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FIRST ORDER APPROACH AND INDEX THEOREMS FOR DISCRETE AND METRIC GRAPHS

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Abstract. The aim of the present paper is to introduce a unified notion of Laplacians on discrete and metric graphs. In order to cover all self-adjoint vertex conditions for the associated metric graph Laplacian, we develop systematically a new type of discrete graph operators acting on a decorated graph. The decoration at each vertex of degree d is given by a subspace of $\mathbb{C}^d$, generalising the fact that a function on the standard vertex space has only a scalar value. We illustrate the abstract concept by giving classical examples throughout the article. Our approach includes infinite graphs as well.

We develop the notion of exterior derivative, differential forms, Dirac and Laplace operators in the discrete and metric case, using a supersymmetric framework. We calculate the (supersymmetric) index of the discrete Dirac operator generalising the standard index formula involving the Euler characteristic of a graph. Finally, we show that for finite graphs, the corresponding index for the metric Dirac operator agrees with the discrete one.

1. Introduction

In the last years, many attention has been payed to the analysis of metric graph Laplacians, i.e., operators acting as second order differential operators on each edge considered as one-dimensional space, with suitable vertex conditions turning the Laplacian into a self-adjoint (unbounded) operator. In most of the works, the *second order* operator is the starting object for the analysis. For more details on Laplacians on metric graphs, also labelled as "quantum graphs", we refer to the articles [KS06, Ku04, Ku05, Kuc08] and the references therein.

In this paper whereas, we want to introduce the metric graph Laplacians with general (non-negative) vertex conditions via *first order* operators, namely via an exterior derivative analogue as in differential geometry. As a by-product, we obtain a new type of discrete graph operators acting on a *decorated* graph. The decoration at each vertex v of degree $\deg v$ is given by a subspace of $\mathbb{C}^{\deg v}$, generalising the fact that a function $F \in \ell_2(V)$ on the standard vertex space on V has only a scalar value $F(v) \in \mathbb{C}$. In addition, we introduce the notion of a discrete exterior derivative, a discrete Dirac and Laplace operator and show an index theorem generalising the standard index formula involving the Euler characteristic of a graph (cf. Theorem 4.6).

In a second part, we define exterior derivatives, Dirac and Laplace operators on a (continuous) metric graph and relate their kernels with the appropriate discrete objects and show that the index agrees with the index of the discrete setting (cf. Theorem 6.1).

We introduce all Laplacians in a supersymmetric setting in the sense of [CFKS87], i.e., by appropriate "exterior derivatives" mimicking the corresponding notion for manifolds. The advantage is the simple structure of these operators; and the use of the abstract supersymmetric setting, e.g., the spectral equality of the Laplacian defined on even and odd "differential forms" (cf. Proposition 1.2). If not stated otherwise, the results hold for graphs with multiple edges and self-loops, as well as for infinite graphs.

One of our motivation for this work is to understand conceptionally whether a metric graph X_0 with Laplacian Δ_{X_0} occurs as limit of a "smooth" space, i.e., a manifold or an open neighbourhood X_ε of X_0 together with a natural Laplacian Δ_{X_ε}. If X_ε is homotopy-equivalent to X_0 then their indices agree due to the stability of the index under homotopies. In particular, the index is the Euler characteristic $\chi(X_\varepsilon) = \chi(X_0)$ in both cases; and therefore, the limit space operator must have the Euler characteristic $\chi(X_0)$ as index. We comment on this observation in Section 6.2.

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Spectral graph theory is an active area of research. We do not attempt to give a complete overview here. Results on spectral theory of discrete or combinatorial Laplacians can be found e.g. in [Dod84, vB85, MW89, CdV98, Chu97]. For continuous (quantum) graph Laplacians we mention the works [Rot84, Nic87b, KS99, Har00, KS03, Ku04, FT04a, Ku05, KS06, Pan06, P08, LP08, P09b] and the recent book [EKK+08]. In particular, a heat equation approach for the index formula for certain metric graph Laplacian (with energy-independent scattering matrix) can be found in [KPS07]. Fulling, Kuchment and Wilson recently proved in [FKW07] related results on indices of metric graph operators. Our method stresses the relation of discrete and metric graph operators using first order operators.

The paper is organised as follows: In the next subsection, we start with a motivating example of standard vertex conditions in order to illustrate the basic results and ideas. In Section 1.2, we develop the abstract setting of supersymmetry. In Section 2, we define a generalisation for the discrete vertex space $\ell_2(V)$, namely, general vertex spaces. In Section 3 we generalise the notion of the coboundary operator ("exterior derivative"), Dirac and Laplace operators in this context. In Section 4 we calculate the index of the discrete Dirac operator for general vertex spaces and generalise the below discrete Gauß-Bonnet formula (1.6). In Section 5 we develop the theory of "exterior derivatives" on a metric graph and introduce the corresponding notion of Dirac and Laplace operators. In particular, we cover all self-adjoint vertex conditions leading to a non-negative Laplacian. Finally, in Section 6 we show that the discrete and continuous Laplacians agree at the bottom of the spectrum, i.e., the index formula (1.8) for the general case. We conclude with a series of examples showing how an index formula can be used to find "smooth" approximations of metric graph Laplacians.

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1.1. **The standard case.** In order to motivate our abstract setting, we start with the standard Laplacian in the discrete and continuous setting. Details can be found in the subsequent sections. Let $X = (V, E)$ be an oriented graph with V the set of vertices and E the set of edges e, where we denote the initial vertex by $\partial_- e$ and the terminal vertex by $\partial_+ e$. Denote by $\ell_2(V)$ the standard vertex space with weight $\deg v$, the degree of the vertex v. We consider a (scalar) function in $\ell_2(V)$ as a "0-form". The *coboundary operator* or *(discrete) exterior derivative* is defined as

$$\mathsf{d}\colon \ell_2(V) \longrightarrow \ell_2(E), \qquad (\mathsf{d}F)_e = F(\partial_+ e) - F(\partial_- e)$$

mapping 0-forms into 1-forms with adjoint operator

$$\mathsf{d}^*\colon \ell_2(E) \longrightarrow \ell_2(V), \qquad (\mathsf{d}^*\eta)(v) = \frac{1}{\deg v} \sum_{e\in E_v} \widehat{\mathbb{1}}_e(v)\eta_e$$

where $\widehat{\mathbb{1}}_e(v) = \pm 1$ if $v = \partial_\pm e$ and E_v is the set of edges adjacent to v. We call the operator

$$\mathsf{D}(F \oplus \eta) = \mathsf{d}^*\eta \oplus \mathsf{d}F, \qquad \text{i.e.,} \qquad \mathsf{D} \cong \begin{pmatrix} 0 & \mathsf{d}^* \\ \mathsf{d} & 0 \end{pmatrix} \tag{1.1}$$

the associated *Dirac* operator on $\ell_2(\Lambda X) := \ell_2(V) \oplus \ell_2(E)$. The associated Laplacian is defined as $\mathsf{\Delta}_{\Lambda X} := \mathsf{D}^2$, and in particular, its component on 0-forms, i.e., on $\ell_2(V)$ is the standard Laplacian of discrete graph theory, namely

$$(\mathsf{d}^*\mathsf{d}F)(v) = (\mathsf{\Delta}^0_X F)(v) = \frac{1}{\deg v} \sum_{e\in E_v} \big(F(v) - F(v_e)\big) \tag{1.2}$$

where v_e denotes the vertex opposite to v on $e \in E_v$. For a *finite* graph X, we define the *index* of D as

$$\operatorname{ind} \mathsf{D} := \dim\ker \mathsf{d} - \dim\ker \mathsf{d}^*, \tag{1.3}$$

i.e., the 0-enlarged case (ii) with $\mathscr{G} = \mathscr{G}^{\mathrm{std}}$ and operator $L = 0$. Again, the index of the associated Dirac operator is $\chi(X)$ (cf. Theorem 6.1 (ii)). The dual operator is the decoupled Neumann operator.

Example 6.7 (The borderline case)**.** In [KuZ03] and [EP05, Sec. 7] there is a special class of approximations $X_\varepsilon \subset \mathbb{R}^2$ where the volume of a vertex neighbourhood $U_{\varepsilon,v}$ is $\operatorname{vol} U_{\varepsilon,v} = \varepsilon \operatorname{vol} U_v$ (i.e., $\alpha = 1/2$). In this case, the limit operator on 0-forms is of the form Proposition 5.19 (ii) with $\mathscr{G} = \mathscr{G}^{\mathrm{std}}$ and $L(v) = (\operatorname{vol} U_v)^{-1}$ (multiplication operator). In particular, the unusual vertex conditions in this case with the enlarged graph space are "natural" in this setting. Again, the index of the associated Dirac operator is $\chi(X)$. Note that the dual operator is a "real" quantum graph Laplacian, namely the domain consists of functions $g \in \mathsf{H}^2_{\max}(X)$ such that

$$g' \quad \text{is continuous,} \qquad (\deg v)(\operatorname{vol} U_v)\underline{g}'(v) = \sum_{e\in E_v} \widehat{\underline{g}}(v),$$

i.e., a type of δ'-condition with strength given by the local volume (and with oriented evaluation, since we are on 1-forms).

Example 6.8 (The Dirichlet decoupling case)**.** In [P05] we proved an approximation result for the Laplacian with Dirichlet boundary conditions on a certain set $X_\varepsilon \subset \mathbb{R}^2$ which is "small" around the vertex neighbourhood. The limit operator on functions in this case is the simple decoupled operator $\bigoplus_e \Delta^{\mathrm{D}}_e$, i.e., the simple case (i) with $\mathscr{G} = 0$. The index formula in this case leads to

$$\operatorname{ind} D = -|E|.$$

Note that the index of the Dirichlet Laplacian on X_ε is the *relative Euler characteristic*

$$\chi(X_\varepsilon, \partial X_\varepsilon) = \chi(X_\varepsilon) - \chi(\partial X_\varepsilon) = \chi(X_\varepsilon) = \chi(X_0) = |V| - |E|$$

in this case. But we have to consider here the Laplacian on functions and closed 1-forms only, leading to a different index. We will treat this question also in a forthcoming publication.

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For example, a dead end e with Dirichlet vertex space at $v \in \partial e$ has negative curvature. In some sense, one could say that high negative curvature forces the function to vanish: If the dead end has length $\ell_e \to 0$ with standard vertex conditions on the other vertex $w \in \partial e$ (of degree ≥ 3), then e has curvature $\kappa_e \to -\infty$ as $\ell \to 0$, and finally forces the function to vanish also on w.

On the other hand a dead end e of length $\ell_e \to 0$ with Neumann vertex space at the endpoint v has curvature tending to ∞, but the curvature at the other point w is negative and remains finite. Therefore there exists a point $x \to w$ as $\ell \to 0$ on the edge e such that $\kappa_e(x) = 0$; and the dead end just "disappears" in the limit.

Remark 6.4. Here, we only discussed the spectral relation for the discrete and metric Laplacian at the *bottom* of the spectrum. The relation for other points in the spectrum is in general complicated. If the metric graph is *equilateral*, i.e., if all length are the same, say $\ell_e = 1$, then there is a simple relation between the spectrum of the metric and discrete graph Laplacian. Such questions are discussed e.g. in [vB85, Nic87a] and in [Pan06, P08] using the notion of *boundary triples*. In [P09a] we also discussed the "enlarged" case. Finally, in [LP08], the "exceptional" Dirichlet eigenvalues arising from eigenfunctions on an equilateral metric graph vanishing at the vertices are described in a geometric way.

