $\lambda \in \sigma(A_0)$ is the rank $\operatorname{rk} P_0 := \dim \operatorname{Rg} P_0$ of the spectral projection

$$P_0 := \frac{1}{2\pi \mathrm{i}} \int_{\partial B(\lambda,r)} R(z, A_0)\, \mathrm{d}z,$$

where $r > 0$ is such that $\overline{B(\lambda, r)} \cap \sigma(A_0) = \{\lambda\}$, compare [ALL01, §1.3]. By Cauchy's integral theorem, this definition does not depend on the particular choice of $r$, and in fact we could replace the circle $\partial B(\lambda, r)$ by any positively oriented, smooth curve that surrounds $\lambda$, but no other point of $\sigma(A_0)$.

*Remark* 3.15. Since $R(z, A_0)$ is locally bounded as a $\mathscr{L}(H_0, V_0)$-valued function, see for example estimate (3.10), the spectral projection $P_0$ is a bounded operator from $H_0$ to $V_0$.

**Lemma 3.16.** *There exist $\delta_0 > 0$ such that $\|J^{\uparrow\varepsilon} f\|_{H_\varepsilon} \geq \frac{1}{2}\|f\|_{H_0}$ for all $f \in \operatorname{Rg} P_0$ if $\delta_\varepsilon \in (0, \delta_0]$.*

*Proof.* For all $f \in V_0$ we have by (2.7b'), (2.7c) and (2.7d) that

$$\begin{aligned} \|f\|^2_{H_\varepsilon} - \|J^{\uparrow\varepsilon} f\|^2_{H_0} &= \langle f|f\rangle_{H_0} - \langle J^{\uparrow\varepsilon} f|J^{\uparrow\varepsilon} f\rangle_{H_\varepsilon} \\ &= \langle f - J^{\downarrow\varepsilon} J^{\uparrow\varepsilon} f|f\rangle_{H_\varepsilon} + \delta_\varepsilon \|J^{\uparrow\varepsilon} f\|_{H_\varepsilon} \|f\|_{H_0} \\ &\leq \delta_\varepsilon \|f\|_{V_0} \|f\|_{H_0} + \delta_\varepsilon \kappa \|f\|^2_{H_0}. \end{aligned}$$

Now if $f \in \operatorname{Rg} P_0$, i.e., $f = P_0 f$, and $f \neq 0$, we obtain that

$$\begin{aligned} \|f\|_{H_\varepsilon} - \|J^{\uparrow\varepsilon} f\|_{H_0} &\leq \delta_\varepsilon \frac{\|P_0\|_{\mathscr{L}(H_0,V_0)} + \kappa}{\|J^{\uparrow\varepsilon} f\|_{H_\varepsilon} + \|f\|_{H_0}} \|f\|^2_{H_0} \\ &\leq \delta_0 \big(\|P_0\|_{\mathscr{L}(H_0,V_0)} + \kappa\big) \|f\|_{H_0} = \frac{1}{2}\|f\|_{H_0} \end{aligned}$$

for

$$\delta_0 := \frac{1}{2}\big(\|P_0\|_{\mathscr{L}(H_0,V_0)} + \kappa\big)^{-1}. \qquad \square$$

We now prove our main theorem about continuous dependence of the discrete spectrum. For simplicity we assume that $\varrho(A_0)$ is connected, even though it would suffice that $\varrho(A_\varepsilon) \cap B(\lambda, r) \neq \emptyset$ for small $\varepsilon$ and all $r > 0$.

**Theorem 3.17.** *Let $\varrho(A_0)$ be connected, let $\lambda$ be an isolated point of $\sigma(A_0)$ with finite algebraic multiplicity $m_0(\lambda) \in \mathbb{N}$, and let $D$ be a bounded, open set such that $\overline{D} \cap \sigma(A_0) = \{\lambda\}$. Then there exists $\delta_0 > 0$ such that if $A_\varepsilon$ is $\delta_\varepsilon$-$\kappa$-quasi-unitarily equivalent to $A_0$ for $\delta_\varepsilon \in (0, \delta_0]$, then there exist eigenvalues $(\lambda_{\varepsilon,i})_{i=1}^{m_0(\lambda)}$ of $A_\varepsilon$ such that*

$$\sigma(A_\varepsilon) \cap D = \big\{\lambda_{\varepsilon,1}, \dots, \lambda_{\varepsilon,m_0(\lambda)}\big\}.$$

*Here, the values $(\lambda_{\varepsilon,i})$ are not necessarily pairwise different, but rather each value is repeated according to its algebraic multiplicity with respect to $A_\varepsilon$.*

*Proof.* We may assume that $D$ has smooth boundary. In fact, otherwise we can replace $D$ by an open set $D_1 \subset D$ with smooth boundary still containing $\lambda$. Since $(\overline{D} \setminus D_1) \cap \sigma(A_\varepsilon) = \emptyset$ for small $\delta_\varepsilon$ by Corollary 3.14, the result carries over from $D_1$ to $D$.

By Corollary 3.14, the integral

$$P_\varepsilon := \frac{1}{2\pi \mathrm{i}} \int_{\partial D} R_\varepsilon(z) \, \mathrm{d}z$$

is defined for sufficiently small $\delta_\varepsilon$, and using Theorem 3.12 we see that there exist $\delta_1 > 0$ and $C_1 \geq 0$ such that

$$\|J^{\downarrow\varepsilon} P_\varepsilon - P_0 J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} < C_1 \delta_\varepsilon$$

if $\delta_\varepsilon \in (0, \delta_1]$.

Now let $u \in \operatorname{Rg}(P_\varepsilon)$, i.e., $P_\varepsilon u = u$. Then by Lemma 3.16 there exists $\delta_2 \in (0, \delta_1)$ such that

$$\|P_0 J^{\downarrow\varepsilon} u\|_{H_0} \geq \|J^{\downarrow\varepsilon} P_\varepsilon u\|_{H_0} - \|(J^{\downarrow\varepsilon} P_\varepsilon - P_0 J^{\downarrow\varepsilon})u\|_{H_0} \geq \frac{1}{2}\|u\|_{H_\varepsilon} - C_1 \delta_\varepsilon \|u\|_{H_\varepsilon} > 0$$

whenever $\delta_\varepsilon \in (0, \delta_2]$. This proves that $P_0 J^{\downarrow\varepsilon}$ is injective on $\operatorname{Rg}(P_\varepsilon)$, showing that $\operatorname{rk} P_0 \geq \operatorname{rk} P_\varepsilon$ whenever $\delta_\varepsilon \in (0, \delta_2]$.

For the converse inequality, we interchange the roles of $P_0$ and $P_\varepsilon$. In fact, The estimate $\|J^{\downarrow\varepsilon} u\|_{H_0} \geq \frac{1}{2}\|u\|_{H_\varepsilon}$ for $u \in \operatorname{Rg}(P_\varepsilon)$ can be obtained as in Lemma 3.16 by exploiting the fact that Lemma 3.11 and (3.10) provide a uniform bound for $\|P_\varepsilon\|_{\mathscr{L}(H_\varepsilon,V_\varepsilon)}$, compare also Remark 3.15. Now it readily follows that $\operatorname{rk} P_\varepsilon \geq \operatorname{rk} P_0$ for sufficiently small $\delta_\varepsilon$.

Thus there exists $\delta_0 > 0$ such that $m_0(\lambda) = \operatorname{rk} P_\varepsilon$ whenever $\delta_\varepsilon \in (0, \delta_0]$. This implies that $\sigma(A_\varepsilon) \cap D$ consists of finitely many eigenvalues whose algebraic multiplicities add up to $m_0(\lambda)$, compare [ALL01, Theorem 1.32]. □

*Remark* 3.18. Theorem 3.17 says that near an isolated eigenvalue $\lambda$ of $A_0$ any sufficiently close operator $A_\varepsilon$ also possess only isolated eigenvalues, whose multiplicities add up to the multiplicity of $\lambda$. This is a version of [Pos06, Corollary A.15] for non-self-adjoint operators, see also [Pos09].

The following corollary is a trivial consequence of Theorem 3.17 and Corollary 3.14.

**Corollary 3.19.** *Let $\varrho(A_0)$ be connected, let $\lambda$ be an isolated point of $\sigma(A_0)$ with finite algebraic multiplicity, and let $(A_\varepsilon)_{\varepsilon>0}$ converge to $A_0$ in the sense of Definition 2.3. Then the eigenvalues $\lambda_{\varepsilon,i}$ in Theorem 3.17 converge to $\lambda$, i.e., $\lim_{\varepsilon\to 0} \lambda_{\varepsilon,i} \to \lambda$ for every $i = 1, \dots, m_0(\lambda)$.*

3.3. **Invariance and extrapolation.** Assume that $(A_\varepsilon)_{\varepsilon>0}$ converges to $A_0$ in the sense of Definition 2.3. We have already shown that the generated semigroups also converge in an appropriate sense, see Example 3.9. It is now natural to ask whether certain properties of the semigroups $(\mathrm{e}^{-tA_\varepsilon})_{\varepsilon>0}$ are inherited by $\mathrm{e}^{-tA_0}$ under appropriate assumptions on the operators $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$.

In this short section, we formulate a simple result of this kind and apply it to obtain convergence of the semigroups in extrapolation spaces under natural assumptions.

**Theorem 3.20.** *Let $(A_\varepsilon)_{\varepsilon>0}$ converge to $A_0$ as $\varepsilon \to 0$ in the sense of Definition 2.3, let $\theta \in (\arctan \frac{M}{\alpha}, \pi]$, and let $\varphi \in H^\infty(\Sigma_\theta - \omega)$. For every $\varepsilon \geq 0$, let $C_\varepsilon$ be a closed subset of $H_\varepsilon$ such that*

$$J^{\uparrow\varepsilon} C_0 \subset C_\varepsilon \quad \textit{and} \quad J^{\downarrow\varepsilon} C_\varepsilon \subset C_0. \tag{3.13}$$

*If $\varphi(A_\varepsilon) C_\varepsilon \subset C_\varepsilon$ for all $\varepsilon > 0$, then $\varphi(A_0) C_0 \subset C_0$.*

*Proof.* By the assumptions,

$$(J^{\downarrow\varepsilon} \varphi(A_\varepsilon) J^{\uparrow\varepsilon}) C_0 \subset C_0 \tag{3.14}$$

for all $\varepsilon > 0$. Thus the result follows from Corollary 3.8 because the invariance of a closed set is stable under strong convergence. □

*Remark* 3.21. In some applications, for example in Section 5, the condition (3.13) is only satisfied up to a rescaling of the identification operators, i.e., we can write the identification operators as

$$J^{\uparrow\varepsilon} = c_\varepsilon \widetilde{J^{\uparrow\varepsilon}}, \qquad J^{\downarrow\varepsilon} = c_\varepsilon^{-1} \widetilde{J^{\downarrow\varepsilon}}$$

for operators $\widetilde{J^{\uparrow\varepsilon}}$ and $\widetilde{J^{\downarrow\varepsilon}}$ that do satisfy (3.13). It is clear that also in this more general situation the inclusion (3.14) is satisfied and hence the assertion of Theorem 3.20 remains valid.

It is well-known how invariance of closed convex subsets under the action of a semigroup on a Hilbert space $H$ generated by an operator associated with a sesquilinear form can be efficiently characterised by a Beurling-Deny-type criterion due to Ouhabaz, see [Ouh05, Thm. 2.2]. Assuming that $H = L^2(\Omega, \mu)$ with a measure space $\Omega$, typical applications of this criterion involve positivity and $L^\infty$-contractivity (i.e., invariance of the subset of those $L^2$-functions taking a.e. values in the interval $[0, \infty)$ or $[-1, 1]$).

A typical application of Theorem 3.20 is the following.

**Corollary 3.22.** *Let $p \in [1,\infty]$. Assume that $H_\varepsilon = L^2(\Omega_\varepsilon)$ for measure spaces $\Omega_\varepsilon$, $\varepsilon \ge 0$. Assume that $(A_\varepsilon)_{\varepsilon>0}$ converges to $A_0$, where the operators $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$ in the definition of convergence are positive ($L^p$-contractive). Assume that the semigroup $(\mathrm{e}^{-tA_\varepsilon})_{t\ge 0}$ is positive ($L^p$-contractive) on $H_\varepsilon$ for every $\varepsilon > 0$. Then $(\mathrm{e}^{-tA_0})_{t\ge 0}$ is positive ($L^p$-contractive) on $H_0$.*

*Proof.* Apply Theorem 3.20 to the closed (and convex) sets

$$C_\varepsilon := \{u \in L^2(\Omega_\varepsilon) : u \ge 0 \text{ a.e.}\}$$

and

$$C_\varepsilon := \{u \in L^2(\Omega_\varepsilon) \cap L^p(\Omega_\varepsilon) : \|u\|_{L^p(\Omega_\varepsilon)} \le 1\}$$

$p \in [1,\infty]$, respectively. □

If we are in the situation that the semigroups on $H_\varepsilon = L^2(\Omega_\varepsilon)$ are $L^\infty$-contractive, we can even establish convergence in $\mathscr{L}(L^p(\Omega_\varepsilon))$. We could also state the result in a more general version for arbitrary interpolation spaces. But this would involve several technical assumption that we prefer to avoid. It is clear that the analogous result for $1 < p < 2$ holds if we assume the semigroups to be $L^1$-contractive.