6.2. **Metric graphs as limits of smooth spaces.** We will give several examples of quantum graph operators on $X_0 = X$ which occur as limits of an appropriate smooth approximation X_ε. A simple example is given if X_0 is embedded in $\mathbb{R}^2$ and if we choose some homotopy-equivalent open neighbourhood X_ε of X_0. Note that

$$\chi(X_\varepsilon) = \chi(X_0),$$

since X_0 and X_ε are homotopy-equivalent.

In [RS01, KuZ01, KuZ03, EP05, P05, P06, EP07, P09c], the convergence of the 0-form operators has been established in various situations. We will show in a forthcoming article, that the result extends also to differential forms on X_ε under suitable conditions. Note that in the three first examples below, the "approximating" Laplacian $\mathrm{d}_\varepsilon^*\mathrm{d}_\varepsilon$ on X_ε (with Neumann boundary conditions on ∂X_ε) and its dual $\mathrm{d}_\varepsilon\mathrm{d}_\varepsilon^*$ on 1-forms have index equal to $\chi(X_0)$ (more precisely, the Dirac operator associated to the exterior derivative $\mathrm{d}_\varepsilon\colon \mathsf{H}^1(X_\varepsilon) \longrightarrow \mathsf{L}_2(\Lambda^1 X_\varepsilon)$ has index equal to $\chi(X_0)$).

We indicate the limit operators acting on a metric graph in several situations:

Example 6.5 (Standard vertex conditions)**.** If the vertex neighbourhoods do not shrink too slow (e.g., the ε-neighbourhood of the embedded metric graph $X_0 \subset \mathbb{R}^2$ is good enough), then the Neumann Laplacian on functions converges to the standard metric graph Laplacian (see the references above), and we will also show that the 1-form Laplacian on X_ε converges to the 1-form metric graph Laplacian. In particular, the vertex space of the limit operator is $\mathscr{G} = \mathscr{G}^{\mathrm{std}}$ and the domains are given by

$$\mathsf{H}^2_{\mathrm{std}}(X) := \big\{\, f \in \mathsf{H}^2_{\max}(X) \,\big|\, f_e(v) \text{ is independent of } e \in E_v,\ \sum_{e\in E_v} \widehat{f}'(v) = 0 \,\big\},$$

$$\mathsf{H}^2_{\widehat{\Sigma}}(X) := \big\{\, g \in \mathsf{H}^2_{\max}(X) \,\big|\, g'_e(v) \text{ is independent of } e \in E_v,\ \sum_{e\in E_v} \widehat{g}(v) = 0 \,\big\}$$

as domains for the Laplacian on 0- and 1-forms, respectively. The associated Dirac operator D has index equal to the Euler characteristic (see (1.8)). The same is true for more general Schrödinger operators on X_ε like magnetic Laplacians (for the convergence, see e.g. [KuZ01, EP07]). Magnetic Laplacians on metric graphs have been studies thoroughly in [KS03].

Example 6.6 (The decoupling case)**.** In [KuZ03] and [EP05, Sec. 6] there is a class of approximations $X_\varepsilon \subset \mathbb{R}^2$ (roughly with slowly decaying vertex neighbourhood volumes of order $\varepsilon^{2\alpha}$ with $0 < \alpha < 1/2$). In this case, the limit operator on 0-forms is

$$\bigoplus_{e\in E} \Delta_e^{\mathrm{D}} \oplus \bigoplus_{v\in V} 0,$$

i.e., the index of D is the Fredholm index of d. It is a classical result from cohomology theory, that the Fredholm-index of the coboundary operator d equals the *Euler characteristic* $\chi(X) := |V| - |E|$, namely,

$$\operatorname{ind} \mathsf{D} = \chi(X). \tag{1.4}$$

If we define the *curvature* at the vertex $v \in V$ as

$$\kappa(v) := 1 - \frac{1}{2}\deg v, \tag{1.5}$$

we can interpret the formula (1.4) as a "discrete Gauß-Bonnet" theorem, namely

$$\operatorname{ind} \mathsf{D} = \sum_{v\in V} \kappa(v) \tag{1.6}$$

using the classical formula $2|E| = \sum_{v\in V} \deg v$. Note that $\kappa(v) < 0$ iff $\deg v \geq 3$.

Considering X as a metric graph, our basic Hilbert space is $\mathsf{L}_2(X)$ (cf. (5.1)). On the metric graph, we consider the "exterior" derivative

$$\mathrm{d}\colon \operatorname{dom}\mathrm{d} \longrightarrow \mathsf{L}_2(X), \qquad \mathrm{d}f = f' = \{f'_e\}_e$$

where $\operatorname{dom}\mathrm{d} = \mathsf{H}^1_{\max}(X) \cap \mathsf{C}(X)$ is the Sobolev space of functions *continuous* at each vertex. Its L_2-adjoint is

$$\mathrm{d}^*\colon \operatorname{dom}\mathrm{d}^* \longrightarrow \mathsf{L}_2(X), \qquad \mathrm{d}^* g = -g' = \{-g'_e\}_e$$

with $g \in \operatorname{dom}\mathrm{d}^*$ iff

$$\sum_{e\in E_v} \widehat{g}_e(v) = 0, \tag{1.7}$$

where $\widehat{g}_e(v)$ is the *oriented* evaluation at v (see Eq. (5.2)). As before, we can define a Dirac operator D on $\mathsf{L}_2(X) \oplus \mathsf{L}_2(X)$ and the associated Laplacian $\Delta_{\Lambda X}$ such that its 0-form component is

$$\Delta^0_X f := \mathrm{d}^*\mathrm{d}f = -f'' = \{-f''_e\}_e$$

with domain

$$\operatorname{dom}\Delta^0_X = \big\{\, f \in \operatorname{dom}\mathrm{d} \,\big|\, f' \in \operatorname{dom}\mathrm{d}^* \,\big\},$$

i.e., the *standard* Laplacian on a metric graph with functions continuous at each vertex and the Kirchhoff sum condition for the derivative at each vertex. Although the 0- and 1-forms are formally the same, they differ in their interpretation: We consider 0-forms as *scalar* functions, whereas a 1-form is a *vector-field* with orientation. Then the Kirchhoff sum condition Eq. (1.7) is just a "flux" conservation for the flux generated by the "vector field" f'.

Again, we define the index $\operatorname{ind} D$ of the metric graph Dirac operator D as the Fredholm-index of d, i.e. in the same way as in Eq. (1.3) and one of our main results in this setting (cf. Theorem 6.1) is

$$\ker D \cong \ker \mathsf{D} \quad \text{and} \quad \operatorname{ind} D = \operatorname{ind} \mathsf{D}\ (= |V| - |E|), \tag{1.8}$$

i.e., an isomorphism between the kernels of the discrete and continuous case.

We want to generalise the above setting to quantum graph Laplacians with *general* self-adjoint operators Δ_X (such that $\Delta_X \geq 0$) and derive a similar index formula.

1.2. **Supersymmetry.** Let us first collect some structural facts shared by our graph operators defined below. Since in our cases we only have p-forms for $p \in \{0, 1\}$, we can identify forms of even and odd degree with the cases $p = 0$ and $p = 1$, respectively. Following [CFKS87], we define *supersymmetry* as follows:

Definition 1.1. Let $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$ be a Hilbert space and $\mathrm{d}\colon \operatorname{dom}\mathrm{d} \longrightarrow \mathscr{H}_1$ a closed operator with $\operatorname{dom}\mathrm{d} \subset \mathscr{H}_0$ (d may be bounded, in this case we have $\operatorname{dom}\mathrm{d} = \mathscr{H}_0$). Then we say that d has *supersymmetry* or that d is an *exterior derivative*. A *p-form* is an element in $\mathscr{H}_p$. Furthermore, we define the associated *Dirac* operator as

$$D(f_0 \oplus f_1) = \mathrm{d}^* f_1 \oplus \mathrm{d} f_0, \quad \text{i.e.,} \quad D \cong \begin{pmatrix} 0 & \mathrm{d}^* \\ \mathrm{d} & 0 \end{pmatrix}$$

with respect to the decomposition $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$. The associated *Laplacian* is given by $\Delta := D^2$. In particular,

$$\Delta \cong \begin{pmatrix} \Delta_0 & 0 \\ 0 & \Delta_1 \end{pmatrix},$$

where $\Delta_0 = \mathrm{d}^*\mathrm{d}$ and $\Delta_1 = \mathrm{d}\mathrm{d}^*$ on their natural domains.

Clearly, Δ and Δ^p are closed, non-negative operators. Note that $\ker \mathrm{d} = \ker \Delta_0$ and $\ker \mathrm{d}^* = \ker \Delta_1$.

We denote the spectral projection of Δ_p by $\mathbb{1}_B(\Delta_p)$. We have the following results on the spectrum away from 0:

Proposition 1.2. *Assume that* d *has supersymmetry and that $B \subset [0,\infty)$ is a bounded Borel set. Then*

$$\mathrm{d}\mathbb{1}_B(\Delta_0) = \mathbb{1}_B(\Delta_1)\mathrm{d} \quad \textit{and} \quad \mathrm{d}^*\mathbb{1}_B(\Delta_1) = \mathbb{1}_B(\Delta_0)\mathrm{d}^*.$$

Furthermore, if 0 *is* not *contained in* B*, then*

$$\mathrm{d}\colon \mathbb{1}_B(\Delta_0)(\mathscr{H}_0) \longrightarrow \mathbb{1}_B(\Delta_1)(\mathscr{H}_1) \quad \textit{and} \quad \mathrm{d}^*\colon \mathbb{1}_B(\Delta_1)(\mathscr{H}_1) \longrightarrow \mathbb{1}_B(\Delta_0)(\mathscr{H}_0)$$

are isomorphisms. In particular,

$$\dim \mathbb{1}_B(\Delta_0) = \dim \mathbb{1}_B(\Delta_1) \quad \textit{and} \quad \sigma(\Delta_0) \setminus \{0\} = \sigma(\Delta_1) \setminus \{0\},$$

i.e., the spectra of Δ_0 *and* Δ_1 *away from* 0 *agree including multiplicity.*

Proof. The first assertion follows from $\mathrm{d}\varphi(\mathrm{d}^*\mathrm{d}) = \varphi(\mathrm{d}\mathrm{d}^*)\mathrm{d}$, first for polynomials φ, then for functions $\varphi(\lambda) = (\lambda+1)^{-k}$, $k \geq 1$, and finally by the spectral calculus also for (fast enough decaying) continuous and measurable functions. The second assertion follows since $\ker \mathrm{d} = \ker \Delta_0 = 0$ and $\ker \mathrm{d}^* = \ker \Delta_1 = 0$. The last statement is a simple consequence of the isomorphisms. □

We have the following result, an abstract version of the Hodge decomposition:

Proposition 1.3. *Assume that* d *has supersymmetry and that the associated Dirac operator D has a spectral gap at* 0*, i.e.,* $\operatorname{dist}(0, \sigma(D) \setminus \{0\}) > 0$*. Then*[1]

$$\mathscr{H} = \ker D \oplus \operatorname{ran} \mathrm{d}^* \oplus \operatorname{ran} \mathrm{d},$$
$$\mathscr{H}_0 = \ker \mathrm{d} \oplus \operatorname{ran} \mathrm{d}^* \quad \textit{and} \quad \mathscr{H}_1 = \ker \mathrm{d}^* \oplus \operatorname{ran} \mathrm{d}.$$

Proof. It is a general fact that $\mathscr{H}_0 = \ker \mathrm{d} \oplus \overline{\operatorname{ran} \mathrm{d}^*}$ and similarly for $\mathscr{H}_1$. It remains to show that $\operatorname{ran} \mathrm{d}$ and $\operatorname{ran} \mathrm{d}^*$ are closed. Let $\widetilde{D}$ be the restriction of D onto $(\ker D)^\perp$. By our assumption, $\widetilde{D}$ has a bounded inverse, namely

$$\widetilde{D}^{-1} \cong \begin{pmatrix} 0 & \widetilde{\mathrm{d}}^{-1} \\ (\widetilde{\mathrm{d}}^*)^{-1} & 0 \end{pmatrix},$$

where $\widetilde{\mathrm{d}}$ and $\widetilde{\mathrm{d}}^*$ are the restrictions of d and d^* to $(\ker \mathrm{d})^\perp$ and $(\ker \mathrm{d}^*)^\perp$, respectively. In particular, $\widetilde{\mathrm{d}}^{-1}$ and $(\widetilde{\mathrm{d}}^*)^{-1}$ are bounded.