**Theorem 3.23.** *Let $(A_\varepsilon)_{\varepsilon>0}$ converge to $A_0$ in the sense of Definition 2.3 as $\varepsilon \to 0$, assume that $H_\varepsilon = L^2(\Omega_\varepsilon)$ for $\varepsilon \ge 0$ with measure spaces $(\Omega_\varepsilon)$, let $\theta \in (\arctan \frac{M}{\alpha}, \pi]$, let $\varphi \in H^\infty(\Sigma_\theta - \omega)$, and let $p \in [2,\infty)$. Assume that there exists a family $(c_\varepsilon)_{\varepsilon>0}$ of positive real numbers such that $c_\varepsilon J^{\uparrow\varepsilon}$, $c_\varepsilon^{-1} J^{\downarrow\varepsilon}$ and $\varphi(A_\varepsilon)$ are $L^\infty$-contractive for all $\varepsilon > 0$.*

*Then $J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon} \to \varphi(A_0)$ strongly as operators on $L^p(\Omega_0)$. If $\varphi \in H^\infty_{00}(\Sigma_\theta - \omega)$, the operators convergence even in operator norm, and in this case we have*

$$\|J^{\uparrow\varepsilon}\varphi(A_0)J^{\downarrow\varepsilon} - \varphi(A_\varepsilon)\|_{\mathscr{L}(L^p(\Omega_\varepsilon))} \to 0.$$

*Proof.* By Corollary 3.22 and Remark 3.21 also $\varphi(A_0)$ is $L^\infty$-contractive. Moreover, by Corollary 3.8,

$$\|\varphi(A_0)f - J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{H_0} \to 0$$

for all $f \in H_0$. Now by the interpolation inequality

$$\begin{aligned}
&\|\varphi(A_0)f - J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{L^p(\Omega_0)} \\
&\quad\le \|\varphi(A_0)f - J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{L^\infty(\Omega_0)}^{(p-2)/p} \;\|\varphi(A_0)f - J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{L^2(\Omega_0)}^{2/p} \\
&\quad\le \big(2\|f\|_{L^\infty(\Omega_0)}\big)^{(p-2)/p} \;\|\varphi(A_0)f - J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{L^2(\Omega_0)}^{2/p} \to 0
\end{aligned}$$

for all $f$ in the dense subspace $L^2(\Omega_0) \cap L^\infty(\Omega_0)$ of $L^p(\Omega_0)$. Since in addition

$$\|J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}\|_{\mathscr{L}(L^p(\Omega_0))} \le 1$$

by the Riesz-Thorin interpolation theorem, this proves strong convergence.

Now if $\psi \in H^\infty_{00}(\Sigma_\theta)$, then as in the proof of Corollary 3.8 there exists $C_\psi \ge 0$ such that

$$\|\psi(A_0) - J^{\downarrow\varepsilon}\psi(A_\varepsilon)J^{\uparrow\varepsilon}\|_{\mathscr{L}(H_0)} \le C_\psi\delta_\varepsilon.$$

Moreover, by assumption and Corollary 3.22, see also Remark 3.21,

$$\|\psi(A_0)f - J^{\downarrow\varepsilon}\psi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{L^\infty(\Omega_0)} \le 2\|f\|_{L^\infty(\Omega_0)}.$$

Thus by the Riesz-Thorin interpolation theorem

$$\|\psi(A_0) - J^{\downarrow\varepsilon}\psi(A_\varepsilon)J^{\uparrow\varepsilon}\|_{\mathscr{L}(L^p(\Omega_0))} \le 2^{(p-2)/p} C_\psi^{p/2}\delta_\varepsilon^{p/2} \to 0.$$

Employing Theorem 3.7 instead of Corollary 3.8, the same reasoning shows that

$$\|J^{\uparrow\varepsilon}\varphi(A_0)J^{\downarrow\varepsilon} - \varphi(A_\varepsilon)\|_{\mathscr{L}(L^p(\Omega))} \to 0.$$

□

## 4. Simple examples

In this section, we collect some examples to which the theory of the previous section can be applied without much effort. On the other hand, our main application, which involves some delicate calculations, is contained in a section by its own.

4.1. **Fourier series.** We start with an almost trivial example. Let $a_0$ be sectorial form with form domain $V_0$ on a Hilbert space $H_0$ as introduced in Section 2, and let $A_0$ be the associated operator. Assume that $V_0$ is compactly embedded into $H_0$, i.e., that $A_0$ has compact resolvent. For simplicity we also assume that $A_0$ is self-adjoint.

It is classical that in this situation there exists an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $H_0$ consisting of eigenvectors of $A_0$ to eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$, and $\lambda_k \to \infty$. We can assume that $\lambda_k \le \lambda_{k+1}$ for all $k \in \mathbb{N}$, and to make the notation simpler we assume that $\lambda_1 > 0$. Passing to an equivalent norm,

$$V_0 = \Big\{ f \in H_0 \, : \, \|f\|_{V_0}^2 = \sum_{k=1}^{\infty} \lambda_k \big|\langle f|e_k\rangle_{H_0}\big|^2 < \infty \Big\}.$$

We explain how this situation can be embedded into our framework. To this aim, it is convenient to index the Hilbert spaces and operators by $n \in \mathbb{N}$ instead of $\varepsilon$. Let $P_n$ denote the orthogonal projection onto $H_n := V_n := \operatorname{span}(e_k)_{k=1}^n$, $J^{\uparrow n} := J_1^{\uparrow n} := P_n$, and $J^{\downarrow n} := J_1^{\downarrow n} := \mathrm{id}$, where $H_n$ and $V_n$ carry the norms induced by $H_0$ and $V_0$, respectively, and let $a_n$ be the restriction of $a_0$ to $V_n$, so that $A_n$ is the restriction of $A_0$ to $H_n = H_n \cap D(A_0)$.

Now (2.7a), (2.7b) and (2.7d) are trivial; in fact, these conditions hold with $\delta_\varepsilon = 0$ and $\kappa = 1$. Moreover,

$$\|f - P_n f\|_{H_0}^2 = \sum_{k=n+1}^{\infty} \big|\langle f|e_k\rangle_{H_0}\big| \le \frac{1}{\lambda_{n+1}} \|f\|_{V_0}^2,$$

which implies both conditions in (2.7c). Finally, (2.7e) follows from the fact that

$$a_0(f,u) - a_n(P_n f, u) = \sum_{k=1}^{\infty} \lambda_k \langle f - P_n f|e_k\rangle_{H_0} \langle e_k|u\rangle_{H_0} = 0$$

for all $f \in V_0$ and $u \in V_n$. Hence the operators $A_n$ converge to $A_0$ in the sense of Definition 2.3.

The results in Section 3 now tell us that

$$\|P_n R(z, A_0) - R(z, A_n)\|_{\mathscr{L}(H_0)} \to 0,$$

as well as that other functions of these operators like the generated semigroup converge in this sense. Convergence of the spectrum as in Corollary 3.14 and Theorem 3.17 is of course built into this approximation.

4.2. **Varying coefficients.** Studying the convergence of elliptic operators with varying coefficients is a very classical topic. In fact, the underlying spaces typically do not change, so that the theory in Kato's book applies. However, sometimes it is convenient to incorporate the coefficients into the measure of the underlying $L^2$-space. Although such problems are still accessible by classical methods if all the norms are uniformly equivalent, it is quite natural to work with scales of Hilbert spaces instead.

The following example is taken from [CFG$^+$08], where the authors proved strong convergence in $\mathrm{C}(\overline{\Omega})$ as well as in $L^p$ for every $p \in [1,\infty)$ for a class of elliptic operators with Wentzell boundary conditions. Applying our results, on the other hand, we obtain convergence in operator norm for all $p \in (1,\infty)$, see Theorem 3.23. Tracing the constants in the proofs, we in addition have explicit error estimates, and in particular we know the order of convergence, which answers the open question that closes [CFG$^+$08]. We also mention the later article [CGG08], where these results are refined by obtaining a detailed estimate on the order of convergence in operator norm in $H^1$.

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Then $\Gamma := \partial\Omega$ becomes an oriented compact Riemannian manifold with Riemannian metric $g$ in a natural way, where the charts are Lipschitz regular, and the metric is bounded and measurable. As in the smooth case, there exists a volume measure $\sigma$ on $\Gamma$, which coincides with the $(n-1)$-dimensional Hausdorff measure. Let $H^1(\Gamma)$ be the completion of Lipschitz-continuous functions $u$ on $\Gamma$ with respect to the norm defined by

$$\|u\|^2_{H^1(\Gamma)} := \int_\Gamma \bigl(|u|^2 + |\mathrm{d}u|^2_g\bigr)\,\mathrm{d}\sigma,$$

where

$$|\mathrm{d}u|^2_g = \sum_{i,j=1}^{n-1} g^{ij}\partial_i u \partial_j \overline{u} \tag{4.1}$$

in a chart $U \subset \Gamma$ with coordinates $x_i \colon U \longrightarrow \mathbb{R}$, $i = 1, \dots, n-1$, and tangential vectors $\partial_i = \partial/\partial x_i$. Moreover, $(g^{ij})$ is the inverse of $(g_{ij}) = (g(\partial_i, \partial_j))$. For an ad hoc definition of Lipschitz-regular manifolds, we refer to [ABtE08].

Now let the families $(\mathscr{A}_\varepsilon)_{\varepsilon\geq 0} \subset L^\infty(\Omega; \mathscr{L}(\mathbb{C}^n))$, $(\beta_\varepsilon)_{\varepsilon\geq 0} \subset L^\infty(\Gamma)$, $(\gamma_\varepsilon)_{\varepsilon\geq 0} \subset L^\infty(\Gamma)$ and $(q_\varepsilon)_{\varepsilon\geq 0} \subset \mathbb{R}$ be bounded in the respective spaces, and assume that there exist $\alpha > 0$ and $b > 0$ such that for all $\varepsilon \geq 0$ we have $q_\varepsilon \geq \alpha$,

$$\mathrm{Re}\langle \mathscr{A}_\varepsilon \xi | \xi \rangle_{\mathbb{C}^n} \geq \alpha |\xi|^2$$

on $\Omega$ for all $\xi \in \mathbb{C}^n$ and $\beta_\varepsilon \geq b$ on $\Gamma$. For $\varepsilon \geq 0$, define

$$H_\varepsilon := L^2(\Omega) \times L^2\Bigl(\Gamma; \frac{\mathrm{d}\sigma}{\beta_\varepsilon}\Bigr)$$

and

$$V_\varepsilon := \bigl\{(u, f) \in H^1(\Omega) \times H^1(\Gamma) : u|_\Gamma = f\bigr\} \subset H_\varepsilon,$$

and equip these spaces with the natural scalar products. Note that the space $V_\varepsilon$ and its norm do in fact not depend on $\varepsilon$.

**Proposition 4.1.** *The family $(a_\varepsilon)_{\varepsilon\geq 0}$ of sesquilinear forms with form domains $V_\varepsilon$ which is defined by*

$$a_\varepsilon\bigl((u, u|_\Gamma), (v, v|_\Gamma)\bigr) := \int_\Omega \langle \mathscr{A}\nabla u | \nabla v\rangle_{\mathbb{C}^n} + \int_\Gamma \gamma_\varepsilon u \overline{v} \frac{\mathrm{d}\sigma}{\beta_\varepsilon} + q_\varepsilon \int_\Gamma \langle \mathrm{d}u | \mathrm{d}v\rangle_g \,\mathrm{d}\sigma$$

*is equi-sectorial.*

*Proof.* By the uniformity conditions on the coefficients,

$$\|u\|^2_{H_\varepsilon} = \|u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Gamma; \frac{\mathrm{d}\sigma}{\beta_\varepsilon})} \leq \|u\|^2_{H^1(\Omega)} + \tfrac{1}{b}\|u\|^2_{H^1(\Gamma)} \leq \bigl(1 + \tfrac{1}{b}\bigr)\|u\|^2_{V_\varepsilon},$$

which shows that the embedding of $V_\varepsilon$ into $H_\varepsilon$ has a uniform constant. Moreover,

$$\begin{aligned} |a_\varepsilon\bigl((u, u|_\Gamma), (v, v|_\Gamma)\bigr)| &\leq \|\mathscr{A}_\varepsilon\|_{L^\infty(\Omega, \mathscr{L}(\mathbb{C}^n))} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &\quad + \|\gamma_\varepsilon\|_\infty \|u\|_{L^2(\Gamma; \frac{\mathrm{d}\sigma}{\beta_\varepsilon})} \|v\|_{L^2(\Gamma; \frac{\mathrm{d}\sigma}{\beta_\varepsilon})} + q_\varepsilon \|u\|_{H^1(\Gamma)} \|v\|_{H^1(\Gamma)}, \end{aligned}$$

which shows that the forms are uniformly bounded, and

$$\begin{aligned} &\mathrm{Re}\, a_\varepsilon\bigl((u, u|_\Gamma), (u, u|_\Gamma)\bigr) \\ &\quad\geq \alpha\|\nabla u\|^2_{L^2(\Omega)} + \alpha\|\mathrm{d}u\|^2_{L^2(\Gamma)} - \|\gamma_\varepsilon\|_\infty \|u\|^2_{L^2(\Gamma; \frac{\mathrm{d}\sigma}{\beta_\varepsilon}))} \\ &\quad\geq \alpha\Bigl(\|u\|^2_{H^1(\Omega)} + \|u\|^2_{H^1(\Gamma)}\Bigr) - \alpha\|u\|^2_{L^2(\Omega)} - \bigl(\|\gamma_\varepsilon\|_\infty + \alpha\|\beta_\varepsilon\|_\infty\bigr)\|u\|^2_{L^2(\Gamma; \frac{\mathrm{d}\sigma}{\beta_\varepsilon})}, \end{aligned}$$

which shows that the ellipticity constants are uniform with respect to $\varepsilon \geq 0$. $\square$

*Remark* 4.2. Integration by parts shows that (at least formally) the operator $A_\varepsilon$ on $H_\varepsilon$ associated with $a_\varepsilon$ acts as

$$A_\varepsilon\big((u,u|_\Gamma)\big) = \Big(-\operatorname{div}(\mathscr{A}\nabla u), (\mathscr{A}\nabla u)\cdot\nu + \gamma_\varepsilon u|_\Gamma) - q_\varepsilon\beta_\varepsilon\Delta_\Gamma u|_\Gamma)\Big),$$

where $\nu$ denotes the outer unit normal of $\Omega$, and $\Delta_\Gamma$ is the Laplace-Beltrami operator on $\Gamma$, i.e., $A_\varepsilon$ is the operator considered in [CFG$^+$08].