Let $g \in \overline{\operatorname{ran} \mathrm{d}}$, then there exists a sequence $\{f_n\}_n \subset \mathscr{H}_0$ such that $\mathrm{d}f_n \to g$ in $\mathscr{H}_1$. Without loss of generality, we may assume that $f_n \in (\ker \mathrm{d})^\perp$. Therefore, $\widetilde{\mathrm{d}}^{-1}\mathrm{d}f_n = f_n \to \widetilde{\mathrm{d}}^{-1}g =: f$. Now, $f_n \to f$, $\mathrm{d}f_n \to g$ and d is closed, so $f \in \operatorname{dom} \mathrm{d}$ and in particular, $\mathrm{d}f = g \in \operatorname{ran} \mathrm{d}$. □

Definition 1.4. If $\ker \mathrm{d}$ and $\ker \mathrm{d}^*$ are both finite dimensional (i.e., $0 \notin \sigma_{\mathrm{ess}}(D)$), we define the *index* of D as

$$\operatorname{ind} D := \dim \ker \mathrm{d} - \dim \ker \mathrm{d}^*.$$

Note that $\operatorname{ind} D$ is the usual Fredholm index of the operator d.

We need the following fact in order to calculate the index in concrete examples:

[1] The spectral gap condition is only need in order to assure that the ranges are closed. If we replace $\operatorname{ran} \mathrm{d}$ by $\overline{\operatorname{ran} \mathrm{d}}$ and $\operatorname{ran} \mathrm{d}^*$ by $\overline{\operatorname{ran} \mathrm{d}^*}$, then we can drop this condition. Here, ran d denotes the *range* of the operator d, i.e., $\operatorname{ran} \mathrm{d} = \{\, \mathrm{d}f \mid f \in \operatorname{dom} \mathrm{d} \,\}$.

where the rows are obviously chain complexes (with bounded maps) and the diagram is commutative. Note that indeed, $\Phi_1\colon \mathsf{L}_2(\Lambda^1 X) \longrightarrow \ell_2(E)$, $g \mapsto \int g$, is a bounded map also on the L_2-space. The commutativity of the diagram follows from the fact that

$$\begin{aligned}\big((\mathsf{d}\Phi_0 - \Phi_1\mathrm{d})(f,F)\big)_e &= (P_1\underline{f})_e(\partial_+ e) - (P_1\underline{f})_e(\partial_- e) - \int_e f_e' \,\mathrm{d}x \\ &= -(P_0\underline{f})_e(\partial_+ e) + (P_0\underline{f})_e(\partial_- e)\end{aligned}$$

for $f \in \mathsf{H}^1(\Lambda^0 X)$. But note that $\underline{f} = L^{1/2}F$ implies $\underline{f} \in (\ker L)^\perp = \mathscr{G}_1$, so that $\mathsf{d}\Phi_0 = \Phi_1\mathrm{d}$, i.e., Φ is a chain morphism. The corresponding homology induces the above isomorphism of the Dirac operator kernels. The other cases can be treated similarly. If one wishes to stress this abstract point of view one can interpret the "enlarged" spaces as twisted chain complexes.

Proof. The boundedness of Φ in the particular cases follows immediately from Proposition 5.2 and Cauchy-Schwarz. We only prove the second case, since the other ones are similar. First, we note that in Case (ii), we have $(f,F,g) \in \ker D$ iff f_e, g_e are constant, $\underline{f} \in \mathscr{G}_1$, $\widehat{\underline{g}} \in \mathscr{G}_1^\perp$ and $\underline{f} = L^{1/2}F$. In particular, $\mathsf{d}\underline{f} = 0$, $\mathsf{d}^* \int g = P_1\widehat{\underline{g}} = 0$ and therefore $\Phi(\ker D) \subset \ker \mathsf{D}$.

In order to show that Φ is injective on $\ker D$, we note that $P_1\underline{f} = 0$ implies $\underline{f} = 0$ (since already $\underline{f} \in \mathscr{G}_1$). Furthermore, $P_0F = 0$ and from $0 = \underline{f} = L^{1/2}F$ we conclude $P_1F = 0$, i.e., $F = 0$. Finally, $\int g = 0$ and $g_e = \text{const}$ implies that $g = 0$.

It remains to show that $\ker \mathsf{D} \subset \Phi(\ker D)$, let $(F_1, F_0, \eta) \in \ker \mathsf{D}$. Set $f_e(x) := F_{1,e}(\partial_\pm e)$ (both values are the same since $\mathrm{d}F_1 = 0$), $F := L^{-1/2}F_1 + F_0$ and $g_e(x) := \eta_e/\ell_e$. Then $(f,F,g) \in \mathsf{H}^1(\Lambda X)$ since $\underline{f} \in \mathscr{G}_1 \leq \mathscr{G}$ and $L^{1/2}F = F_1 = \underline{f}$. Next, $D(f,F,g) = (-g', L^{1/2}P\widehat{g}, f') = 0$ since $L^{1/2}P_0\widehat{g} = 0$ and $L^{1/2}P_1\widehat{g} = L^{1/2}\mathrm{d}^*\eta = 0$, and f_e, g_e are constant. Finally, $\Phi(f,F,g) = (P_e\underline{f}, P_0F, \int g) = (F_1, F_0, \eta)$ and the assertion is proven.

The index formulas follow from Theorem 4.6 and Eqs. (1.9) and (2.2). □

Remark 6.3. We would like to interpret the above index formula together with the discrete index formula of Theorem 4.6 as a "Gauß-Bonnet theorem" on quantum graphs. To do so, we define the *curvature* $\kappa := \kappa_{(X,\mathscr{G},L)}$ of the quantum graph $(X,\mathscr{G},L)$ as

$$\kappa_e(x) = 2\Big(\frac{\kappa_{\widetilde{\mathscr{G}}}(\partial_- e)}{\sum_{e' \in E_{\partial_- e}} \ell_{e'}}(\ell_e - x) + \frac{\kappa_{\widetilde{\mathscr{G}}}(\partial_+ e)}{\sum_{e' \in E_{\partial_+ e}} \ell_{e'}} x\Big) \tag{6.1}$$

where $\kappa_{\widetilde{\mathscr{G}}}(v)$ is the *discrete* curvature defined in Definition 4.2, but for the vertex space $\widetilde{\mathscr{G}}$ given by $\widetilde{\mathscr{G}}_v = \mathscr{G}_v$ in the cases (i)–(ii), by $\widetilde{\mathscr{G}}_v = \mathscr{G}_v^{\max}$ in the case (iii), by $\widetilde{\mathscr{G}}_v = \mathscr{G}_v^\perp$ in the case (iv), and by $\widetilde{\mathscr{G}}_v = 0$ in the last case (v) (cf. the index formulas in Theorem 6.1). In particular, we can interpret the index formula $\operatorname{ind} D = \operatorname{ind} \mathsf{D}$ as

$$\operatorname{ind} D = \int_X \kappa \,\mathrm{d}x \tag{6.2}$$

since

$$\int_X \kappa \,\mathrm{d}x = \sum_{v \in V} \kappa_{\widetilde{\mathscr{G}}}(v) = \operatorname{ind} \mathsf{D} \tag{6.3}$$

by an obvious calculation and Theorem 4.6. Note that the choice of κ_e is somehow arbitrary, but it is the unique way to define it if we require that (6.3) holds, that $\kappa_e(v) = c(v)\kappa_{\widetilde{G}}(v)$ for some sequence $c(v) > 0$ and that $\kappa_e'' = 0$.

In particular, we have

$$\kappa(v) = \kappa_{(X,\mathscr{G},L)}(v) = \frac{2\kappa_{\widetilde{\mathscr{G}}}(v)}{\sum_{e' \in E_v} \ell_{e'}}.$$

If we have a continuous vertex space, i.e., $\dim \mathscr{G}_v = 1$ like the standard vertex space, then $\kappa(v) = 0$ iff $\deg v = 2$. This reflects the fact that a vertex of degree 2 is invisible. Furthermore if $\deg v = 1$ (i.e., a "dead end" with Neumann vertex space), then $\kappa(v) > 0$. Furthermore, if $\deg v \geq 3$, then $\kappa(v) < 0$. Moreover, shorter lengths ℓ_e at a vertex v mean a higher absolute value of the curvature.

with norm bounded by $2/\sqrt{\ell_0}$ *such that* $\Phi(\ker D) = \ker \mathsf{D}$ *is an isomorphism respecting the supersymmetry (i.e.,* $\Phi = \Phi_0 \oplus \Phi_1$*, cf. Definition 1.6). In particular,* $\Phi_0(\ker \mathrm{d}) = \ker \mathsf{d}$ *and* $\Phi_1(\ker \mathrm{d}^*) = \ker \mathsf{d}^*$ *are isomorphisms. If in addition,* X *is a finite graph, then*

$$\operatorname{ind} D = \operatorname{ind} \mathsf{D},$$

The operators d *and* Φ *are given in the different cases as follows:*

(i) ***The simple case:*** *Here,* $\mathsf{d}\colon \mathscr{G} \longrightarrow \ell_2(E)$,

$$\Phi\colon \mathsf{H}^1_{\mathscr{G}}(X) \oplus \mathsf{H}^1_{\mathscr{G}}(X) \longrightarrow \mathscr{G} \oplus \ell_2(E), \qquad (f,g) \mapsto (\underline{f}, \textstyle\int g),$$
$$\operatorname{ind} D = \operatorname{ind} \mathsf{D} = \dim \mathscr{G} - |E|.$$

(ii) ***The*** 0***-enlarged space:*** *Here,*

$$\mathsf{d}\colon \mathscr{G}_1 \oplus \mathscr{G}_0 \longrightarrow \ell_2(E), \qquad F_1 \oplus F_0 \mapsto \mathsf{d}_1 F_1,$$
$$\Phi\colon \mathsf{H}^1(\Lambda X) \longrightarrow (\mathscr{G}_1 \oplus \mathscr{G}_0) \oplus \ell_2(E), \qquad (f,F,g) \mapsto (P_1\underline{f}, P_0 F, \textstyle\int g),$$
$$\operatorname{ind} D = \operatorname{ind} \mathsf{D} = \dim \mathscr{G} - |E|.$$

(iii) ***The*** 0***-enlarged space with projection:*** *We have*

$$\mathsf{d}\colon \mathscr{G}_0^\perp \oplus \mathscr{G}_0 \longrightarrow \ell_2(E), \qquad F_0^\perp \oplus F_0 \mapsto \mathsf{d}_0^\perp F_0^\perp,$$
$$\Phi\colon \mathsf{H}^1(\Lambda X) \longrightarrow (\mathscr{G}_0^\perp \oplus \mathscr{G}_0) \oplus \ell_2(E), \qquad (f,F,g) \mapsto (P_0^\perp \underline{f}, P_0 F, \textstyle\int g),$$
$$\operatorname{ind} D = \operatorname{ind} \mathsf{D} = |E|.$$

(iv) ***The*** 1***-enlarged space:*** *We have*

$$\mathsf{d}\colon \mathscr{G}_1^\perp \longrightarrow \mathscr{G}_0 \oplus \ell_2(E), \qquad F_1^\perp \mapsto 0 \oplus \mathsf{d}_1^\perp F_1^\perp,$$
$$\Phi\colon \mathsf{H}^1(\Lambda X) \longrightarrow \mathscr{G}_1^\perp \oplus (\mathscr{G}_0 \oplus \ell_2(E)), \qquad (f,g,G) \mapsto (P_1^\perp \underline{f}, P_0 G, \textstyle\int g),$$
$$\operatorname{ind} D = \operatorname{ind} \mathsf{D} = |E| - \dim \mathscr{G}.$$

(v) ***The*** 1***-enlarged space with projection:*** *We have*

$$\mathsf{d}\colon \mathscr{G}_0 \longrightarrow \mathscr{G}_0 \oplus \ell_2(E), \qquad F_0 \mapsto 0 \oplus \mathsf{d}_0 F_0,$$
$$\Phi\colon \mathsf{H}^1(\Lambda X) \longrightarrow \mathscr{G}_0 \oplus (\mathscr{G}_0 \oplus \ell_2(E)), \qquad (f,g,G) \mapsto (P_0\underline{f}, P_0 G, \textstyle\int g),$$
$$\operatorname{ind} D = \operatorname{ind} \mathsf{D} = -|E|.$$

Remark 6.2.