**Proposition 4.3.** *There exists a constant $K$ depending only on $b$ and $\|\beta_0\|_\infty$ and $\|\gamma_0\|_\infty$ such that the operator $A_\varepsilon$ is $\delta_\varepsilon$-$\kappa$-quasi-unitarily equivalent to $A_0$ for $\kappa = 1$ and*

$$\delta_\varepsilon = \mathrm{O}\Big(\|\mathscr{A}_\varepsilon - \mathscr{A}_0\|_\infty + \|\beta_\varepsilon - \beta_0\|_\infty + \|\gamma_\varepsilon - \gamma_0\|_\infty + |q_\varepsilon - q_0|\Big).$$

*Moreover, this equivalence can be realised by taking the identification operators to be the identity operators.*

*Proof.* Let $J^{\uparrow\varepsilon}$, $J_1^{\uparrow\varepsilon}$, $J^{\downarrow\varepsilon}$ and $J_1^{\downarrow\varepsilon}$ be the identity operators between the respective spaces. Then (2.7a), (2.7c) and (2.7d) hold trivially with $\delta_\varepsilon = 0$ and $\kappa = 1$.

To check (2.7b), fix $(u,f) \in H_0$ and $(v,g) \in H_\varepsilon$. Then

$$\begin{aligned}
&\big|\langle J^{\uparrow\varepsilon}(u,f)|(v,g)\rangle_{H_\varepsilon} - \langle (u,f)|J^{\downarrow\varepsilon}(v,g)\rangle_{H_0}\big| \\
&\quad\le \int_\Gamma \big|\tfrac{1}{\beta_\varepsilon} - \tfrac{1}{\beta_0}\big|\,|f||g|\,\mathrm{d}\sigma \le \|\beta_\varepsilon - \beta_0\|_\infty \int_\Gamma \frac{|fg|}{b\sqrt{\beta_0\beta_\varepsilon}}\,\mathrm{d}\sigma \\
&\quad\le \frac{\|\beta_\varepsilon - \beta_0\|_\infty}{b}\|(u,f)\|_{H_0}\|(v,g)\|_{H_\varepsilon},
\end{aligned}$$

i.e., (2.7b) holds with $\delta_\varepsilon = b^{-1}\|\beta_\varepsilon - \beta_0\|_\infty$, compare Remark 2.4.

Finally, to check (2.7e), fix $(u,u|_\Gamma) \in V_0$ and $(v,v|_\Gamma) \in V_\varepsilon$. Then

$$\begin{aligned}
&\big|a_0\big((u,u|_\Gamma), J_1^{\downarrow\varepsilon}(v,v|_\Gamma)\big) - a_\varepsilon\big(J_1^{\uparrow\varepsilon}(u,u|_\Gamma),(v,v|_\Gamma)\big)\big| \\
&\quad\le \|\mathscr{A}_\varepsilon - \mathscr{A}_0\|_\infty \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\
&\qquad + \Big\|\frac{\gamma_0}{\beta_0} - \frac{\gamma_\varepsilon}{\beta_\varepsilon}\Big\|_\infty \|u\|_{L^2(\Gamma)}\|v\|_{L^2(\Gamma)} + |q_0 - q_\varepsilon|\|u\|_{H^1(\Gamma)}\|v\|_{H^1(\Gamma)}.
\end{aligned}$$

Finally, note that

$$\Big\|\frac{\gamma_0}{\beta_0} - \frac{\gamma_\varepsilon}{\beta_\varepsilon}\Big\|_\infty \le \frac{\|\beta_\varepsilon\gamma_0 - \beta_0\gamma_\varepsilon\|_\infty}{b^2} \le \frac{\|\gamma_0\|_\infty}{b^2}\|\beta_\varepsilon - \beta_0\|_\infty + \frac{\|\beta_0\|_\infty}{b^2}\|\gamma_0 - \gamma_\varepsilon\|_\infty,$$

which concludes the proof. □

It is easy to check using the Beurling-Deny criteria that the semigroups $(\mathrm{e}^{-tA_\varepsilon})$ are positive and quasi-contractive in the norm of

$$L^\infty(\Omega) \times L^\infty\Big(\Gamma; \frac{\mathrm{d}\sigma}{\beta_\varepsilon}\Big),$$

i.e.,

$$\|\mathrm{e}^{-tA_\varepsilon}\|_{L^\infty(\Omega)\times L^\infty(\Gamma;\frac{\mathrm{d}\sigma}{\beta_\varepsilon})} \le \mathrm{e}^{rt},$$

where $r \in \mathbb{R}$ depends only on a lower bound of $(\gamma_\varepsilon)_{\varepsilon\ge 0}$. By duality, we obtain uniform quasi-contractivity also in the space

$$L^1(\Omega) \times L^1\Big(\Gamma; \frac{\mathrm{d}\sigma}{\beta_\varepsilon}\Big).$$

In fact, the adjoint operator satisfies the same conditions as $A_\varepsilon$ itself.

Thus, by Theorem 3.23 (and its dual version for $p < 2$) we obtain the following result. In fact, the proof of Theorem 3.23 provides an explicit error estimate.

**Theorem 4.4.** *Assume that $\mathscr{A}_\varepsilon \to \mathscr{A}_0$, $\beta_\varepsilon \to \beta_0$, $\gamma_\varepsilon \to \gamma_0$ and $q_\varepsilon \to q_0$ uniformly on $\Omega$ or $\Gamma$, respectively. Then for every $t \ge 0$ and $p \in (1,\infty)$ the operators $\mathrm{e}^{-tA_\varepsilon}$ converge to $\mathrm{e}^{-tA_0}$ as $\varepsilon \to 0$ in the operator norm of $L^p(\Omega) \times L^p\big(\Gamma; \mathrm{d}\sigma\big)$.*

4.3. **Degenerate equations in non-divergence form.** Now we show that our machinery also applies to the approximation of degenerate elliptic operators in non-divergence form. More precisely, we study the operator $m\Delta$ with Dirichlet boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^N$ for a possibly degenerate function $m$. This operator has been studied for example by Arendt and Chovanec [AC10], who investigated under which conditions its part in $\mathrm{C}_0(\Omega)$ generates a $\mathrm{C}_0$-semigroup.

Let $m_0 \colon \Omega \to (0,\infty)$ be a bounded, measurable function and assume that $\frac{1}{m_0} \in L^q(\Omega)$, where $q = 1$ if $N = 1$, $q > 1$ if $N = 2$, and $q = \frac{N}{2}$ if $N \geq 3$. Define $m_\varepsilon := \max\{m_0, \varepsilon\}$ Our goal is to show that the uniformly elliptic operators $m_\varepsilon\Delta$ converge to the (possibly degenerate) operator $m_0\Delta$ in the sense of our abstract framework as $\varepsilon \to 0$, where

$$\begin{aligned} \operatorname{dom}(m_\varepsilon\Delta) := \bigl\{\, & u \in H_0^1(\Omega) \cap L^2(\Omega; \tfrac{\mathrm{d}x}{m_\varepsilon(x)}) \; : \\ & \exists f \in L^2(\Omega; \tfrac{\mathrm{d}x}{m_\varepsilon(x)}) \text{ such that } \Delta u = \tfrac{f}{m_\varepsilon} \,\bigr\}, \qquad (m_\varepsilon\Delta)u := f. \end{aligned}$$

Here, in the definition of $D(m_\varepsilon\Delta)$, the expression $\Delta u$ has to be understood as a distribution.

We start by introducing the forms that give rise to these operators. Define

$$H_\varepsilon := L^2(\Omega; \tfrac{\mathrm{d}x}{m_\varepsilon(x)}) \quad \text{and} \quad V_\varepsilon := H_0^1(\Omega).$$

Note that $V_\varepsilon \subset H_\varepsilon$ even for $\varepsilon = 0$ by the Sobolev embedding theorem and Hölder's inequality due to the integrability assumption $\frac{1}{m_0} \in L^q(\Omega)$. Thus the natural inner product

$$\langle u|v\rangle_{V_\varepsilon} := \int_\Omega \nabla u \overline{\nabla v} + \langle u|v\rangle_{H_\varepsilon}, \tag{4.2}$$

turns $V_\varepsilon \cap H_\varepsilon = V_\varepsilon$ into a Hilbert space. Here we used the equivalent norm $u \mapsto \|\nabla u\|$ on $H_0^1(\Omega)$. For the norm associated to (4.2), the embedding constant of $V_\varepsilon$ into $H_\varepsilon$ is at most 1. We emphasise that in general the Hilbert spaces $H_\varepsilon$ do not agree with $H_0$, not even as sets. We define the form $a_\varepsilon \colon V_\varepsilon \times V_\varepsilon \to \mathbb{C}$ by

$$a_\varepsilon(f,g) := \int_\Omega \nabla u \overline{\nabla v}.$$

Then $a_\varepsilon$ is bounded with constant $M = 1$, and $a_\varepsilon$ is elliptic with constants $\omega = 1$ and $\alpha = 1$. Hence the family $(a_\varepsilon)_{\varepsilon \geq 0}$ is equi-sectorial in the sense of Definition 2.1. The form $a_\varepsilon$ is associated with the operator $-m_\varepsilon\Delta$ as defined above, compare [AC10].

**Theorem 4.5.** *The operators $m_\varepsilon\Delta$ and $m_0\Delta$ are $\delta_\varepsilon$-$\kappa$-quasi unitarily equivalent for $\kappa = 1$ and a family $(\delta_\varepsilon)_{\varepsilon>0}$ of real numbers such that $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$.*

*Proof.* For simplicity, we assume that $N \geq 3$. Define

$$J^{\downarrow\varepsilon} u := \sqrt{\frac{m_0}{m_\varepsilon}}\, u \quad \text{and} \quad J^{\uparrow\varepsilon} f := \sqrt{\frac{m_\varepsilon}{m_0}}\, f.$$

Then $J^{\uparrow\varepsilon} \colon H_\varepsilon \to H_0$ and $J^{\downarrow\varepsilon} \colon H_0 \to H_\varepsilon$ are isometric isomorphisms, hence unitary. Moreover, $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$ are inverse to each other, so (2.7b), (2.7c) and (2.7d) are satisfied with $\delta_\varepsilon = 0$ and $\kappa = 1$.

We take $J_1^{\uparrow\varepsilon}$ and $J_1^{\downarrow\varepsilon}$ to be the identity. Then (2.7e) is fulfilled with $\delta_\varepsilon = 0$. Moreover, by Hölder's inequality

$$\begin{aligned} \|J^{\uparrow\varepsilon} u - J_1^{\uparrow\varepsilon} u\|_{H_\varepsilon}^2 &\leq \int_\Omega \Bigl| \sqrt{\frac{m_\varepsilon}{m_0}} - 1 \Bigr|^2 \frac{|u|^2}{m_\varepsilon} \leq \|\tfrac{1}{\sqrt{m_0}} - \tfrac{1}{\sqrt{m_\varepsilon}}\|_{L^N(\Omega)}^2 \|u\|_{L^{\frac{2N}{N-2}}(\Omega)}^2 \\ &\leq c^2 \delta_\varepsilon^{2/N} \|u\|_{V_0}^2 \end{aligned}$$

for all $u \in V_0$ with the embedding constant $c$ of $H^1_0(\Omega)$ into $L^{\frac{2N}{N-2}}(\Omega)$, and for

$$\delta_\varepsilon := \int_\Omega \Big| \frac{1}{\sqrt{m_0}} - \frac{1}{\sqrt{m_\varepsilon}} \Big|^N.$$

Since

$$\Big| \frac{1}{\sqrt{m_0}} - \frac{1}{\sqrt{m_\varepsilon}} \Big|^N \le \frac{1}{m_0^{N/2}} \in L^1(\Omega)$$

by assumption and in addition $m_\varepsilon(x) \to m_0(x)$ for all $x \in \Omega$, we obtain that $\delta_\varepsilon \to 0$ by the dominated convergence theorem. The other inequality in (2.7b) is proved in a similar way; in fact, the calculations are symmetric in $m_0$ and $m_\varepsilon$. □

## 5. Shrinking tubes with Robin boundary conditions

In this section we present our main example of convergence of Laplacians acting in different Hilbert spaces. We consider a family of manifolds $X_\varepsilon$ with boundary together with the corresponding Laplacian $A_\varepsilon$ with (in general) non-local boundary conditions.

We use Robin-type boundary condition of a certain scaling. In Remark 5.2 we compare our approach with the ones used in [Gri08] and [CF08].

5.1. **The metric graph model.** As an example of our approximation scheme, we consider a diffusive process on a family of $(m+1)$-dimensional manifolds $(X, g_\varepsilon)$ converging to a limit space given by a metric graph $X_0$. We will now present the construction in detail. We consider compact spaces only. For the non-compact case, see Remark 5.10.

Let $(\mathsf{V}, \mathsf{E}, \partial)$ be a directed graph where $\mathsf{V}$ and $\mathsf{E}$ are finite sets, the set of *vertices* and *edges*. Furthermore, $\partial \colon \mathsf{E} \longrightarrow \mathsf{V} \times \mathsf{V}$ encodes the graph structure and orientation by associating to an edge $\mathsf{e} \in \mathsf{E}$ the pair $(\partial_- \mathsf{e}, \partial_+ \mathsf{e})$ of its *initial* and *terminal* vertex. The orientation is only introduced for convenience. The definition of $A_0$ below does not depend on the choice of orientation. We denote by

$$\mathsf{E}^\pm_\mathsf{v} := \{\, \mathsf{e} \in \mathsf{E} \,:\, \partial_\pm \mathsf{e} = \mathsf{v} \,\} \qquad \text{and} \qquad \mathsf{E}_\mathsf{v} := \mathsf{E}^-_\mathsf{v} \mathbin{\dot\cup} \mathsf{E}^+_\mathsf{v}$$

the set of edges *terminating in* $\mathsf{v}$ $(+)$, *starting in* $\mathsf{v}$ $(-)$ resp. *adjacent with* $\mathsf{v}$. We denote by $\deg \mathsf{v} := |\mathsf{E}_\mathsf{v}|$ the *degree* of a vertex $v$, i.e., the number of edges terminating and starting in $\mathsf{v}$.

Let $X_0$ be the topological graph associated to $(\mathsf{V}, \mathsf{E}, \partial)$, i.e., the edges are 1-dimensional intervals meeting in the vertices according to the graph structure. The *metric* structure of $X_0$ is defined by a function $\ell \colon \mathsf{E} \longrightarrow (0, \infty)$ associating to each edge $\mathsf{e}$ a length $\ell_\mathsf{e}$. We parametrise each edge with a coordinate $s = s_\mathsf{e}$, i.e., we identify the directed edge $\mathsf{e}$ with the associated *metric edge* $I_\mathsf{e} := [0, \ell_\mathsf{e}]$ in such a way that $\partial_- \mathsf{e}$ corresponds to $s = 0$ and $\partial_+ \mathsf{e}$ corresponds to $s = \ell_\mathsf{e}$. Introducing the obvious distance function now turns the topological graph $X_0$ into a metric space, the *metric graph*. Similarly, we have a natural measure on $X_0$ given by the Lebesque measure on each edge $\mathrm{d}s = \mathrm{d}s_\mathsf{e}$.

The basic Hilbert space is

$$H_0 := L^2(X_0) := \bigoplus_{\mathsf{e} \in \mathsf{E}} L^2(I_\mathsf{e}),$$

with norm[1] $\|f\|^2_{X_0} = \sum_{\mathsf{e} \in \mathsf{E}} \|f_\mathsf{e}\|^2_{I_\mathsf{e}}$, where $L^2(I_\mathsf{e})$ carries the usual norm given by $\|f_\mathsf{e}\|^2_{I_\mathsf{e}} = \int_0^{\ell_\mathsf{e}} |f_\mathsf{e}|^2 \, \mathrm{d}s$.

[1] Here and in the sequel, we use the notation $\|f\|_M$ for the $L^2$-norm of a measurable function $u \colon M \longrightarrow \mathbb{C}$ on a measure space $M$.

The Hilbert space, which will serve as domain of the sesquilinear form defined below, is given by

$$V_0 := H^1(X_0) := C(X_0) \cap \bigoplus_{\mathsf{e}\in\mathsf{E}} H^1(I_\mathsf{e})$$

with norm defined by

$$\|f\|_{V_0}^2 := \sum_{\mathsf{e}\in\mathsf{E}} \bigl(\|f'\|_{I_\mathsf{e}}^2 + \|f\|_{I_\mathsf{e}}^2\bigr),$$

i.e., a function in $V_0$ is of class $H^1$ along the edges and also continuous at each vertex. Trivially, $\|f\|_{H_0} \le \|f\|_{V_0}$, i.e., we can choose $C_V = 1$ in (2.3).

We define the operator governing an evolution process via the sesquilinear form

$$a_0(f,g) := a_{0,\mathsf{V}}(f,g) + a_{0,\mathsf{E}}(f,g) \tag{5.1}$$

for functions $f, g \in V_0$, where

$$a_{0,\mathsf{V}}(f,g) := \sum_{\mathsf{v}\in\mathsf{V}} \sum_{\mathsf{w}\in\mathsf{V}} \gamma_{\mathsf{vw}} f(\mathsf{w}) \overline{g(\mathsf{v})} \deg \mathsf{v} \quad \text{and}$$

$$a_{0,\mathsf{E}}(f,g) := \langle f' | g' \rangle_{X_0} = \sum_{\mathsf{e}\in\mathsf{E}} \int_0^{\ell_\mathsf{e}} f'_\mathsf{e} \overline{g}'_\mathsf{e} \, \mathrm{d}s$$

for a given coefficient matrix $(\gamma_{\mathsf{vw}})_{\mathsf{v},\mathsf{w}\in\mathsf{V}}$.

The following estimate follows easily from a standard Sobolev estimate on an interval. In particular, we have

$$\|\underline{f}\|_{\mathsf{V}}^2 := \sum_{\mathsf{v}\in\mathsf{V}} |f(\mathsf{v})|^2 \deg \mathsf{v} \le 4b\|f'\|_{X_0}^2 + \frac{8}{b}\|f\|_{X_0}^2 \tag{5.2}$$

for $f \in H^1(X_0)$, where $\underline{f} = (f(\mathsf{v}))_{\mathsf{v}\in\mathsf{V}}$ and $0 < b \le \min_\mathsf{e} \ell_\mathsf{e}$, see e.g. [EP09]. The next proposition is an easy consequence of (5.2).

**Proposition 5.1.** *The sesquilinear form $a_0$ is well-defined on $V_0 = H^1(X_0)$. Moreover, given $\alpha \in (0,1)$, there exists $\omega \ge 0$ such that*

$$\operatorname{Re} a_0(f,f) + \omega\|f\|_{H_0}^2 \ge \alpha\|f\|_{V_0}^2$$

*for all $f \in V_0 = H^1(X_0)$. In particular, $a_0$ is $H_0$-elliptic.*

It is easily seen that the corresponding operator $A_0$ acts as $(A_0 f)_\mathsf{e} = -f''_\mathsf{e}$ on each edge for $f \in \operatorname{dom} A_0$, where $f \in \operatorname{dom} A_0$ iff $f \in C(X_0) \cap \bigoplus_{\mathsf{e}\in\mathsf{E}} H^2(I_\mathsf{e})$ and

$$\frac{1}{\deg \mathsf{v}} \sum_{\mathsf{e}\in\mathsf{E}_\mathsf{v}} f'_\mathsf{e}(\mathsf{v}) + \sum_{\mathsf{w}\in\mathsf{V}} \gamma_{\mathsf{vw}} f(\mathsf{w}) = 0,$$

where $f'_\mathsf{e}(\mathsf{v}) = -f'_\mathsf{e}(0)$ if $\mathsf{v} = \partial_-\mathsf{e}$ and $f'_\mathsf{e}(\mathsf{v}) = f'_\mathsf{e}(\ell_\mathsf{e})$ if $\mathsf{v} = \partial_+\mathsf{e}$. Observe that for a non-diagonal matrix $\gamma$ the vertex conditions defined above turn out to be non-local.

5.2. **The manifold model.** In the sequel, we will construct a manifold $X$ according to the graph $(\mathsf{V}, \mathsf{E}, \partial)$ together with a family of metrics $g_\varepsilon$ such that $(X, g_\varepsilon)$ shrinks to the metric graph $X_0$ in a suitable sense (see Figure 1).

Let $X$ be an $(m+1)$-dimensional connected manifold with boundary $\partial X$. We assume that $X$ decomposes as

$$X = \dot{\bigcup_{\mathsf{v}\in\mathsf{V}}} X_\mathsf{v} \cup \dot{\bigcup_{\mathsf{e}\in\mathsf{E}}} X_\mathsf{e}, \tag{5.3}$$

where the *vertex* and *edge manifolds*, $X_\mathsf{v}$ and $X_\mathsf{e}$, are compact connected subsets with non-empty interior. Moreover, we assume that $\{X_\mathsf{v}\}_{\mathsf{v}\in\mathsf{V}}$ and $\{X_\mathsf{e}\}_{\mathsf{e}\in\mathsf{E}}$ are families of pairwise disjoint sets, respectively (indicated by $\dot\cup$), and that

$$X_\mathsf{e} \cong I_\mathsf{e} \times Y_\mathsf{e},$$

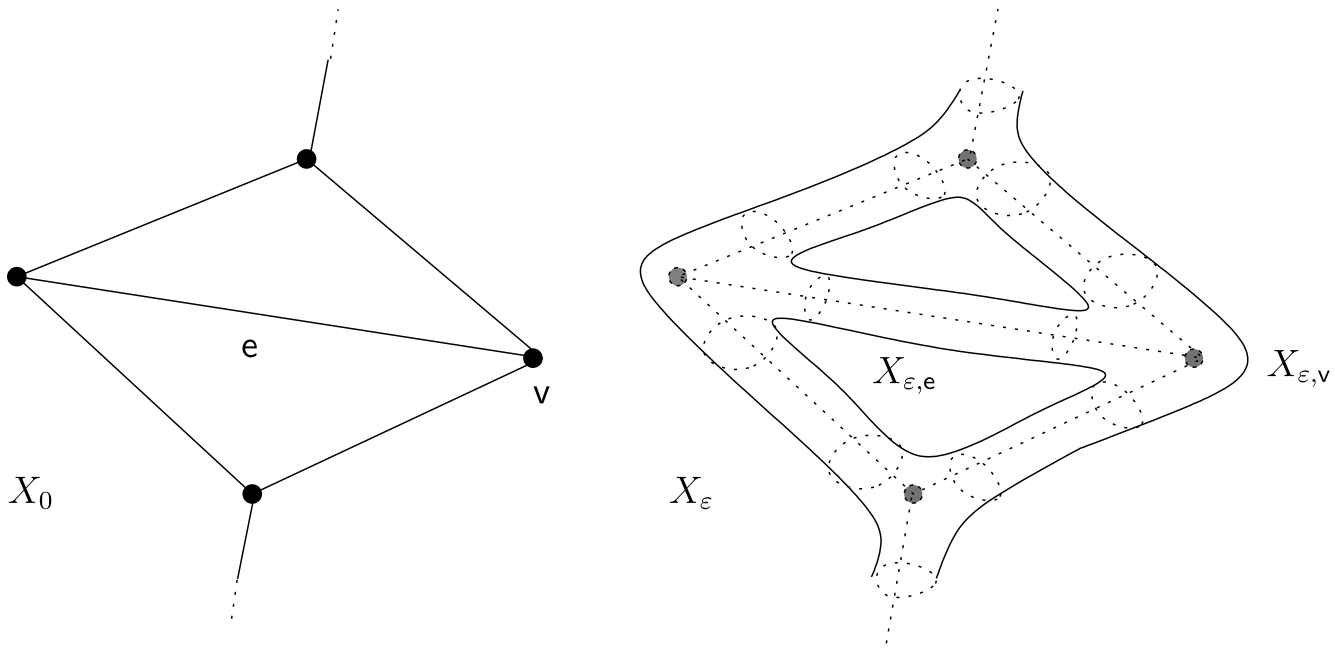


FIGURE 1. The metric graph $X_0$ and the family of manifolds $(X, g_\varepsilon)$ shrinking to the metric graph. Here, $(X, g_\varepsilon)$ can be considered as a subset of $\mathbb{R}^3$, i.e., as a full cylinder with boundary consisting of the surface of the pipeline network.

where $Y_{\mathsf{e}}$ is a compact, connected, $m$-dimensional manifold, the *transversal* or *cross-section manifold* at the edge $\mathsf{e}$. Note that $Y_{\mathsf{e}}$ has a boundary (as far as $\partial X \cap X_{\mathsf{e}}$ is non-empty). In the sequel, we will identify $X_{\mathsf{e}}$ with the product $I_{\mathsf{e}} \times Y_{\mathsf{e}}$. Finally, we assume that

$$Y_{\mathsf{v},\mathsf{e}} := X_{\mathsf{v}} \cap X_{\mathsf{e}} \cong \begin{cases} Y_{\mathsf{e}}, & \text{if } \mathsf{v} \in \partial\mathsf{e}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $g$ be a smooth Riemannian metric on $X$ having product structure on $X_{\mathsf{e}}$, i.e.,

$$g_{\mathsf{e}} = \mathrm{d}s_{\mathsf{e}}^2 + h_{\mathsf{e}}$$

on $X_{\mathsf{e}}$, where $h_{\mathsf{e}}$ is a Riemannian metric on $Y_{\mathsf{e}}$. Here, and in the sequel, we use the subscripts $(\cdot)_{\mathsf{v}}$ and $(\cdot)_{\mathsf{e}}$ to indicate the restriction to $X_{\mathsf{v}}$ and $X_{\mathsf{e}}$ for objects on the manifold. By assumption, $X_{\mathsf{v}}$ is a manifold with boundary in which the disjoint union of transversal manifolds

$$Y_{\mathsf{v}} = \dot{\bigcup_{\mathsf{e} \in \mathsf{E}_{\mathsf{v}}}} Y_{\mathsf{v},\mathsf{e}}$$

is embedded. In addition, the embedding is isometric. We can think of $X$ as being constructed from the graph $(\mathsf{V}, \mathsf{E}, \partial)$ and the family of transversal manifolds $\{Y_{\mathsf{e}}\}_{\mathsf{e} \in \mathsf{E}}$ and vertex manifolds $\{X_{\mathsf{v}}\}_{\mathsf{v} \in \mathsf{V}}$ according to the graph.