(i) Note that in all cases, the index is independent of L, i.e., of the decomposition of $\mathscr{G}$ into $\mathscr{G}_0 = \ker L$ and $\mathscr{G}_1 = \mathscr{G} \ominus \ker L$ as one expects since the index should be constant passing to the limit $L \to 0$.

(ii) In the first two cases, we obtain the Euler characteristic as index (if $\mathscr{G} = \mathscr{G}^{\mathrm{std}}$). These two cases are the ones we obtain by a limit argument where the metric graph is approached by a manifold (cf. [EP05, P09c] and a forthcoming paper) provided the transversal manifold F is simply connected (see also Examples 6.5–6.7).

(iii) If we assume that L is invertible, then the index in each case remains the same when passing to the limit $L \to \infty$ (cf. Remark 5.17).

(iv) We can interpret Φ in the above theorem as a sort of Hilbert chain morphism (cf. [Lüc02, Ch. 1]). For example, in the 0-enlarged case (ii), we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{H}^1(\Lambda^0 X) & \xrightarrow{\mathrm{d}} & \mathsf{L}_2(\Lambda^1 X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi_0} & & \downarrow{\scriptstyle \Phi_1} & & \\
0 & \longrightarrow & \mathscr{G}_1 \oplus \mathscr{G}_0 & \xrightarrow{\mathsf{d}_1 \overset{\curlywedge}{\oplus} 0} & \ell_2(E) & \longrightarrow & 0
\end{array}$$

Proposition 1.5. *Assume that* $\{D_t\}_{t\in\mathbb{R}}$ *is a family of bounded Dirac operators such that* $t \mapsto D_t$ *is norm-continuous. Then* $\operatorname{ind} D_t$ *is constant.*

Proof. This follows from the fact that the Fredholm index depends continuously on the operator and that a continuous function into $\mathbb{Z}$ is locally constant (see e.g. [Gil95, Lem. 1.4.3]). □

We need the notion of a morphism of this structure.

Definition 1.6. Suppose that $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$ with operator d and $\widetilde{\mathscr{H}} = \widetilde{\mathscr{H}_0} \oplus \widetilde{\mathscr{H}_1}$ with operator $\widetilde{\mathrm{d}}$ and associated Dirac operators D and $\widetilde{D}$, respectively, have supersymmetry. We say that a linear map $\Phi\colon \operatorname{dom} D \longrightarrow \operatorname{dom} \widetilde{D}$ *respects supersymmetry* iff Φ decomposes into $\Phi = \Phi_0 \oplus \Phi_1$ where Φ_p maps p-forms onto p-forms.

In some cases we need to enlarge the Hilbert space $\mathscr{H}$ by a space $\mathscr{N}$ on which the exterior derivative acts trivially:

Definition 1.7. Let $\mathscr{N}$ be a Hilbert space. We set $\mathscr{H}_{\mathscr{N}} := \mathscr{H}_0 \oplus \mathscr{N} \oplus \mathscr{H}_1$. Assume that d is an exterior derivative on $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$. Then we call

$$\mathrm{d}_{\mathscr{N}^0} = \mathrm{d} \overset{\curlywedge}{\oplus} 0\colon \operatorname{dom} \mathrm{d} \oplus \mathscr{N} \longrightarrow \mathscr{H}_1, \qquad f \oplus h \mapsto \mathrm{d}f$$

the exterior derivative *trivially* 0*-enlarged by* $\mathscr{N}$. The associated Dirac operator will be denoted by $D_{\mathscr{N}^0}$.

Similarly, we call

$$\mathrm{d}_{\mathscr{N}^1} = \mathrm{d} \overset{\curlyvee}{\oplus} 0\colon \operatorname{dom} \mathrm{d} \longrightarrow \mathscr{H}_1 \oplus \mathscr{N}, \qquad f \mapsto \mathrm{d}f \oplus 0$$

the exterior derivative *trivially* 1*-enlarged by* $\mathscr{N}$. The associated Dirac operator will be denoted by $D_{\mathscr{N}^1}$.

Note that $\mathrm{d}^*_{\mathscr{N}^0} = (\mathrm{d} \overset{\curlywedge}{\oplus} 0)^* = \mathrm{d}^* \overset{\curlyvee}{\oplus} 0$ and $\mathrm{d}^*_{\mathscr{N}^1} = (\mathrm{d} \overset{\curlyvee}{\oplus} 0)^* = \mathrm{d}^* \overset{\curlywedge}{\oplus} 0$. Furthermore, $\ker \mathrm{d}_{\mathscr{N}^0} = \ker \mathrm{d} \oplus \mathscr{N}$, $\ker \mathrm{d}^*_{\mathscr{N}^0} = \ker \mathrm{d}^*$ and $\ker \mathrm{d}_{\mathscr{N}^1} = \ker \mathrm{d}$, $\ker \mathrm{d}^*_{\mathscr{N}^1} = \ker \mathrm{d}^* \oplus \mathscr{N}$. In particular, we have

$$\operatorname{ind} D_{\mathscr{N}^0} = \operatorname{ind} D + \dim \mathscr{N} \quad \text{and} \quad \operatorname{ind} D_{\mathscr{N}^1} = \operatorname{ind} D - \dim \mathscr{N}. \tag{1.9}$$

2. Vertex spaces on discrete graphs

2.1. **Discrete graphs.** Suppose X is a discrete weighted graph given by (V, E, ∂, ℓ) where (V, E, ∂) is a usual graph, i.e., V denotes the set of vertices, E denotes the set of edges, $\partial\colon E \longrightarrow V \times V$ associates to each edge e the pair $(\partial_- e, \partial_+ e)$ of its *initial* and *terminal* vertex (and therefore an orientation). That X is an *(edge-)weighted* graph means that there is a *length* or *(inverse) edge weight function* $\ell\colon E \longrightarrow (0,\infty)$ associating to each edge e a length ℓ_e. For simplicity, we consider *internal* edges only, i.e., edges of *finite* length $\ell_e < \infty$.

For each vertex $v \in V$ we set

$$E_v^\pm := \{\, e \in E \,|\, \partial_\pm e = v \,\} \qquad \text{and} \qquad E_v := E_v^+ \cup E_v^-,$$

i.e., $E_v^\pm$ consists of all edges starting $(-)$ resp. ending $(+)$ at v and E_v their *disjoint* union. Note that the *disjoint* union is necessary in order to allow self-loops, i.e., edges having the same initial and terminal point. The *(in/out-)*degree of $v \in V$ is defined as

$$\deg^+ v := |E_v^+|, \qquad \deg^- v := |E_v^-|, \qquad \deg v := |E_v| = \deg^+ v + \deg^- v,$$

respectively. In order to avoid trivial cases, we assume that $\deg v \geq 1$, i.e., no vertex is isolated. On the vertices, we usually consider the canonical (vertex-)weight $\deg v$ (see e.g. the norm definition of $\ell_2(V)$ in (2.5)).

We say that the graph X is *d-regular*, iff $\deg v = d$ for all $v \in V$. Furthermore, X is *bipartite*, if there is a decomposition $V = V_- \cup V_+$ such that no vertex in V_- is joined with a vertex in V_- by an edge and similar for V_+.

We have the following equalities

$$\dot{\bigcup_{v\in V}} E_v^+ = \dot{\bigcup_{v\in V}} E_v^- = E \quad \text{and} \quad \dot{\bigcup_{v\in V}} E_v = E \dot\cup E, \tag{2.1}$$

since each (internal) edge has exactly one terminal vertex and one initial vertex. In addition, a self-loop edge e is counted twice in E_v. In particular,

$$\sum_{v\in V} \deg v = 2|E|. \tag{2.2}$$

2.2. **General vertex spaces.** We want to introduce a vertex space allowing us to define Laplace-like operators coming from general vertex conditions for quantum graphs. The standard discrete Laplacian is defined on 0-forms and 1-forms, namely, on sections in the trivial bundles

$$\Lambda^0 X = V \times \mathbb{C} \quad \text{and} \quad \Lambda^1 X = E \times \mathbb{C}.$$

In order to allow more general vertex conditions in the quantum graph case later on, we need to enlarge the space at each vertex v. We denote $\mathscr{G}_v^{\max} := \mathbb{C}^{E_v}$ the *maximal vertex space* at the vertex $v \in V$, i.e., a value $\underline{F}(v) \in \mathscr{G}_v^{\max}$ has $\deg v$ components, one for each adjacent edge. A (general) *vertex space* is a family $\{\mathscr{G}_v\}_v$ of subspaces $\mathscr{G}_v$ of $\mathscr{G}_v^{\max}$ for each vertex v. We can consider a vertex space as a vector bundle

$$\Lambda^0 X := \dot{\bigcup_{v\in V}} \mathscr{G}_v$$

over the discrete base space V with fibres $\mathscr{G}_v$ of mixed rank generalising the standard setting above, where $\mathscr{G}_v \cong \mathbb{C}$ at each vertex. An element of $\mathscr{G}_v^{\max}$ will generally be denoted by $\underline{F}(v) = \{F_e(v)\}_{e\in E_v}$. Note that

$$\mathscr{G}^{\max} := \bigoplus_v \mathscr{G}_v^{\max} \cong \bigoplus_{e\in E} \mathbb{C}^2 \tag{2.3}$$

since each edge occurs twice in the E_v, $v \in V$ (cf. Eq. (2.1)).

We denote by

$$\ell_2(\Lambda^0 X) = \mathscr{G} := \bigoplus_{v\in V} \mathscr{G}_v \quad \text{and} \quad \ell_2(\Lambda^1 X) = \ell_2(E) \tag{2.4}$$

the associated Hilbert spaces of 0- and 1-forms with norms defined by

$$\|F\|_{\mathscr{G}}^2 := \sum_{v\in V} |\underline{F}(v)|^2 = \sum_{v\in V}\sum_{e\in E_v} |F_e(v)|^2 \quad \text{and} \quad \|\eta\|_{\ell_2(E)}^2 := \sum_{e\in E} |\eta_e|^2 \frac{1}{\ell_e}.$$

Abusing the notation, we also call the section space $\mathscr{G}$ a *vertex space*.