Let us now define the family of $\varepsilon$-depending metrics on $X$ via

$$g_{\varepsilon,\mathsf{v}} := \varepsilon^2 g_{\mathsf{v}} \qquad \text{and} \qquad g_{\varepsilon,\mathsf{e}} := \mathrm{d}s_{\mathsf{e}}^2 + \varepsilon^2 h_{\mathsf{e}},$$

i.e., $(X, g_\varepsilon)$ is obtained from the manifold $(X, g)$ by $\varepsilon$-homothetically shrinking of the vertex manifold $X_{\mathsf{v}}$ and the transversal manifold $Y_{\mathsf{e}}$ of the edge manifold $X_{\mathsf{e}}$ (thin tube). Note that $(X, g_\varepsilon)$ defines a smooth Riemannian manifold. The smoothness of the metric along the passage from $X_{\mathsf{v}}$ to $X_{\mathsf{e}}$ is assured since the original metric $g = g_1$ is assumed to be smooth on $X$.

If the metric graph $X_0$ is embedded in $\mathbb{R}^{m+1}$, then one can choose a closed neighbourhood $X_\varepsilon$ of $X_0$ in $\mathbb{R}^{m+1}$ with smooth boundary and thickness of order $\varepsilon$. The smoothness is assumed only for simplicity; a Lipschitz boundary would be enough. Note that a decomposition as in (5.3) does not give an *isometric* decomposition, since the edge neighbourhood $X_{\varepsilon,\mathsf{e}}$ is slightly shorter than $\ell_\mathsf{e}$ due to the presence of the vertex neighbourhoods. Nevertheless, this example can be treated in the same way after a longitudinal rescaling of the edge variable. The error made is only of order $\varepsilon$. For details, we refer to [EP09, Lem. 2.7] or [Pos09, Prop. 5.3.10].

The decomposition (5.3) induces a decomposition of the boundary $\Gamma = \partial X$, an $m$-dimensional Riemannian manifold,

$$\Gamma = \dot{\bigcup_{\mathsf{v}\in\mathsf{V}}} \Gamma_\mathsf{v} \cup \dot{\bigcup_{\mathsf{e}\in\mathsf{E}}} \Gamma_\mathsf{e},$$

where $\Gamma_\mathsf{v} \subset \partial X_\mathsf{v}$ and $\Gamma_\mathsf{e} = I_\mathsf{e} \times \partial Y_\mathsf{e} \subset \partial X_\mathsf{e}$ are pairwise disjoint (or intersect only in sets of $m$-dimensional measure 0). Moreover, we have

$$\partial X_\mathsf{v} = \Gamma_\mathsf{v} \cup \bigcup_{\mathsf{e}\in\mathsf{E}_\mathsf{v}} Y_{\mathsf{v},\mathsf{e}} \qquad \text{and} \qquad \partial X_\mathsf{e} = \Gamma_\mathsf{e} \cup \bigcup_{\mathsf{v}\in\partial\mathsf{e}} Y_{\mathsf{v},\mathsf{e}}.$$

The Riemannian measure associated with a Riemannian manifold $(M, g_\varepsilon)$ is denoted by $\mathrm{d}M_\varepsilon$. In particular, we have

$$\mathrm{d}X_{\varepsilon,\mathsf{v}} = \varepsilon^{m+1}\,\mathrm{d}X_\mathsf{v}, \qquad \mathrm{d}\Gamma_{\varepsilon,\mathsf{v}} = \varepsilon^{m}\,\mathrm{d}\Gamma_\mathsf{v}, \tag{5.4a}$$

$$\mathrm{d}X_{\varepsilon,\mathsf{e}} = \varepsilon^{m}\,\mathrm{d}X_\mathsf{e} = \varepsilon^{m}\,\mathrm{d}s_\mathsf{e}\,\mathrm{d}Y_\mathsf{e}, \qquad \mathrm{d}\Gamma_{\varepsilon,\mathsf{e}} = \varepsilon^{m-1}\,\mathrm{d}\Gamma_\mathsf{e} = \varepsilon^{m-1}\,\mathrm{d}s_\mathsf{e}\,\mathrm{d}\partial Y_\mathsf{e}. \tag{5.4b}$$

We will use the abbreviation $X_\varepsilon$, $X_{\varepsilon,\mathsf{v}}$ etc. for the measure spaces $(X, \mathrm{d}X_\varepsilon)$, $(X_\mathsf{v}, \mathrm{d}X_{\varepsilon,\mathsf{v}})$, etc.

Here and in the sequel, we use the notation

$$X_\mathsf{E} := \bigcup_{\mathsf{e}\in\mathsf{E}} X_\mathsf{e}, \qquad X_\mathsf{V} := \bigcup_{\mathsf{v}\in\mathsf{V}} X_\mathsf{v}, \qquad \Gamma_\mathsf{E} := \bigcup_{\mathsf{e}\in\mathsf{E}} \Gamma_\mathsf{e}$$

etc. for the (disjoint) union of the corresponding manifolds. Similarly, $X_{\varepsilon,\mathsf{E}}$, $X_{\varepsilon,\mathsf{V}}$, $\Gamma_{\varepsilon,\mathsf{E}}$ etc. denote the corresponding Riemannian manifolds with the $\varepsilon$-depending metric.

The basic Hilbert space we are working in is $H_\varepsilon := L^2(X_\varepsilon)$. We often write $\|u\|_{X_\varepsilon}$ instead of $\|u\|_{L^2(X_\varepsilon)}$ for the corresponding norm. The $\varepsilon$-dependence of the norms for the scaled spaces can easily calculated using (5.4); e.g. for $X_{\varepsilon,\mathsf{v}}$ and $X_{\varepsilon,\mathsf{e}}$ we have

$$\|u\|^2_{X_{\varepsilon,\mathsf{v}}} = \varepsilon^{m+1} \int_{X_\mathsf{v}} |u|^2\,\mathrm{d}X_\mathsf{v} \qquad \text{and} \qquad \|u\|^2_{X_{\varepsilon,\mathsf{e}}} = \varepsilon^{m} \int_{I_\mathsf{e}\times Y_\mathsf{e}} |u|^2\,\mathrm{d}Y_\mathsf{e}\,\mathrm{d}x_\mathsf{e}.$$

Let $V_\varepsilon := H^1(X_\varepsilon)$ be the Sobolev space of first order defined as the completion of smooth functions on $X_\varepsilon$ with respect to the norm defined by

$$\|u\|^2_{H^1(X_\varepsilon)} = \|u\|^2_{L^2(X_\varepsilon)} + \|\mathrm{d}u\|^2_{L^2(X_\varepsilon)},$$

where $\|\mathrm{d}u\|^2_{L^2(X_\varepsilon)} = \int_X |\mathrm{d}u|^2_{g_\varepsilon}\,\mathrm{d}X_\varepsilon$, and $|\mathrm{d}u|^2_{g_\varepsilon}$ is given in (4.1). Trivially, $\|u\|_{H_\varepsilon} \le \|u\|_{V_\varepsilon}$, i.e., we can choose $C_V = 1$ in (2.3).

We define a sesquilinear form by

$$a_\varepsilon(u,v) = \int_X \langle \mathrm{d}u | \mathrm{d}v\rangle_{g_\varepsilon}\,\mathrm{d}X_\varepsilon + \int_\Gamma \beta_\varepsilon u\overline{v}\,\mathrm{d}\Gamma_\varepsilon + \int_{\Gamma\times\Gamma} \gamma_\varepsilon(u\otimes\overline{v})\,\mathrm{d}\Gamma_\varepsilon\otimes\mathrm{d}\Gamma_\varepsilon \tag{5.5}$$

for functions $u \in V_\varepsilon = H^1(X_\varepsilon)$. Here $\langle\cdot|\cdot\rangle_{g_\varepsilon}$ is the (pointwise) inner product on $T^*X$ defined via the Riemannian metric $g_\varepsilon$. Moreover, $(u\otimes\overline{v})(x_1,x_2) := u(x_1)\overline{v(x_2)}$. We assume that $\beta_\varepsilon \in L_\infty(\Gamma)$ and $\gamma_\varepsilon \in L^2(\Gamma\times\Gamma)$. In this case, $a_\varepsilon$ is indeed well-defined for all $u \in H^1(X_\varepsilon)$ (see Proposition 5.7).

The associated operator is given by $A_\varepsilon u = -\Delta u = \mathrm{d}^*\mathrm{d}u$ for functions $u \in \operatorname{dom} A_\varepsilon$. Moreover, $u \in \operatorname{dom} A_\varepsilon$ iff $u \in H^2(X_\varepsilon)$ and

$$\partial_\mathrm{n} u + \beta_\varepsilon u + \int_\Gamma \gamma_\varepsilon u\,\mathrm{d}\Gamma_\varepsilon = 0,$$

where the integral is taken with respect to the first variable of $\gamma_\varepsilon\colon \Gamma\times\Gamma \longrightarrow \mathbb{C}$.

*Remark* 5.2. Let us illustrate the effect of scaling the underlying space for Robin boundary conditions in a simple example: Assume that the transversal manifold $Y_{\varepsilon,\mathsf{e}}$ is isometric to the interval $[0, \varepsilon]$ and that there is no non-local contribution, i.e., $\gamma_\varepsilon = 0$. We consider the Laplacian with Neumann boundary conditions at 0 and Robin boundary conditions at $\varepsilon$, i.e.,

$$v'(\varepsilon) + \beta_\varepsilon v(\varepsilon) = 0,$$

where $\beta_\varepsilon \in \mathbb{R} \setminus \{0\}$ is the coupling constant. An eigenfunction for the Laplacian $\Delta v = -v''$ with Neumann boundary conditions at 0 is of the form

$$v(s) = v(0)\cos(\omega s) \qquad \text{and} \qquad v(s) = v(0)\cosh(\omega s),$$

with eigenvalue $\omega^2$ and $-\omega^2$ if $\beta_\varepsilon > 0$ and $\beta_\varepsilon < 0$, respectively. The lowest eigenvalue $\mu(\varepsilon)$ is then of the same order as $\varepsilon^{-1}\beta_\varepsilon$ for $\varepsilon \to 0$.

If one chooses *scale-invariant* Robin boundary conditions, i.e., $\beta_\varepsilon = \varepsilon^{-1}\beta$ for some $\beta \neq 0$, as e.g. in [Gri08] and [CF08], then the lowest eigenvalue $\mu(\varepsilon)$ is of order $\varepsilon^{-2}$ and one has to substract the divergent term $\mu(\varepsilon)$ in order to expect convergence to a limit.

Here, we use a different approach. We assume that the coupling constant $\beta_\varepsilon$ is of order $\varepsilon^{3/2}$ on the edge neighbourhoods, see (5.13) (actually, $\varepsilon^{1+\eta}$ would be enough for some $\eta > 0$). In this case, the lowest (transversal) eigenvalue $\mu(\varepsilon)$ is of order $\varepsilon^{1/2}$ (resp. $\varepsilon^{\eta}$), and converges to 0. We are then in the situation, where the Robin Laplacian is close to the Neumann Laplacian. This is the reason why we are in the same setting as in the (simpler) Neumann boundary condition case treated already in [Pos06].

5.3. **Some estimates on the manifold.** Let us collect some estimates needed later on. Basically, we need a Sobolev trace estimate. Let $M$ be a compact Riemannian manifold of dimension $n$ with metric $g$ and boundary $\partial M$, and $B$ a compact $(n-1)$-dimensional submanifold of $\partial M$ carrying the induced metric. It follows that there is a constant $C^{\mathrm{tr}}_{B,M} > 0$ such that

$$\|u\|^2_B \le C^{\mathrm{tr}}_{B,M}\bigl(\|\mathrm{d}u\|^2_M + \|u\|^2_M\bigr) \tag{5.6a}$$

for all $u \in H^1(M)$. The constant $C^{\mathrm{tr}}_{B,M}$ geometrically depends on the shape of $B$ embedded in $M$.

If we scale the metric by a factor, $g_\varepsilon = \varepsilon^2 g$, then the estimate changes to

$$\|u\|^2_{B_\varepsilon} \le C^{\mathrm{tr}}_{B,M}\Bigl(\varepsilon\|\mathrm{d}u\|^2_{M_\varepsilon} + \frac{1}{\varepsilon}\|u\|^2_{M_\varepsilon}\Bigr), \tag{5.6b}$$

using $\mathrm{d}M_\varepsilon = \varepsilon^n\,\mathrm{d}M$, $\mathrm{d}B_\varepsilon = \varepsilon^{n-1}\,\mathrm{d}B$ and the fact that $|\mathrm{d}u|^2_{g_\varepsilon} = \varepsilon^{-2}|\mathrm{d}u|^2_g$. Here, $B_\varepsilon$ and $M_\varepsilon$ denote the corresponding Riemannian manifolds with the $\varepsilon$-depending metric.

We will apply this trace estimate basically in the situations $(\Gamma_{\mathsf{v}}, X_{\mathsf{v}})$, $(Y_{\mathsf{v}}, X_{\mathsf{v}})$ and $(\partial Y_{\mathsf{e}}, Y_{\mathsf{e}})$. Let us first prove the following lemma, which shows that the trace estimate for $(\partial Y_{\mathsf{e}}, Y_{\mathsf{e}})$ gives a trace estimate for the product $(\Gamma_{\mathsf{e}}, X_{\mathsf{e}}) = (I_{\mathsf{e}} \times \partial Y_{\mathsf{e}}, I_{\mathsf{e}} \times Y_{\mathsf{e}})$:

**Lemma 5.3.** *We have*

$$\|u\|^2_{\Gamma_{\varepsilon,\mathsf{e}}} \le C^{\mathrm{tr}}_{\partial Y_{\mathsf{e}},Y_{\mathsf{e}}}\Bigl(\varepsilon\|\mathrm{d}_{Y_{\mathsf{e}}}u\|^2_{X_{\varepsilon,\mathsf{e}}} + \frac{1}{\varepsilon}\|u\|^2_{X_{\varepsilon,\mathsf{e}}}\Bigr)$$

*for all $u \in H^1(X_{\varepsilon,\mathsf{e}})$, where $\mathrm{d}_{Y_{\mathsf{e}}}u$ denotes the exterior derivative with respect to the second variable of $I_{\mathsf{e}} \times Y_{\mathsf{e}}$.*

*Proof.* Let $u \in H^1(X_\varepsilon)$ be smooth, then

$$\|u\|^2_{\Gamma_{\varepsilon,\mathsf{e}}} = \int_0^{\ell_\mathsf{e}} \|u(s)\|^2_{\partial Y_{\varepsilon,\mathsf{e}}} \,\mathrm{d}s \le \int_0^{\ell_\mathsf{e}} C^{\mathrm{tr}}_{\partial Y_\mathsf{e}, Y_\mathsf{e}} \Big(\varepsilon \|\mathrm{d}_{Y_\mathsf{e}} u(s)\|^2_{Y_{\varepsilon,\mathsf{e}}} + \frac{1}{\varepsilon}\|u(s)\|^2_{Y_{\varepsilon,\mathsf{e}}}\Big)\,\mathrm{d}s$$
$$= C^{\mathrm{tr}}_{\partial Y_\mathsf{e}, Y_\mathsf{e}} \Big(\varepsilon \|\mathrm{d}_{Y_\mathsf{e}} u\|^2_{X_{\varepsilon,\mathsf{e}}} + \frac{1}{\varepsilon}\|u\|^2_{X_{\varepsilon,\mathsf{e}}}\Big)$$

using the Sobolev trace estimate (5.6b) for $\partial Y_\mathsf{e} \subset Y_\mathsf{e}$ pointwise. Since smooth functions are dense in $H^1(X_\mathsf{e})$ and since the operators $H^1(X_\mathsf{e}) \to L^2(\Gamma_\mathsf{e})$, $u \mapsto u\restriction_{\Gamma_\mathsf{e}}$ and $H^1(X_\mathsf{e}) \to L^2(X_\mathsf{e}, T^*X_\mathsf{e})$, $u \mapsto \mathrm{d}_{Y_\mathsf{e}} u$ are bounded, the estimate also holds for $u \in H^1(X_\mathsf{e})$. □

It follows from these trace estimates that the global trace operator $u \mapsto u\restriction_\Gamma$ is bounded, either as operator $H^1(X) \to L^2(\Gamma)$ or $H^1(X_\varepsilon) \to L^2(\Gamma_\varepsilon)$ (see also Proposition 5.7 below).

In the following, we need several averaging operators. Let

$$\fint_\mathsf{v} u := \fint_{X_\mathsf{v}} u \,\mathrm{d}X_\mathsf{v}. \tag{5.7}$$

denote the average value of $u$ on $X_\mathsf{v}$ (and also the corresponding constant function on $X_\mathsf{v}$). Here, $\fint_M := (\operatorname{vol} M)^{-1} \int$ is the normalised volume integral. Denote by $\lambda_2(X_\mathsf{v})$ the second (first non-vanishing) eigenvalue of the Neumann problem on $X_\mathsf{v}$. Let us now compare the average of $u$ on $\Gamma_\mathsf{v}$ with the average of $u$ on $X_\mathsf{v}$:

**Lemma 5.4.** *For all $u \in H^1(X_\mathsf{v})$, we have*

$$\varepsilon^m \|u - \fint_\mathsf{v} u\|^2_{\Gamma_\mathsf{v}} \le \varepsilon C^{\mathrm{tr}}_{\Gamma_\mathsf{v}, X_\mathsf{v}} \Big(\frac{1}{\lambda_2^{\mathrm{N}}(X_\mathsf{v})} + 1\Big) \|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{v}}}.$$

*Proof.* Interpreting $\widetilde{u} := u - \fint_\mathsf{v} u$ as a function on $\Gamma_\mathsf{v}$ we have

$$\varepsilon^m \|\widetilde{u}\|^2_{\Gamma_\mathsf{v}} \le \varepsilon^m C^{\mathrm{tr}}_{\Gamma_\mathsf{v}, X_\mathsf{v}} \big(\|\widetilde{u}\|^2_{X_\mathsf{v}} + \|\mathrm{d}u\|^2_{X_\mathsf{v}}\big) \le \varepsilon^m C^{\mathrm{tr}}_{\Gamma_\mathsf{v}, X_\mathsf{v}} \Big(\frac{1}{\lambda_2^{\mathrm{N}}(X_\mathsf{v})} + 1\Big) \|\mathrm{d}u\|^2_{X_\mathsf{v}}$$

using Cauchy-Schwarz, the Sobolev trace estimate (5.6a) for $\Gamma_\mathsf{v} \subset X_\mathsf{v}$ and the min-max principle

$$\lambda_2^{\mathrm{N}}(X_\mathsf{v}) \|\widetilde{u}\|^2_{X_\mathsf{v}} \le \|\mathrm{d}\widetilde{u}\|^2_{X_\mathsf{v}} = \|\mathrm{d}u\|^2_{X_\mathsf{v}},$$

since $\widetilde{u}$ is orthogonal to the first (constant) eigenfunction of the Neumann Laplacian on $X_\mathsf{v}$. The scaling property $\varepsilon^{m-1}\|\mathrm{d}u\|^2_{X_\mathsf{v}} = \|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{v}}}$ now gives the result. □

Next, we compare the average of $u$ on $Y_{\mathsf{v},\mathsf{e}}$ with the average of $u$ on $X_\mathsf{v}$. To do so, we introduce a partial averaging operator also needed later on for the identification operators. For simplicity, we assume that

$$\operatorname{vol}_m Y_\mathsf{e} = 1 \tag{5.8}$$

for all $\mathsf{e} \in \mathsf{E}$. We set

$$(\fint_\mathsf{e} u)(s) := \fint_{Y_\mathsf{e}} u(s, y) \,\mathrm{d}Y_\mathsf{e}(y), \tag{5.9}$$

Note that the integral exists for almost every $s$ and $\fint_\mathsf{e} u$ defines a function in $L^2(I_\mathsf{e})$. If $s = 0$ or $s = \ell_\mathsf{e}$ denotes the vertex $\mathsf{v} = \partial_-\mathsf{e}$ or $\mathsf{v} = \partial_+\mathsf{e}$, respectively, we also write $(\fint_\mathsf{e} u)(\mathsf{v})$.

The proof of the following lemma is similar to the proof of Lemma 5.4 (see also [EP09, Lem. 2.8]):

**Lemma 5.5.** *We have*

$$\varepsilon^m \sum_{\mathsf{e} \in \mathsf{E}_\mathsf{v}} |\fint_\mathsf{v} u - \fint_\mathsf{e} u(\mathsf{v})|^2 \le \varepsilon C^{\mathrm{tr}}_{Y_\mathsf{v}, X_\mathsf{v}} \Big(\frac{1}{\lambda_2^{\mathrm{N}}(X_\mathsf{v})} + 1\Big) \|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{v}}}$$

*for all $u \in H^1(X_\mathsf{v})$.*

We finally need an estimate over the vertex neighbourhoods. It will assure that in the limit $\varepsilon \to 0$, no family of normalised eigenfunctions $(u_\varepsilon)_\varepsilon$ with uniformly bounded eigenvalues can concentrate on $X_{\varepsilon,\mathsf{v}}$, i.e., $\|u\|_{X_{\varepsilon,\mathsf{v}}}/\|u\|_{X_\varepsilon} \to 0$. A proof of the following estimate was given e.g. in [EP09, Lem. 2.9]:

**Lemma 5.6.** *We have*

$$\|u\|^2_{X_{\varepsilon,\mathsf{v}}} \le \varepsilon^2 C_{\mathsf{v}} \|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{v}}} + 8\varepsilon c_{\mathrm{vol},\mathsf{v}} \Big[ b\|u'\|^2_{X_{\varepsilon,\mathsf{E}_\mathsf{v}}} + \frac{2}{b}\|u\|^2_{X_{\varepsilon,\mathsf{E}_\mathsf{v}}}\Big]$$

*for* $0 < b \le \min_{\mathsf{e}} \ell_{\mathsf{e}}$*, where*

(5.10)

$$C_{\mathsf{v}} := 4\Big[\frac{1}{\lambda_2(X_{\mathsf{v}})} + c_{\mathrm{vol},\mathsf{v}} C^{\mathrm{tr}}_{Y_{\mathsf{v}},X_{\mathsf{v}}}\big(\frac{1}{\lambda_2^{\mathrm{N}}(X_{\mathsf{v}})} + 1\big)\Big] \qquad \text{and} \qquad c_{\mathrm{vol},\mathsf{v}} := \frac{\mathrm{vol}_{m+1} X_{\mathsf{v}}}{\deg \mathsf{v}}.$$

*Moreover,* $u'$ *denotes the derivative with respect to the longitudinal variable* $s \in I_{\mathsf{e}}$ *on each component* $X_{\mathsf{e}} = I_{\mathsf{e}} \times Y_{\mathsf{e}}$ *of* $X_{\mathsf{E}_\mathsf{v}}$*.*

5.4. **Equi-ellipticity.** Let us now show that the family of sesquilinear forms $(a_\varepsilon)_\varepsilon$ is equi-elliptic. To do so, we need assumptions on $\beta_\varepsilon$ and $\gamma_\varepsilon$. We assume that

$$\gamma_\varepsilon \in \big(L^2(\Gamma_{\varepsilon,\mathsf{V}}) \otimes L^2(\Gamma_{\varepsilon,\mathsf{V}})\big) \oplus \big(L^2(\Gamma_{\varepsilon,\mathsf{E}}) \otimes L^2(\Gamma_{\varepsilon,\mathsf{E}})\big) \subset L^2(\Gamma_\varepsilon) \otimes L^2(\Gamma_\varepsilon), \tag{5.11a}$$

$$\|\beta_{\varepsilon,\mathsf{V}}\|_\infty + \|\gamma_\varepsilon\|_{\Gamma_{\varepsilon,\mathsf{V}}\times\Gamma_{\varepsilon,\mathsf{V}}} \le C_{\beta,\gamma,\mathsf{V}}, \quad \|\beta_{\varepsilon,\mathsf{E}}\|_\infty + \|\gamma_\varepsilon\|_{\Gamma_{\varepsilon,\mathsf{E}}\times\Gamma_{\varepsilon,\mathsf{E}}} \le \varepsilon C_{\beta,\gamma,\mathsf{E}} \tag{5.11b}$$

for all $\varepsilon > 0$ small enough, where $\beta_{\varepsilon,\mathsf{V}}$ is the restriction of $\beta_\varepsilon$ to $\Gamma_{\mathsf{V}}$ etc. Note that we assumed for simplicity that $\gamma_\varepsilon$ only couples edge neighbourhoods with edge neighbourhoods and vertex neighbourhoods with vertex neighbourhoods.