Definition 2.1. We say that an operator A on $\mathscr{G}$ is *local* iff A decomposes with respect to $\mathscr{G} = \bigoplus_{v\in V} \mathscr{G}_v$, i.e., $A = \bigoplus_{v\in V} A_v$ where A_v is an operator on $\mathscr{G}_v$.

Associated to a vertex space is an orthogonal projection $P = \bigoplus_{v\in V} P_v$ in $\mathscr{G}^{\max}$, where P_v is the orthogonal projection in $\mathscr{G}_v^{\max}$ onto $\mathscr{G}_v$. Alternatively, a vertex space is characterised by fixing an orthogonal projection P in $\mathscr{G}$ which is local.

Remark 2.2. If X is finite, we can assume without loss of generality that P is local. If this is not the case, we can pass to a new graph $\widetilde{X}$ by identifying vertices $v \in V$ for which P does not decompose with respect to $\mathscr{G}_v^{\max} \oplus \bigoplus_{w\neq v} \mathscr{G}_w^{\max}$. In the worst case, the new graph $\widetilde{X}$ is a rose, i.e., $\widetilde{X}$ consists of only one vertex with $|E|$ self-loops attached.

The following notation will be useful:

Definition 2.3. The linear operator $\tau = \widehat{(\cdot)} \colon \mathscr{G}^{\max} \longrightarrow \mathscr{G}^{\max}$, $F \mapsto \widehat{F}$, defined by $\tau := \bigoplus_{v\in V} \tau_v$ and

$$\tau_v(\underline{F}(v)) := \widehat{\underline{F}}(v) = \{\widehat{F}_e(v)\}_{e\in E_v}, \qquad \widehat{F}_e(v) := \pm F_e(v), \quad \text{if } v = \partial_\pm e,$$

is called *orientation map*. We say that τ *switches from an unoriented evaluation to an oriented evaluation* and vice-versa.

(ii) ***The* 0*-enlarged space:*** *Here we have*

$$\mathsf{H}^2(\Lambda^0 X) = \big\{ (f,F) \in \mathsf{H}^2_{\max}(X) \oplus \mathscr{G} \,\big|\, \underline{f} \in \mathscr{G}, \underline{f} = L^{1/2} F \big\},$$
$$\mathsf{H}^2(\Lambda^1 X) = \big\{ g \in \mathsf{H}^2_{\max}(X) \,\big|\, \underline{g}' \in \mathscr{G}, \quad \underline{g}' + LP\widehat{\underline{g}} = 0 \big\},$$
$$\Delta_{\Lambda^0 X}(f,F) = (-f'', L^{1/2} P \widehat{\underline{f}}'), \qquad \Delta_{\Lambda^1 X} g = -g''.$$

The 1*-form space* $\mathsf{H}^2(\Lambda^1 X)$ *equals* $\mathsf{H}^2(X, \widehat{\mathscr{G}}^{\max}, \widetilde{L})$ *where* $\widetilde{L}(F, F^\perp) := (LF, 0)$ *with respect to the decomposition* $\mathscr{G}^{\max} = \mathscr{G} \oplus \mathscr{G}^\perp$. *In particular, the* 1*-form space represents the quantum graph* $(X, \widehat{\mathscr{G}}^{\max}, \widetilde{L})$.

(iii) ***The* 0*-enlarged space with projection:*** *We have*

$$\mathsf{H}^2(\Lambda^0 X) = \big\{ (f,F) \in \mathsf{H}^2_{\max}(X) \oplus \mathscr{G} \,\big|\, P\underline{f} = L^{1/2} F, \quad \widehat{\underline{f}}' \in \mathscr{G} \big\}$$
$$\mathsf{H}^2(\Lambda^1 X) = \big\{ g \in \mathsf{H}^2_{\max}(X) \,\big|\, \widehat{\underline{g}} \in \mathscr{G}, \quad P\underline{g}' + L\widehat{\underline{g}} = 0 \big\},$$
$$\Delta_{\Lambda^0 X}(f,F) = (-f'', L^{1/2} \widehat{\underline{f}}') \quad \textit{and} \quad \Delta_{\Lambda^1 X} g = -g''.$$

The 1*-form Laplacian defines a quantum graph* $(X, \widehat{\mathscr{G}}, L)$.

(iv) ***The* 1*-enlarged space:*** *We have*

$$\mathsf{H}^2(\Lambda^0 X) = \big\{ f \in \mathsf{H}^2_{\max}(X) \,\big|\, \widehat{\underline{f}}' \in \mathscr{G}, \quad \widehat{\underline{f}}' + LP\underline{f} = 0 \big\}$$
$$\mathsf{H}^2(\Lambda^1 X) = \big\{ (g,G) \in \mathsf{H}^2_{\max}(X) \oplus \mathscr{G} \,\big|\, \widehat{\underline{g}} \in \mathscr{G}, \quad \widehat{\underline{g}} + L^{1/2} G = 0 \big\},$$
$$\Delta_{\Lambda^0 X} f = -f'' \quad \textit{and} \quad \Delta_{\Lambda^1 X}(g,G) = (-g'', -L^{1/2} P\underline{g}').$$

We have $\mathsf{H}^2(\Lambda^0 X) = \mathsf{H}^2(X, \mathscr{G}^{\max}, \widetilde{L})$ *where* $\widetilde{L}$ *is defined as in Case* (ii). *In particular, the* 0*-form Laplacian defines the quantum graph* $(X, \mathscr{G}^{\max}, \widetilde{L})$.

(v) ***The* 1*-enlarged space with projection:*** *We have*

$$\mathsf{H}^2(\Lambda^0 X) = \big\{ f \in \mathsf{H}^2_{\max}(X) \,\big|\, \underline{f} \in \mathscr{G}, \quad P\widehat{\underline{f}}' + L\underline{f} = 0 \big\}$$
$$\mathsf{H}^2(\Lambda^1 X) = \big\{ (g,G) \in \mathsf{H}^2_{\max}(X) \oplus \mathscr{G} \,\big|\, \underline{g}' \in \mathscr{G}, \quad P\widehat{\underline{g}} + L^{1/2} G = 0 \big\},$$
$$\Delta_{\Lambda^0 X} f = -f'' \quad \textit{and} \quad \Delta_{\Lambda^1 X}(g,G) = (-g'', -L^{1/2} \underline{g}').$$

The 0*-form Laplacian defines the "classical" quantum graph* $(X, \mathscr{G}, L)$.

Note that the case (ii) becomes the case (iv) and the case (iii) becomes the case (v) when interchanging 0- and 1-forms, replacing F by $-G$, and exchanging $\mathscr{G}$ by $\widehat{\mathscr{G}}$ and L by $\widehat{L} = \tau L \tau$.

6. Index formulas for metric graphs

6.1. **Isomorphism between kernels of discrete and quantum graph Dirac operators.** We will now present one of the main results of this article, namely we establish an isomorphism between $\ker D$ and $\ker \mathsf{D}$ in the five cases of differential forms mentioned above respecting the supersymmetric space decomposition.

We need some notation: For a bounded operator L in $\mathscr{G}$, we set $\mathscr{G}_0 := \ker L$ and $\mathscr{G}_1 := (\ker L)^\perp$. Furthermore, $\mathscr{G}_i^\perp := \mathscr{G}^{\max} \ominus \mathscr{G}_i$. In addition, we denote the projections corresponding to $\mathscr{G}_i$ and $\mathscr{G}_i^\perp$ by P_i and $P_i^\perp$, respectively. Similarly, we denote the corresponding exterior derivatives by $\mathsf{d}_i \colon \mathscr{G}_i \longrightarrow \ell_2(E)$ and $\mathsf{d}_i^\perp \colon \mathscr{G}_i^\perp \longrightarrow \ell_2(E)$.

The discrete Dirac operator needs to be trivially enlarged by the space $\mathscr{N} = \mathscr{G}_0$ in the cases (ii)–(v) (cf. Definition 1.7). Finally, $\eta := \int g$ is defined by $\eta_e := \int_0^{\ell_e} g_e(x)\,\mathrm{d}x$.

Recall that our graphs may be infinite and may have multiple edges or self-loops.

Theorem 6.1. *Let* X *be a metric graph fulfilling the global lower bound condition on the edge lengths* (3.1)*, i.e.,* $\ell_e \geq \ell_0 > 0$. *Denote by* D *and* D *the Dirac operators associated to the metric and discrete exterior derivatives* d *and* d*, respectively, described below. Then there exists a bounded operator*

$$\Phi \colon \mathsf{H}^1(\Lambda X) \longrightarrow \ell_2(\Lambda X)$$

Remark 5.16. By splitting the vertex space $\mathscr{G}$ we may assume that L is invertible on a smaller vertex space: For example, in the 0-enlarged case (ii), the condition $\underline{f} = L^{1/2}F$ implies that $\underline{f} \in (\ker L^{1/2})^\perp =: \mathscr{G}_1$. Then we have

$$\mathsf{H}^1(\Lambda^0 X) = \big\{ (f,F) \in \mathsf{H}^1_{\mathscr{G}_1}(X) \oplus \mathscr{G}_1 \,\big|\, \underline{f} = L_1^{1/2} F_1 \big\} \oplus \mathscr{G}_0$$

where $\mathscr{G}_0 := \ker L$ and $L_1 := L\restriction_{\mathscr{G}_1}$ is invertible.

Remark 5.17. We assume here that L is invertible on $\mathscr{G}$. Then we can pass to the limit $L \to \infty$ in the equation $G = LF$ in the following sense: We consider the limit $L^{-1} \to 0$ in $L^{-1}G = F$, i.e., $F = 0$ and no restriction on G. We use this interpretation in order to show how the above different cases are related in the limit case:

(ii) **The 0-enlarged space:** Here, the condition in $\mathsf{H}^1(\Lambda^0 X)$ is $\underline{f} = L^{1/2}F$. The limit $L \to \infty$ leads to $F = 0$, i.e., $\mathscr{G}^0 = 0$. Moreover, the second component in $\mathrm{d}^* g$ has to vanish, i.e., $P\widehat{\underline{g}} = 0$. In particular, the added space $\mathscr{G}^0 = 0$ vanishes and we arrive at the simple case (i).

(iii) **The 0-enlarged space with projection:** The condition in $\mathsf{H}^1(\Lambda^0 X)$ is $P\underline{f} = L^{1/2}F$. The limit $L \to \infty$ leads again to $F = 0$, i.e., $\mathscr{G} = 0$. Furthermore, the second component in $\mathrm{d}^* g$ has to vanish, i.e., $\widehat{\underline{g}} = 0$. In particular, we arrive at the simple case (i) with Neumann vertex space $\mathscr{G} = \mathscr{G}^{\max}$.

(iv) **The 1-enlarged space:** The condition in $\mathsf{H}^1(\Lambda^1 X)$ is $\widehat{\underline{g}} + L^{1/2}G = 0$. The limit $L \to \infty$ here leads to $G = 0$, i.e., $\mathscr{G}^1 = 0$ and therefore $P\underline{f} = 0$. In particular, we arrive at the simple case (i) with the roles of $\mathscr{G}$ and $\mathscr{G}^\perp$ interchanged.

(v) **The 1-enlarged space with projection:** Finally, in this case, we arrive at the simple case (i) with Dirichlet vertex space $\mathscr{G} = \mathscr{G}^{\min} = 0$.