**Proposition 5.7.** *Assume that* (5.11a)–(5.11b) *are fulfilled. Then,* $a_\varepsilon(u,u)$ *is well-defined for* $u \in V_\varepsilon = H^1(X_\varepsilon)$*. Moreover, given* $\alpha \in (0,1)$*, there exists* $\omega \ge 0$ *and* $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ *such that*

$$\operatorname{Re} a_\varepsilon(u,u) + \omega\|u\|^2_{H_\varepsilon} \ge \alpha\|u\|^2_{V_\varepsilon}$$

*for all* $u \in V_\varepsilon$ *and all* $\varepsilon \in (0,\varepsilon_0]$*. In particular,* $(a_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ *is an* $(H_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$*-equi-sectorial family and* $H_\varepsilon = L^2(X_\varepsilon)$*.*

*Proof.* Let us show that (2.2) holds with uniform constants $\omega$ and $\alpha$. Estimate (2.1) can be seen similarly; and (2.3) is fulfilled with $c_V = 1$.

We start estimating the difference $a_\varepsilon(u,u) - \|\mathrm{d}u\|^2_{X_\varepsilon}$. We have

$$\big|a_\varepsilon(u,u) - \|\mathrm{d}u\|^2_{X_\varepsilon}\big| \le C_{\beta,\gamma,\mathsf{V}}\|u\|^2_{\Gamma_{\varepsilon,\mathsf{V}}} + \varepsilon C_{\beta,\gamma,\mathsf{E}}\|u\|^2_{\Gamma_{\varepsilon,\mathsf{E}}}$$

by Cauchy-Schwarz, Fubini, (5.11a)–(5.11b). It follows from the Sobolev trace estimate (5.6b) and Lemma 5.6 that

$$\begin{aligned}\|u\|^2_{\Gamma_{\varepsilon,\mathsf{V}}} &\le \max_{\mathsf{v}} C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}}\Big(\varepsilon\|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{V}}} + \frac{1}{\varepsilon}\|u\|^2_{X_{\varepsilon,\mathsf{V}}}\Big)\\ &\le \max_{\mathsf{v}}(C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}} + C_{\mathsf{v}})\varepsilon\|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{V}}} + 16\max_{\mathsf{v}} C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}} c_{\mathrm{vol},\mathsf{v}}\Big[b\|u'\|^2_{X_{\varepsilon,\mathsf{E}}} + \frac{2}{b}\|u\|^2_{X_{\varepsilon,\mathsf{E}}}\Big]\end{aligned}$$

for $0 < b \le \min_{\mathsf{e}} \ell_{\mathsf{e}}$. For $\|u\|^2_{\Gamma_{\varepsilon,\mathsf{E}}}$, we have the estimate

$$\|u\|^2_{\Gamma_{\varepsilon,\mathsf{E}}} \le \max_{\mathsf{e}} C^{\mathrm{tr}}_{\partial Y_{\mathsf{e}},Y_{\mathsf{e}}}\Big(\varepsilon\|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{E}}} + \frac{1}{\varepsilon}\|u\|^2_{X_{\varepsilon,\mathsf{E}}}\Big)$$

by Lemma 5.3. It follows that

$$\big|a_\varepsilon(u,u) - \|\mathrm{d}u\|^2_{X_\varepsilon}\big| \le C(\varepsilon,b)\|\mathrm{d}u\|^2_{X_\varepsilon} + \omega(a)\|u\|^2_{X_\varepsilon},$$

where

$$\begin{aligned}C(\varepsilon,b) &:= \max_{\mathsf{v},\mathsf{e}}\big\{\varepsilon C_{\beta,\gamma,\mathsf{V}}(C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}} + C_{\mathsf{v}}),\ 16bC_{\beta,\gamma,\mathsf{V}}C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}}c_{\mathrm{vol},\mathsf{v}},\ \varepsilon^2 C_{\beta,\gamma,\mathsf{E}}C^{\mathrm{tr}}_{\Gamma_{\mathsf{e}},X_{\mathsf{e}}}\big\},\\ \omega(b) &:= \max_{\mathsf{v},\mathsf{e}}\big\{16b^{-1}C_{\beta,\gamma,\mathsf{V}}C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}}c_{\mathrm{vol},\mathsf{v}},\ C_{\beta,\gamma,\mathsf{E}}C^{\mathrm{tr}}_{\Gamma_{\mathsf{e}},X_{\mathsf{e}}}\big\}.\end{aligned}$$

For $\alpha \in (0,1)$ we choose

$$\varepsilon_0 := \min_{\mathsf{v},\mathsf{e}} \Big\{ 1,\ \frac{1-\alpha}{C_{\beta,\gamma,\mathsf{v}}(C^{\mathrm{tr}}_{\Gamma_\mathsf{v},X_\mathsf{v}} + C_\mathsf{v})},\ \frac{1-\alpha}{C_{\beta,\gamma,\mathsf{e}} C^{\mathrm{tr}}_{\Gamma_\mathsf{e},X_\mathsf{e}}} \Big\}$$

and

$$b := \min_{\mathsf{v},\mathsf{e}} \Big\{ \ell_\mathsf{e},\ \frac{1-\alpha}{16 C_{\beta,\gamma,\mathsf{V}} C^{\mathrm{tr}}_{\Gamma_\mathsf{v},X_\mathsf{v}} c_{\mathrm{vol},\mathsf{v}}} \Big\}.$$

Then $C(\varepsilon,b) \le C(\varepsilon_0,b) \le 1-\alpha$ and we have

$$\begin{aligned} \operatorname{Re} a_\varepsilon(u) &\ge \|\mathrm{d}u\|^2_{X_\varepsilon} - \big| a_\varepsilon(u) - \|\mathrm{d}u\|^2_{X_\varepsilon} \big| \\ &\ge \big(1 - C(\varepsilon_0,b)\big) \|\mathrm{d}u\|^2_{X_\varepsilon} - \omega(b)\|u\|^2_{X_\varepsilon} \ge \alpha \|\mathrm{d}u\|^2_{X_\varepsilon} - \omega \|u\|^2_{X_\varepsilon} \end{aligned}$$

for all $\varepsilon \in (0,\varepsilon_0]$ with $\omega := \omega(b)$. In particular, the family $(a_\varepsilon)_\varepsilon$ is equi-sectorial. □

5.5. **The identification operators.** We now fix the identification operators $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$ similar as in [Pos06] (see also Remark 5.2). In particular, we set

$$J^{\uparrow\varepsilon} \colon L^2(X) \longrightarrow L^2(X_\varepsilon), \qquad (J^{\uparrow\varepsilon} f)_\mathsf{v} := 0, \quad (J^{\uparrow\varepsilon} f)_\mathsf{e} := f_\mathsf{e} \otimes \mathbb{1}_{\varepsilon,\mathsf{e}}, \tag{5.12}$$

where we use the decomposition of $u = J^{\uparrow\varepsilon} f$ with respect to (5.3). Here $\mathbb{1}_{\varepsilon,\mathsf{e}}(y) := \varepsilon^{-m/2}$ for all $y \in Y_\mathsf{e}$, thus

$$(J^{\uparrow\varepsilon} f)_\mathsf{e}(s,y) = \varepsilon^{-m/2} f_\mathsf{e}(s).$$

Note that $\|J^{\uparrow\varepsilon} f\|_{H_\varepsilon} \le \|f\|_{H_0}$. For $J^{\downarrow\varepsilon}$, we just choose the adjoint, i.e., $J^{\downarrow\varepsilon} := (J^{\uparrow\varepsilon})^*$. In particular, (2.7b) is fulfilled and we have

$$(J^{\downarrow\varepsilon} u)_\mathsf{e} = \varepsilon^{m/2} ⨍_\mathsf{e} u.$$

Moreover, we need the corresponding identification operators on the level of quadratic form domains. As in [Pos06], we define

$$(J_1^{\uparrow\varepsilon} f)_\mathsf{e} := (J^{\uparrow\varepsilon} f)_\mathsf{e}, \qquad (J_1^{\uparrow\varepsilon} f)_\mathsf{v} := \varepsilon^{-m/2} f(\mathsf{v})$$

(see (5.9) for the notation). Note that $f(\mathsf{v})$ is well defined for $f \in V_0$, and that $J_1^{\uparrow\varepsilon} f \in V_\varepsilon$. For the operator in the opposite direction, we choose

$$\begin{aligned} (J_1^{\downarrow\varepsilon} u)_\mathsf{e}(s) :=& (J^{\downarrow\varepsilon} u)_\mathsf{e}(s) + \varepsilon^{m/2} \sum_{\mathsf{v} \in \partial \mathsf{e}} \chi_{\mathsf{v},\mathsf{e}}(s) \big( ⨍_\mathsf{v} u - ⨍_\mathsf{e} u(\mathsf{v}) \big) \\ =& \varepsilon^{m/2} \Big( ⨍_\mathsf{e} u(s) + \sum_{\mathsf{v} \in \partial \mathsf{e}} \chi_{\mathsf{v},\mathsf{e}}(s) \big( ⨍_\mathsf{v} u - ⨍_\mathsf{e} u(\mathsf{v}) \big) \Big) \end{aligned}$$

where $\chi_{\mathsf{v},\mathsf{e}}$ is the continuous function on the metric edge $I_\mathsf{e}$ with $\chi_{\mathsf{v},\mathsf{e}}(\mathsf{v}) = 1$, $\chi_{\mathsf{v},\mathsf{e}}$ being affine linear on $I_{\mathsf{v},\mathsf{e}} := \{\, s \in I_\mathsf{e} \,:\, d(s,v) \le \ell_0 \,\}$ and $\chi_{\mathsf{v},\mathsf{e}}(s) = 0$ for $s \in I_{\mathsf{v},\mathsf{e}}$. Recall the definition of $⨍_\mathsf{v} u$ in (5.7). In particular, it is easy to see that $(J_1^{\downarrow\varepsilon} u)_\mathsf{e}(\mathsf{v}) = \varepsilon^{m/2} ⨍_\mathsf{v} u$, independently of the edge $\mathsf{e} \in \mathsf{E}_\mathsf{v}$, i.e., $J_1^{\downarrow\varepsilon} u \in \mathscr{V}$.

Let $\gamma$ be the matrix of Section 5.1. We additionally need that

$$\|\widetilde{\gamma}_\varepsilon - \gamma\|_{\mathscr{L}(\ell^2(\mathsf{V}))} \le \varepsilon^{1/2} C'_{\beta,\gamma,\mathsf{V}} \quad \text{and} \quad \|\beta_{\varepsilon,\mathsf{E}}\|_\infty + \|\gamma_\varepsilon\|_{\Gamma_{\varepsilon,\mathsf{E}} \times \Gamma_{\varepsilon,\mathsf{E}}} \le \varepsilon^{3/2} C'_{\beta,\gamma,\mathsf{E}}, \tag{5.13}$$

where $\tilde{\gamma}_\varepsilon$ is the $|\mathsf{V}| \times |\mathsf{V}|$-matrix defined by

$$\widetilde{\gamma}_{\varepsilon,\mathsf{vw}} := \frac{1}{\deg \mathsf{v}} \Big( \delta_{\mathsf{vw}} \int_{\Gamma_\mathsf{v}} \beta_\varepsilon \, \mathrm{d}\Gamma_\mathsf{v} + \varepsilon^m \int_{\Gamma_\mathsf{v} \times \Gamma_\mathsf{w}} \gamma_\varepsilon \, \mathrm{d}\Gamma_\mathsf{v} \otimes \mathrm{d}\Gamma_\mathsf{w} \Big).$$

Moreover, $(\widetilde{\gamma}_\varepsilon \varphi)(\mathsf{v}) := \sum_{\mathsf{w} \in \mathsf{V}} \widetilde{\gamma}_{\varepsilon,\mathsf{vw}} \varphi(\mathsf{w})$ denotes the corresponding operator in the Hilbert space $\ell^2(\mathsf{V})$ with weighted norm $\|\varphi\|^2_\mathsf{V} := \sum_\mathsf{v} |\varphi(\mathsf{v})|^2 \deg \mathsf{v}$. Note that the second condition in (5.13) is stronger than the second condition in (5.11b).