As in Section 1.2 we can associate a *Laplacian* $\Delta_{\Lambda X} := D^2$ to the Dirac operator D on the metric graph X with differential form space $\mathsf{H}^1(\Lambda X) \subset \mathsf{L}_2(\Lambda X)$ with natural domain

$$\mathsf{H}^2(\Lambda X) = \big\{ \omega \in \mathsf{H}^1(\Lambda X) \,\big|\, D\omega \in \mathsf{H}^1(\Lambda X) \big\}.$$

Furthermore, the operator decomposes into the components

$$\Delta_{\Lambda X} = \Delta_{\Lambda^0 X} \oplus \Delta_{\Lambda^1 X} = \mathrm{d}^*\mathrm{d} \oplus \mathrm{d}\mathrm{d}^*$$

with natural domains

$$\mathsf{H}^2(\Lambda^0 X) := \big\{ f \in \mathsf{H}^1(\Lambda^0 X) \,\big|\, \mathrm{d} f \in \mathsf{H}^1(\Lambda^1 X) \big\}$$
$$\mathsf{H}^2(\Lambda^1 X) := \big\{ g \in \mathsf{H}^1(\Lambda^1 X) \,\big|\, \mathrm{d}^* g \in \mathsf{H}^1(\Lambda^0 X) \big\}.$$

Lemma 5.18. *Assume that D is a Dirac operator on the metric graph X and that $\Delta_{\Lambda X} := D^2$ is its associated Laplacian. Then the components $\Delta_{\Lambda^p X}$ act as Laplacians on the metric graph part, i.e.,*

$$(\iota^0)^* \Delta_{\Lambda^0 X} \iota^0 = \mathrm{d}_0^*\mathrm{d}_0 \colon \mathsf{H}^2_{\mathrm{D}}(X) \longrightarrow \mathsf{L}_2(X)$$
$$(\iota^1)^* \Delta_{\Lambda^1 X} \iota^1 = \mathrm{d}_0^*\mathrm{d}_0 \colon \mathsf{H}^2_{\mathrm{D}}(X) \longrightarrow \mathsf{L}_2(X),$$

where $\mathrm{d}_0^*\mathrm{d}_0 = \bigoplus_{e\in E} \Delta_e^{\mathrm{D}}$ is the sum of the Dirichlet Laplacian on each edge.

Proof. The assertion follows directly from Definition 5.10 (ii)–(iii). □

We give the concrete domains of the Laplacians in each of the above cases:

Proposition 5.19. *Assume the condition* (3.1) *on the edge lengths, namely $\ell_e \geq \ell_0 > 0$.*

(i) ***The simple case:*** *Here we have*

$$\mathsf{H}^2(\Lambda^0 X) = \big\{ f \in \mathsf{H}^2_{\max}(X) \,\big|\, \underline{f} \in \mathscr{G}, \quad \widehat{\underline{f}}' \in \mathscr{G}^\perp \big\}, \qquad \Delta_{\Lambda^0 X} f = -f''$$
$$\mathsf{H}^2(\Lambda^1 X) = \big\{ g \in \mathsf{H}^2_{\max}(X) \,\big|\, \widehat{\underline{g}} \in \mathscr{G}^\perp, \quad \underline{g}' \in \mathscr{G} \big\}, \qquad \Delta_{\Lambda^1 X} g = -g''.$$

In particular, on 0-forms, we have the quantum graph $(X, \mathscr{G}, 0)$, and on 1-forms the quantum graph $(X, \widehat{\mathscr{G}}^\perp, 0)$.

Clearly, τ is a unitary local involution and given by the multiplication with $\widehat{\mathbb{1}}(v)$ on $\mathscr{G}_v^{\max}$ where $\widehat{\mathbb{1}}_e(v) = \pm 1$ if $v = \partial_\pm e$.

Definition 2.4. Let $\mathscr{G} = \bigoplus_{v\in V} \mathscr{G}_v$ be a vertex space with associated projection P. The *dual* vertex space is defined by[2] $\mathscr{G}^\perp = \mathscr{G}^{\max} \ominus \mathscr{G}$ with projection $P^\perp = \mathbb{1} - P$. The *oriented* version of the vertex space $\mathscr{G}$ is defined by $\widehat{\mathscr{G}} := \tau\mathscr{G}$ with projection $\widehat{P} = \tau P \tau$.

It can easily be seen that $\widehat{\mathscr{G}} = \mathscr{G}$ iff $\widehat{\mathbb{1}}(v) = \pm\mathbb{1}(v)$ for all $v \in V$, i.e., iff the graph X is bipartite (with partition $V = V_- \cup V_+$) and the orientation is chosen in such a way that $\partial_\pm e \in V_\pm$ for all $e \in E$.

In the following we give several examples of vertex spaces. We will see later on that these spaces are closely related to quantum graph Laplacians where the names come from. We start with two trivial vertex spaces:

Example 2.5.

(i) We call the trivial subspace $\mathscr{G}_v = \mathscr{G}_v^{\min} = 0$ the *minimal* or *Dirichlet vertex space*. The corresponding projection is $P_v = 0$.

(ii) We call the maximal subspace $\mathscr{G}_v = \mathscr{G}_v^{\max}$ the *maximal* or *Neumann vertex space*. The corresponding projection is $P_v = \mathbb{1}$. Clearly, $\mathscr{G}^{\max}$ is dual to $\mathscr{G}^{\min}$.

These examples are trivial, since every edge decouples from the others:

Definition 2.6. Let $\mathscr{G}_v$ be a vertex space at v with projection P_v.

(i) We say that $e_1 \in E_v$ *interacts with* $e_2 \in E_v$ *in* $\mathscr{G}_v$ iff

$$p_{e_1,e_2}(v) := \langle \delta_{e_1}(v), P_v \delta_{e_2}(v) \rangle \neq 0$$

where $(\delta_{e_1})_e(v) = 1$ if $e = e_1$ and 0 otherwise. If $p_{e_1,e_2}(v) = 0$, we say that $e_1, e_2 \in E_v$ *decouple in* $\mathscr{G}_v$.

(ii) We say that $\mathscr{G}_v$ *decouples along* $E_1 \cup E_2 \subset E_v$ iff e_1 and e_2 decouple in $\mathscr{G}_v$ for all $e_1 \in E_1$ and $e_2 \in E_2$.

(iii) We say that $\mathscr{G}_v$ is *completely interacting* iff e_1 and e_2 are interacting for any $e_1, e_2 \in E_v$, $e_1 \neq e_2$.

Lemma 2.7. *The edges $e_1, e_2 \in E_v$ ($e_1 \neq e_2$) are interacting (resp. decoupling) in $\mathscr{G}_v$ iff they are in $\mathscr{G}_v^\perp$. In particular, $\mathscr{G}_v$ is completely interacting iff $\mathscr{G}_v^\perp$ is.*

Proof. The claim follows immediately from

$$\big\langle \delta_{e_1}, P_v^\perp \delta_{e_2} \big\rangle = -\big\langle \delta_{e_1}, P_v \delta_{e_2} \big\rangle$$

since $e_1 \neq e_2$. □

Remark 2.8. Let $\mathscr{G}$ be a vertex space associated to the graph X such that $\mathscr{G}_v$ decouples along $E_1 \cup E_2 = E_v$, then $\mathscr{G}_v = \mathscr{G}_{1,v} \oplus \mathscr{G}_{2,v}$. Passing to a new graph $\widetilde{X}$ with the same edge set $E(\widetilde{X}) = E(X)$ but replacing $v \in V(X)$ by two vertices v_1, v_2 with $E_{v_1} = E_1$ and $E_{v_2} = v_2$, we obtain a new graph with one more vertex. Repeating this procedure, we can always assume that no vertex space $\mathscr{G}_v$ decouple along a non-trivial decomposition $E_v = E_1 \cup E_2$. It would be interesting to understand the "irreducible" building blocks of this decomposition procedure (see also [P09b, P09c]).

We will define now our main example, since it covers many of classically defined discrete Laplacians on a graph, as we will see later on:

Definition 2.9. We say that a vertex space $\mathscr{G}_v$ is *(weighted) continuous* if $\dim \mathscr{G}_v = 1$, i.e.,

$$\mathscr{G}_v = \mathbb{C}\underline{p}(v), \qquad |\underline{p}(v)|^2 = \deg v,$$

and $\mathscr{G}_v$ is completely interacting, i.e., $p_e(v) \neq 0$ for all $e \in E_v$ where $\underline{p}(v) = \{p_e(v)\}_e$.

[2] Here and in the sequel, $\mathscr{H} \ominus \mathscr{H}_1$ denotes the orthogonal complement of the (closed) subspace $\mathscr{H}_1$ in $\mathscr{H}$.

A vertex space $\mathscr{G}$ is called *(weighted) continuous* if all its components $\mathscr{G}_v$ are (weighted) continuous and if there are uniform constants $p_\pm \in (0,\infty)$ such that

$$p_- \le |p_e(v)| \le p_+, \qquad e \in E_v, \quad V \in V.$$

The dual of a continuous vertex space is called an *(unoriented weighted) sum vertex space.*

Applying the procedure of Remark 2.8, any vertex space $\mathscr{G}_v$ of dimension 1 with generating vector $\underline{p}(v)$ has a decomposition of $\mathscr{G}_v$ along $E_1 := \{\, e \in E_v \,|\, p_e(v) \neq 0 \,\}$ and $E_2 := E_v \setminus E_1$. The corresponding space $\mathscr{G}_{1,v}$ is now a continuous vertex space.

In all of the following examples, we can choose $p_\pm = 1$ as uniform bounds.

Example 2.10.

(iii) Choosing $\underline{p}(v) = \underline{\mathbb{1}}(v)$, i.e., $\mathscr{G}_v := \mathscr{G}_v^{\mathrm{std}} := \mathbb{C}\underline{\mathbb{1}}(v) = \mathbb{C}(1,\dots,1)$, we obtain the *(uniform) continuous* or *standard* vertex space denoted by $\mathscr{G}_v^{\mathrm{std}}$ where all coefficients $p_e(v) = 1$. The associated projection is

$$P_v = \frac{1}{\deg v}\mathbb{E}$$

where $\mathbb{E}$ denotes the square matrix of rank $\deg v$ where all entries equal 1.

(iv) We also have an *oriented* version of the standard vertex space, in particular $\widehat{\mathscr{G}^{\mathrm{std}}} = \mathbb{C}\widehat{\mathbb{1}}$ where $\widehat{\mathbb{1}}$ is defined in Definition 2.3. In particular,

$$p_e(v) = \pm 1 \quad \text{if} \quad v = \partial_\pm e.$$

(v) We call the dual $\mathscr{G}_v^\Sigma := (\mathscr{G}_v^{\mathrm{std}})^\perp = \mathscr{G}_v^{\max} \ominus \mathbb{C}(1,\dots,1)$ of the continuous vertex space the *(unoriented uniform) sum* or Σ-vertex space. Its associated projection is

$$P_v = \mathbb{1} - \frac{1}{\deg v}\mathbb{E}.$$

(vi) The *oriented sum vertex space* is the dual of the oriented continuous vertex space, i.e., $\widehat{\mathscr{G}^\Sigma} := (\widehat{\mathscr{G}^{\mathrm{std}}})^\perp$.

(vii) A more general case of continuous vertex spaces is given by vectors $\underline{p}(v)$ such that $|p_e(v)| = 1$, we call such continuous vertex spaces *magnetic*. An example is given in the following way: Let $\alpha \in \mathbb{R}^E$ be a function associating to each edge e the *magnetic vector potential* $\alpha_e \in \mathbb{R}$ and set

$$p_e(v) = \mathrm{e}^{-\mathrm{i}\widehat{\alpha}_e(v)/2}$$

where $\widehat{\alpha}_e(v) := \pm\alpha_e$ if $v = \partial_\pm e$ as in Definition 2.3. We call the associated vertex space $\mathscr{G}_v^{\mathrm{mag},\alpha}$ *magnetic* (see also Subsection 3.3.1).

Remark 2.11. Obviously, for the standard vertex space $\mathscr{G}_v^{\mathrm{std}} = \mathscr{G}_v^{\mathrm{mag},0}$. Furthermore, the oriented standard vertex space $\widehat{\mathscr{G}^{\mathrm{std}}}$ of (iv) is unitary equivalent to a special case of magnetic vertex spaces in (vii): Choose $\alpha_e = \pi$ for all $e \in E$ then $p_e(\partial_\pm e) = \mp\mathrm{i}$, i.e., $\underline{p}(v) = -\mathrm{i}\widehat{\underline{\mathbb{1}}}(v)$ and therefore $\widehat{\mathscr{G}^{\mathrm{std}}} = \mathrm{i}\mathscr{G}^{\mathrm{mag},\pi}$.

We want to express continuous vertex spaces with respect to the standard space $\ell_2(V)$, the "classical" space of 0-forms $\widetilde{F}\colon V \longrightarrow \mathbb{C}$ with norm defined by

$$\|\widetilde{F}\|^2_{\ell_2(V)} := \sum_{v\in V} |\widetilde{F}(v)|^2 \deg v. \tag{2.5}$$

In particular, the next lemma shows, that the vertex-weight $\deg v$ is canonical in the sense of (iii):

Lemma 2.12. *Let $\mathscr{G}$ be a continuous vertex space with projection P and denote by $[p^{-1}]$ the operator*

$$\big[p^{-1}\big]\colon \mathscr{G}^{\max} \longrightarrow \mathscr{G}^{\max}, \qquad F \mapsto \widetilde{F} = \{\underline{\widetilde{F}}(v)\}_v, \qquad \widetilde{F}_e(v) = \frac{F_e(v)}{p_e(v)}.$$

(i) *The multiplication operators $[p^{-1}]$ and $[p] = [p^{-1}]^{-1}$ are bounded on $\mathscr{G}^{\max}$*

We will now give concrete examples of Dirac operators. Since at this stage it is not clear what definition is "natural" we list some reasonable possibilities how to define the H^1-spaces:

Proposition 5.15. *Assume the condition* (3.1) *on the edge lengths, namely $\ell_e \ge \ell_0 > 0$.*

(i) ***The simple case:*** *We set*

$$\mathscr{G}^0 := 0, \quad \mathscr{G}^1 := 0, \quad \mathsf{H}^1(\Lambda^0 X) := \mathsf{H}^1_{\mathscr{G}}(X), \quad \mathrm{d}f := f'.$$

Then

$$\mathsf{H}^1(\Lambda^1 X) = \mathsf{H}^1_{\widehat{\mathscr{G}^\perp}}(X) \quad \textit{and} \quad \mathrm{d}^* g = -g'.$$

(ii) ***The 0-enlarged space:*** *Let $\mathscr{G}$ be a vertex space with bounded operator $L \ge 0$ on $\mathscr{G}$. We set $\mathscr{G}^0 := \mathscr{G}$, $\mathscr{G}^1 := 0$,*

$$\mathsf{H}^1(\Lambda^0 X) := \big\{\, (f,F) \in \mathsf{H}^1_{\mathscr{G}}(X) \oplus \mathscr{G} \,\big|\, \underline{f} = L^{1/2}F \,\big\}$$

and $\mathrm{d}(f,F) = f'$. Then we have

$$\mathsf{H}^1(\Lambda^1 X) = \mathsf{H}^1_{\max}(X) \quad \textit{and} \quad \mathrm{d}^* g = (-g', L^{1/2}P\widehat{\underline{g}}).$$

(iii) ***The 0-enlarged space with projection:*** *Let $\mathscr{G}$ be a vertex space with associated projection P and bounded operator $L \ge 0$ on $\mathscr{G}$. We set $\mathscr{G}^0 := \mathscr{G}$, $\mathscr{G}^1 := 0$,*

$$\mathsf{H}^1(\Lambda^0 X) := \big\{\, (f,F) \in \mathsf{H}^1_{\max}(X) \oplus \mathscr{G} \,\big|\, P\underline{f} = L^{1/2}F \,\big\}$$

and $\mathrm{d}(f,F) = f'$. Then

$$\mathsf{H}^1(\Lambda^1 X) = \mathsf{H}^1_{\widehat{\mathscr{G}}}(X) \quad \textit{and} \quad \mathrm{d}^* g = (-g', L^{1/2}\widehat{\underline{g}}).$$

(iv) ***The 1-enlarged space:*** *Let $\mathscr{G}$ be a vertex space with bounded operator $L \ge 0$. We set*

$$\mathscr{G}^0 = 0, \quad \mathscr{G}^1 := \mathscr{G}, \quad \mathsf{H}^1(\Lambda^0 X) := \mathsf{H}^1_{\max}(X), \quad \mathrm{d}f = (f, L^{1/2}P\underline{f}).$$

Then

$$\mathsf{H}^1(\Lambda^1 X) = \big\{\, (g,G) \in \mathsf{H}^1_{\max}(X) \oplus \mathscr{G} \,\big|\, \widehat{\underline{g}} \in \mathscr{G}, \quad \widehat{\underline{g}} + L^{1/2}G = 0 \,\big\}$$

and $\mathrm{d}^*(g,G) = -g'$.

(v) ***The 1-enlarged space with projection:*** *Let $\mathscr{G}$ be a vertex space with associated projection P and bounded operator $L \ge 0$ on $\mathscr{G}$. We set $\mathscr{G}^0 := 0$, $\mathscr{G}^1 := \mathscr{G}$,*

$$\mathsf{H}^1(\Lambda^0_{\mathscr{G}} X) := \mathsf{H}^1_{\mathscr{G}}(X), \quad \mathrm{d}f = (f', L^{1/2}\underline{f}).$$

Then

$$\mathsf{H}^1(\Lambda^1 X) = \big\{\, (g,G) \in \mathsf{H}^1_{\max}(X) \oplus \mathscr{G} \,\big|\, P\widehat{\underline{g}} + L^{1/2}G = 0 \,\big\}$$

and $\mathrm{d}^*(g,G) = -g'$.

Proof. We only check the conditions for (ii) since the other cases are similar. We apply Lemma 5.11 and have to show first that $\mathsf{H}^1(\Lambda^0 X)$ is an H^1-space of 0-forms and second, that d is an exterior derivative. In order to show the first, note that $\mathsf{H}^1(\Lambda^0 X)$ is dense in $\mathsf{L}_2(\Lambda^0 X)$: Let $(f,F) \in \mathsf{L}_2(X) \oplus \mathscr{G}$ and $\varepsilon > 0$. By density of $\mathsf{H}^1_{\max}(X)$ we can find a function $f_1 \in \mathsf{H}^1_{\max}(X)$ such that $\|f - f_1\|_{\mathsf{L}_2(X)} \le \varepsilon/2$. Furthermore, we can change f_1 to f_2 near a vertex v in such a way that $\underline{f}_2(v) = F(v)$ and that their norm difference does not exceed $\varepsilon/2$. Then (f_2, F) has distance at most ε from (f,F) in $\mathsf{L}_2(\Lambda^0 X)$.

The second and third condition of Definition 5.9 are obviously fulfilled. In order to show that d is an exterior derivative we have to check the conditions of Definition 5.10. For the closeness of d note that the graph norm of d defined by

$$\|(f,F)\|^2_{\mathrm{d}} := \|\mathrm{d}f\|^2_{\mathsf{L}_2(\Lambda^1)} + \|(f,F)\|^2_{\mathsf{L}_2(\Lambda^0 X)} = \|f'\|^2_{\mathsf{L}_2(X)} + \|f\|^2_{\mathsf{L}_2(X)} + \|F\|^2_{\mathscr{G}}$$

is the Sobolev norm. It remains to show that $\mathsf{H}^1(\Lambda^0 X)$ is closed in $\mathsf{H}^1_{\max}(X) \oplus \mathscr{G}$: Note that $(f,F) \mapsto P\underline{f} - L^{1/2}F$ is continuous by Proposition 5.2 and since L is bounded. Furthermore, $\mathsf{H}^1(\Lambda^0 X)$ is the kernel of this map and therefore closed. The second and third condition of Definition 5.10 follow by an immediate calculation. □

Note that $(\iota^p)^*$ is the projection onto the first factor.

For the definition of an exterior derivative, we need the following decoupled operator

$$\mathrm{d}_0\colon \mathsf{H}^1_\circ(X) \longrightarrow \mathsf{L}_2(X), \qquad \mathrm{d}_0 f = f'.$$

Note that d_0 is a closed operator with adjoint $\mathrm{d}_1 g = -g'$ and $\operatorname{dom}\mathrm{d}^* = \mathsf{H}^1_{\max}(X)$. We now define an exterior derivative associated to spaces of p-forms. Examples are given below.

Definition 5.10. Let $\mathsf{L}_2(\Lambda^p X)$ be an L_2-space of p-forms and $\mathsf{H}^1(\Lambda^0 X)$ be an H^1-space of 0-forms. We call an operator

$$\mathrm{d}\colon \mathsf{H}^1(\Lambda^0 X) \longrightarrow \mathsf{L}_2(\Lambda^1 X)$$

an *exterior derivative* on the metric graph X iff the following conditions are fulfilled:

(i) The operator d as unbounded operator from the 0-form space $\mathsf{L}_2(\Lambda^0 X)$ into the 1-form space $\mathsf{L}_2(\Lambda^0 X)$ is closed.
(ii) We have $\mathrm{d}\iota^0 = \iota^1 \mathrm{d}_0$, i.e., $\mathrm{d}(\iota^0 f) = \iota^1 f'$ for all $f \in \mathsf{H}^1_\circ(X)$.
(iii) We have $(\iota^1)^*\mathrm{d} = -\mathrm{d}_0^*(\iota^0)^*$, i.e., $(\iota^1)^*\mathrm{d}\widetilde{f} = ((\iota^0)^*\widetilde{f})'$.

Note that the closeness of d ensures that we choose the "right" norm on $\mathsf{H}^1(\Lambda^p X)$ (and not an artificially smaller space).

Lemma 5.11. *Given the p-form spaces $\mathsf{L}_2(\Lambda^p X)$, $p \in \{0,1\}$, the 0-form space $\mathsf{H}^1(\Lambda^0 X)$ and an exterior derivative $\mathrm{d}\colon \mathsf{H}^1(\Lambda^0 X) \longrightarrow \mathsf{L}_2(\Lambda^0 X)$, then the adjoint d^* is uniquely defined and closed as operator from $\mathsf{L}_2(\Lambda^1 X)$ into $\mathsf{L}_2(\Lambda^0 X)$. Its domain*

$$\mathsf{H}^1(\Lambda^1 X) := \operatorname{dom}\mathrm{d}^*$$

is an H^1-space of 1-forms (cf. Definition 5.9 (i)–(iii)).