**Proposition 5.8.** *Assume that* (5.11a)–(5.11b) *and* (5.13) *hold, then we have*

$$\left|a_\varepsilon\big(J_1^{\uparrow\varepsilon} f, u\big) - a_0\big(f, J_1^{\downarrow\varepsilon} u\big)\right| \le \delta_\varepsilon \|f\|_{V_0} \|u\|_{V_\varepsilon}$$

*for all* $f \in V_0 = H^1(X_0)$*,* $u \in V_\varepsilon = H^1(X_\varepsilon)$ *and* $\varepsilon \in (0, \varepsilon_0]$*, where* $\delta_\varepsilon = \mathrm{O}(\varepsilon^{1/2})$ *depends only on the geometry of the unscaled manifold* $X$ *and the metric graph.*

*Proof.* In order to verify the estimate, we will split the estimate in its vertex and edge part. For the edge contribution, we have

$$\begin{aligned}
&\left|a_{\varepsilon,\mathsf{E}}\big(J_1^{\uparrow\varepsilon} f, u\big) - a_{0,\mathsf{E}}\big(f, J_1^{\downarrow\varepsilon} u\big)\right| \\
&\qquad = \varepsilon^{m/2}\Big|\varepsilon^{-1}\Big(\int_{\Gamma_\mathsf{E}} \beta_\varepsilon f \overline{u}\,\mathrm{d}\Gamma + \varepsilon^{m-1}\int_{\Gamma_\mathsf{E}\times\Gamma_\mathsf{E}} \gamma_\varepsilon(f\otimes\overline{u})\,\mathrm{d}\Gamma\otimes\mathrm{d}\Gamma\Big) \\
&\qquad\qquad + \sum_{\mathsf{e}\in\mathsf{E}}\sum_{\mathsf{v}\in\partial\mathsf{e}}\int_{I_\mathsf{e}} \chi'_{\mathsf{v},\mathsf{e}} f'_\mathsf{e}\big(⨍_\mathsf{v}\overline{u} - ⨍_\mathsf{e}\overline{u}(\mathsf{v})\big)\,\mathrm{d}s\Big|.
\end{aligned}$$

The first two integrals can be estimated by

$$\begin{aligned}
&\varepsilon C'_{\beta,\gamma,\mathsf{E}}(c_{\mathrm{vol},\mathsf{E}})^{1/2}\|f\|_{X_0}\|u\|_{\Gamma_{\varepsilon,\mathsf{E}}} \\
&\quad\le \varepsilon^{1/2} C'_{\beta,\gamma,\mathsf{E}}(c_{\mathrm{vol},\mathsf{E}})^{1/2}\|f\|_{X_0}\big(\max_\mathsf{e} C^{\mathrm{tr}}_{\partial Y_\mathsf{e},Y_\mathsf{e}}(\varepsilon^2\|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{E}}} + \|u\|^2_{X_{\varepsilon,\mathsf{E}}})\big)^{1/2} \\
&\qquad\qquad \le \varepsilon^{1/2} C'_{\beta,\gamma,\mathsf{E}}\big(c_{\mathrm{vol},\mathsf{E}}\max_\mathsf{e} C^{\mathrm{tr}}_{\partial Y_\mathsf{e},Y_\mathsf{e}}\big)^{1/2}\|f\|_{X_0}\|u\|_{H^1(X_\varepsilon)}
\end{aligned}$$

using (5.13), Cauchy-Schwarz, Lemma 5.3 and $\varepsilon \le 1$, where $c_{\mathrm{vol},\mathsf{E}} := \max_\mathsf{e}(\mathrm{vol}_{m-1}\partial Y_\mathsf{e})$. The last term of the edge contribution is small since

$$\begin{aligned}
&\varepsilon^{m/2}\Big|\sum_{\mathsf{e}\in\mathsf{E}}\sum_{\mathsf{v}\in\partial\mathsf{e}}\int_{I_\mathsf{e}} \chi'_{\mathsf{v},\mathsf{e}} f'_\mathsf{e}\big(⨍_\mathsf{v}\overline{u} - ⨍_\mathsf{e}\overline{u}(\mathsf{v})\big)\,\mathrm{d}s\Big| \\
&\qquad \le 2\varepsilon^{m/2}\ell_0^{-1/2}\|f'\|_{X_0}\Big(\sum_{\mathsf{v}\in\mathsf{V}}\sum_{\mathsf{e}\in\mathsf{E}_\mathsf{v}}\big|⨍_\mathsf{v} u - ⨍_\mathsf{e} u(\mathsf{v})\big|^2\Big)^{1/2} \\
&\qquad\qquad \le 2\varepsilon^{1/2}\max_\mathsf{v}\Big(\frac{C^{\mathrm{tr}}_{Y_\mathsf{v},X_\mathsf{v}}}{\ell_0}\Big)^{1/2}\Big(\frac{1}{\lambda_2^{\mathrm{N}}(X_\mathsf{v})} + 1\Big)^{1/2}\|f'\|_{X_0}\|\mathrm{d}u\|_{X_{\varepsilon,\mathsf{V}}}
\end{aligned}$$

using Cauchy-Schwarz again, the fact that $\|\chi'_{\mathsf{v},\mathsf{e}}\|^2_{I_\mathsf{e}} = 1/\ell_0$, where $\ell_0 = \min_\mathsf{e}\{\ell_\mathsf{e}, 1\}$, and Lemma 5.5.

For the vertex contribution, we have

$$\begin{aligned}
(5.14)\quad &\left|a_{\varepsilon,\mathsf{V}}\big(J_1^{\uparrow\varepsilon} f, u\big) - a_{0,\mathsf{V}}\big(f, J_1^{\downarrow\varepsilon} u\big)\right| = \varepsilon^{m/2}\Big|\sum_{\mathsf{v}\in\mathsf{V}} f(\mathsf{v})\int_{\Gamma_\mathsf{v}} \beta_\varepsilon\overline{u}\,\mathrm{d}\Gamma_\mathsf{v} \\
&\quad + \sum_{\mathsf{v},\mathsf{w}\in\mathsf{V}} f(\mathsf{w})\Big(\varepsilon^m\int_{\Gamma_\mathsf{v}\times\Gamma_\mathsf{w}} \gamma_\varepsilon(\mathbb{1}\otimes\overline{u})\,\mathrm{d}\Gamma_\mathsf{v}\otimes\mathrm{d}\Gamma_\mathsf{w} - \gamma_{\mathsf{vw}}(\deg\mathsf{v})⨍_\mathsf{v}\overline{u}\Big)\Big| \\
&\quad \le \varepsilon^{m/2}\Big|\sum_{\mathsf{v},\mathsf{w}\in\mathsf{V}} f(\mathsf{w})(\widetilde{\gamma}_{\varepsilon,\mathsf{vw}} - \gamma_{\mathsf{vw}})⨍_\mathsf{v}\overline{u})\deg\mathsf{v}\Big| \\
&\quad\quad + \varepsilon^{m/2}\sum_{\mathsf{v}\in\mathsf{V}}\Big(|f(\mathsf{v})|\|\beta_\varepsilon\|_{\Gamma_\mathsf{v}} + \sum_{\mathsf{w}\in\mathsf{V}}|f(\mathsf{w})|\|\gamma_\varepsilon\|_{\Gamma_{\varepsilon,\mathsf{v}}\times\Gamma_{\varepsilon,\mathsf{w}}}\|\mathbb{1}\|_{\Gamma_\mathsf{v}}\Big)\|u - ⨍_\mathsf{v} u\|_{\Gamma_\mathsf{v}}
\end{aligned}$$

since the derivative vanishes as $J_1^{\uparrow\varepsilon} f$ is constant on $X_\mathsf{v}$, and where we replaced $u$ by $⨍_\mathsf{v} u + (u - ⨍_\mathsf{v} u)$ in the first two integrals. The first sum of the last estimate can be estimated by

$$\varepsilon^{m/2}\big|\langle \underline{f}|(\widetilde{\gamma}_\varepsilon - \gamma)\underline{u}\rangle_\mathsf{V}\big| \le \|\widetilde{\gamma}_\varepsilon - \gamma\|_{\mathscr{L}(\ell^2(\mathsf{V}))}\|\underline{f}\|_\mathsf{V}(\varepsilon^{m/2}\|\underline{u}\|_\mathsf{V}),$$

where $\underline{f} = (f(\mathsf{v}))_{\mathsf{v}\in\mathsf{V}}$ and $\underline{u} = (⨍_{\mathsf{v}} u)_{\mathsf{v}\in\mathsf{V}}$. Now,

$$\begin{aligned}\varepsilon^m \|\underline{u}\|^2_{\mathsf{V}} &\le \varepsilon^m \sum_{\mathsf{v}\in\mathsf{V}} \frac{\deg \mathsf{v}}{\mathrm{vol}_{m+1} X_{\mathsf{v}}} \|u\|^2_{X_{\mathsf{v}}} \\ &\le c'_{\mathrm{vol},\mathsf{V}} \cdot \Big(\varepsilon \max_{\mathsf{v}} C_{\mathsf{v}} \|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{V}}} + \frac{16 \max_{\mathsf{v}} c_{\mathrm{vol},\mathsf{v}}}{\ell_0} \big(\|u'\|^2_{X_{\varepsilon,\mathsf{E}}} + \|u\|^2_{X_{\varepsilon,\mathsf{E}}}\big)\Big)\end{aligned}$$

by Lemma 5.6, where $c'_{\mathrm{vol},\mathsf{V}} := \max_{\mathsf{v}}(\deg \mathsf{v})/(\mathrm{vol}_{m+1} X_{\mathsf{v}})$. In particular, the first sum equals $\varepsilon^{m/2}\big|\langle \underline{f}|(\widetilde{\gamma}_\varepsilon - \gamma)\underline{u}\rangle_{\mathsf{V}}\big|$ and can be estimated from above by

$$\varepsilon^{1/2} C'_{\beta,\gamma,\mathsf{V}} \Big(\frac{8c'_{\mathrm{vol},\mathsf{V}}}{\ell_0} \max_{\mathsf{v}} \Big\{C_{\mathsf{v}},\ \frac{16 c_{\mathrm{vol},\mathsf{v}}}{\ell_0}\Big\}\Big)^{1/2} \|f\|_{H^1(X_0)} \|u\|_{H^1(X_\varepsilon)}$$

by (5.2) and since $\varepsilon \le 1$. The second summand of the right hand side of (5.14) can be estimated by

$$\begin{aligned}&\varepsilon^{m/2}\big(\|\beta_\varepsilon\|_\infty + \|\gamma_\varepsilon\|_{\Gamma_{\varepsilon,\mathsf{V}}\times\Gamma_{\varepsilon,\mathsf{V}}}\big)\|\underline{f}\|_{\mathsf{V}} \Big(\sum_{\mathsf{v}\in\mathsf{V}} \frac{\mathrm{vol}\,\Gamma_{\mathsf{v}}}{\deg \mathsf{v}} \|u - ⨍_{\mathsf{v}} u\|^2_{\Gamma_{\mathsf{v}}}\Big)^{1/2} \\ &\qquad \le \varepsilon^{1/2} C_{\beta,\gamma,\mathsf{V}} \max_{\mathsf{v},\mathsf{e}} \Big(\frac{8c''_{\mathrm{vol},\mathsf{v}} C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}}}{\ell_{\mathsf{e}}} \Big(\frac{1}{\lambda_2^{\mathrm{N}}(X_{\mathsf{v}})} + 1\Big)\Big)^{1/2} \|f\|_{H^1(X_0)} \|\mathrm{d}u\|_{X_{\varepsilon,\mathsf{V}}}\end{aligned}$$

using (5.11b), (5.2) and Lemma 5.4, where $c''_{\mathrm{vol},\mathsf{v}} := (\mathrm{vol}_m \Gamma_{\mathsf{v}})/(\deg \mathsf{v})$. □

Let us now formulate the main theorem of this section.

**Theorem 5.9.** *Assume that* (5.11a)–(5.11b) *and* (5.13) *are fulfilled. Then the sesquilinear forms* $(a_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *form an equi-sectorial family for some* $\varepsilon_0 > 0$. *Moreover, the corresponding operator* $A_\varepsilon$ *is* $\delta_\varepsilon$*-*$\kappa$*-quasi-unitarily equivalent to* $A_0$ *for* $\delta_\varepsilon = \mathrm{O}(\varepsilon^{1/2})$ *and* $\kappa = 1$.

*In particular, the convergence results of Section 2 apply, e.g., the spectrum of* $\sigma(A_\varepsilon)$ *converges to the spectrum of* $\sigma(A_0)$ *in the sense of Definition 3.10.*

*Proof.* Condition (2.7e) has been shown in Proposition 5.8. The other conditions have already been shown in [Pos06] or [EP09, Prp. 3.2]. Note that the spectrum of $A_\varepsilon$ and $A_0$ is purely discrete, since the underlying spaces are compact. □

*Remark* 5.10. If the graph $X_0$ and the corresponding manifold $X_\varepsilon$ are not compact, the corresponding forms $a_0$ and $a_\varepsilon$ are still (equi-)sectorial and fulfil Definition 2.3 provided we have a uniform control of the geometry of the graph and the manifold building blocks (see the constants in the proofs). For example, we need a positive lower bound on the edge length, i.e., $\inf_{\mathsf{e}} \ell_{\mathsf{e}} > 0$ and a uniform finite upper bound on the Sobolev trace constants like $\sup_{\mathsf{v}} C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}} < \infty$. The uniform control of the geometry is in particular fulfilled if there is a finite set of of manifolds $\mathscr{M}$ such that the building blocks $X_{\mathsf{v}}$ and $Y_{\mathsf{e}}$ of the manifold $X$, constructed according to the graph $(\mathsf{V}, \mathsf{E}, \partial)$, are isometric to a member in $\mathscr{M}$. Coverings of compact spaces provide such examples.

*Remark* 5.11. Under suitable conditions on the coefficients we can apply Theorem 3.23 in the context of the approximation results of this section. More precisely, assume e.g. that $\beta_\varepsilon \ge 0$ and $\gamma_\varepsilon = 0$. Then it can easily be verified that the forms $a_\varepsilon$ satisfy the Beurling-Deny conditions for all $\varepsilon \ge 0$. Thus the associated semigroups are positive and $L^\infty$-contractive. Thus for $\varphi(z) = \mathrm{e}^{-tz}$ the assumptions of Theorem 3.23 are satisfied with $c_\varepsilon = \varepsilon^{m/2}$. Hence

$$\|J^{\uparrow\varepsilon} \mathrm{e}^{-tA_0} J^{\downarrow\varepsilon} - \mathrm{e}^{tA_\varepsilon}\|_{\mathscr{L}(L^p(X_\varepsilon))} \to 0$$

as $\varepsilon \to 0$.

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