Proof. Since d is densely defined by Definition 5.10 (i), it follows, that d^* is uniquely determined, closed and densely defined, i.e., Definition 5.9 (i) is fulfilled. In order to verify Definition 5.9 (ii), we have to show that for $f \in \mathsf{H}^1_\circ(X)$, the 1-form $\iota^1 f$ is in $\operatorname{dom}\mathrm{d}^*$: Set $h := -\iota^0\mathrm{d}_0 f$. Then $h \in \mathsf{L}_2(\Lambda^0 X)$ and we have

$$\langle h, g\rangle = \langle -\mathrm{d}_0^*(\iota^0)^*\widetilde{f}, f\rangle = \langle (\iota^1)^*\mathrm{d}\widetilde{f}, f\rangle = \langle d\widetilde{f}, \iota^1 f\rangle$$

for all $\widetilde{f} \in \operatorname{dom} d$, i.e., $\iota^1 f \in \operatorname{dom}\mathrm{d}^*$, where we used Definition 5.10 (iii). Condition (iii) follows similarly from Definition 5.10 (ii). □

Definition 5.12. We call the operator

$$D\colon \mathsf{H}^1(\Lambda X) \longrightarrow \mathsf{L}_2(\Lambda X), \qquad D(f,g) := (\mathrm{d}^* g, \mathrm{d} f)$$

the *Dirac-operator* associated to the p-form spaces $\mathsf{H}^1(\Lambda X) \subset \mathsf{L}_2(\Lambda X)$ and the exterior derivative d.

Remark 5.13.

(i) Obviously, D is a closed and self-adjoint operator with the matrix representation

$$D \cong \begin{pmatrix} 0 & \mathrm{d}^* \\ \mathrm{d} & 0 \end{pmatrix}$$

where we split the space in its 0- and 1-form component.

(ii) In order to define a Dirac operator D, it suffices — due to Lemma 5.11 — to determine

$$\mathscr{G}^0, \quad \mathscr{G}^1, \quad \mathsf{H}^1(\Lambda^0 X), \quad \mathrm{d}\colon \mathsf{H}^1(\Lambda^0 X) \longrightarrow \mathsf{L}_2(\Lambda^1 X),$$

and to ensure that the conditions of Definition 5.9 (i)–(iii) for $p = 0$ and Definition 5.10 (i)–(iii) are fulfilled. In this case, the Dirac operator is uniquely determined.

Again, we have a "baby" version of the Hodge decomposition theorem (see Proposition 1.3):

Proposition 5.14. *Assume that D has a spectral gap at 0, i.e., that $\operatorname{dist}(0, \sigma(D) \setminus \{0\}) > 0$ (e.g., X compact is sufficient). Then*

$$\mathsf{L}_2(\Lambda X) = \ker D \oplus \operatorname{ran}\mathrm{d}^* \oplus \operatorname{ran}\mathrm{d}, \qquad \textit{i.e.,}$$
$$\mathsf{L}_2(\Lambda^0 X) = \ker\mathrm{d} \oplus \operatorname{ran}\mathrm{d}^* \quad \textit{and} \quad \mathsf{L}_2(\Lambda^1 X) = \ker\mathrm{d}^* \oplus \operatorname{ran}\mathrm{d}.$$

(ii) *We have $[p^{-1}](\mathscr{G}) = \mathscr{G}^{\mathrm{std}}$ and $[p^{-1}](\mathscr{G}^\perp) = \mathscr{G}^{\Sigma|p|^2}$ where*

$$\mathscr{G}^{\Sigma|p|^2} := \Big\{ \widetilde{F} \in \mathscr{G}^{\max} \Big| \sum_{e\in E_v} |p_e(v)|^2 \widetilde{F}_e(v) = 0 \quad \forall v \in V \Big\}$$

for the dual.

(iii) *Denote $\widetilde{U}\colon \mathscr{G}^{\mathrm{std}} \longrightarrow \ell_2(V)$ the local operator mapping $\underline{\widetilde{F}}(v) = \widetilde{F}(v)(1,\dots,1)$ onto $\widetilde{F}(v) \in \mathbb{C}$, then $\widetilde{U}$ is unitary. Furthermore,*

$$U\colon \mathscr{G} \longrightarrow \ell_2(V), \qquad U := \widetilde{U} \circ \big[p^{-1}\big]$$

is unitary.

(iv) *The transformed projection $\widetilde{P} := UP\colon \mathscr{G}^{\max} \longrightarrow \ell_2(V)$ is given by*

$$(\widetilde{P}_v F)(v) = \frac{1}{\deg v} \sum_{e\in E_v} \overline{p_e(v)} F_e(v) \in \mathbb{C}$$

and no coefficient $p_e(v)$ vanishes.

Proof. (i) The boundedness follows from the global bounds $p_\pm$ on $|p_e(v)|$ (cf. Definition 2.9). (ii) $[p^{-1}]$ restricted to $\mathscr{G}^{\max}_v$ maps the vector $\underline{p}(v)$ onto $(1,\dots,1)$, i.e., $\mathscr{G}_v$ onto $\mathscr{G}^{\mathrm{std}}_v$; a vector $\underline{F}(v) \in \mathscr{G}^\perp$ satisfies $\sum_{e\in E_v} \overline{p_e(v)} F_e(v) = 0$, and therefore $\underline{\widetilde{F}}(v) \in \mathscr{G}^{\Sigma|p|^2}_v$. (iii) We have

$$|\underline{\widetilde{F}}(v)|^2_{\mathbb{C}^{E_v}} = |\widetilde{F}(v)|^2 |(1,\dots,1)|^2_{\mathbb{C}^{E_v}} = |\widetilde{F}(v)|^2 \deg v$$

and therefore, $\widetilde{U}$ is unitary. Furthermore,

$$\|\widetilde{F}\|^2_{\ell_2(V)} = \sum_{v\in V} |\widetilde{F}(v)|^2 \deg v = \sum_{v\in V}\sum_{e\in E_v} |\widetilde{F}(v) p_e(v)|^2 = \sum_{v\in V}\sum_{e\in E_v} |F_e(v)|^2 = \|F\|^2_{\mathscr{G}}$$

since $|\underline{p}(v)|^2 = \deg v$. The last assertion follows by a straightforward calculation. □

Note that the decomposition into $\mathscr{G}^{\mathrm{std}}$ and $\mathscr{G}^{\Sigma|p|^2}$ is no longer orthogonal if $[p^{-1}]$ is not unitary (i.e., $|p_e(v)| \neq 1$ for some $e \in E_v$).

The trivial, the uniform continuous and the sum vertex spaces are obviously invariant under permutation of the edges in E_v. Indeed, these are the only possibilities for such an invariance:

Proposition 2.13. *A vertex space $\mathscr{G}_v$ is invariant under permutation of the coordinates $e \in E_v$ iff $\mathscr{G}_v$ is either maximal ($\mathscr{G}^{\max}_v = \mathbb{C}^{E_v}$), minimal ($\mathscr{G}^{\min}_v = 0$), uniform continuous ($\mathscr{G}^{\mathrm{std}}_v = \mathbb{C}(1,\dots,1)$) or the sum vertex space ($\mathscr{G}^{\Sigma}_v = \mathbb{C}^{E_v} \ominus \mathbb{C}(1,\dots,1)$).*

Proof. It can be shown, that a square matrix P of dimension $d = \deg v$ is invariant under the symmetric group S_d of order d iff P has the form

$$P = a\mathbb{1} + b\mathbb{E},$$

since the only subspaces invariant under S_d are $\mathbb{C}(1,\dots,1)$ and its orthogonal complement, and the representation of S_d on the orthogonal complement is irreducible (see e.g. the references in [Ku04]). Using the relations $P = P^*$ and $P^2 = P$ for an orthogonal projection, we obtain that a and b must be real and satisfy the relations $a^2 = a$ and $2ab + (\deg v)b = b$, from which the four cases follow. □

3. Operators on vertex spaces

In this section, we define a generalised coboundary operator or exterior derivative associated to a vertex space. We use this exterior derivative for the definition of an associated Dirac and Laplace operator as in the supersymmetric setting of Section 1.2.

3.1. **Discrete exterior derivatives.** On the maximal vertex space $\mathscr{G}^{\max}$, we define a general coboundary operator or *exterior derivative* as

$$\mathsf{d} = \mathsf{d}^{\max}\colon \mathscr{G}^{\max} \longrightarrow \ell_2(E), \qquad (\mathsf{d}F)_e := F_e(\partial_+ e) - F_e(\partial_- e),$$

Definition 3.1. Let $\mathscr{G}$ be a vertex space of the graph X. The exterior derivative on $\mathscr{G}$ is defined as

$$\mathsf{d}_{\mathscr{G}} := \mathsf{d}^{\max}|_{\mathscr{G}}\colon \mathscr{G} \longrightarrow \ell_2(E), \qquad (\mathsf{d}F)_e := F_e(\partial_+ e) - F_e(\partial_- e),$$

mapping 0-forms onto 1-forms.

We often drop the subscript $\mathscr{G}$ for the vertex space, or use other intuitive notation in order to indicate the vertex space.

We define a multiplication operator $[\ell^{-1}]$ on $\mathscr{G}^{\max}$ and $\ell_2(E)$ by

$$\big([\ell^{-1}]F\big)_e(v) = \frac{1}{\ell_e}F_e(v) \quad \text{and} \quad \big([\ell^{-1}]\eta\big)_e = \frac{1}{\ell_e}\eta_e,$$

respectively. Clearly, $[\ell^{-1}]$ is bounded on both spaces iff there exists $\ell_0 > 0$ such that

$$\ell_e \geq \ell_0, \qquad e \in E. \tag{3.1}$$

Without loss of generality, we assume that $\ell_0 \leq 1$. On a vertex space $\mathscr{G} \leq \mathscr{G}^{\max}$ with associated projection P, we can relax the condition slightly, namely, we assume that $P[\ell^{-1}]$ is bounded, i.e., that

$$\kappa := \sup_{v\in V}\big|P_v[\ell^{-1}]_v\big|_v < \infty \tag{3.2}$$

where $|\cdot|_v$ denotes the operator norm for matrices on $\mathbb{C}^{E_v}$.

Remark 3.2.

(i) If (3.1) is fulfilled, then $\kappa \leq 1/\ell_0$. In particular, if $\ell_e = \ell_0$ for all $e \in E$ then $\kappa = 1/\ell_0$.
(ii) For the (uniform) continuous vertex space $\mathscr{G}^{\mathrm{std}}$, we have

$$|P_v[\ell^{-1}]_v|_v = \frac{1}{\deg v}\sum_{e\in E_v}\frac{1}{\ell_e}.$$

(iii) If we assume that (3.2) holds for P and $P^\perp$, then (3.1) is also fulfilled. For simplicity, we assume therefore that (3.1) holds (if not stated otherwise).

Lemma 3.3. *Assume* (3.2)*, then* d *is norm-bounded by* $\sqrt{2\kappa}$*. The adjoint*

$$\mathsf{d}^*\colon \ell_2(E) \longrightarrow \mathscr{G}$$

fulfils the same norm bound and is given by

$$(\mathsf{d}^*\eta)(v) = P_v\Big(\Big\{\frac{1}{\ell}_e\,\widehat{\eta}_e(v)\Big\}\Big) \in \mathscr{G}_v,$$

where $\widehat{\eta}_e(v) := \pm\eta_e$ *if* $v = \partial_\pm e$ *denotes the* oriented *evaluation of* η_e *at the vertex* v.

Definition 5.7. A *quantum graph* X is a metric graph together with a self-adjoint Laplacian $\Delta_{(X,\mathscr{G},L)}$ where $\mathscr{G}$ is a vertex space and L a self-adjoint, bounded operator on $\mathscr{G}$. A quantum graph X is therefore determined by the data $(V, E, \partial, \ell, \mathscr{G}, L)$, i.e., by the metric graph (V, E, ∂, ℓ) and the vertex conditions determined by $(\mathscr{G}, L)$.

Remark 5.8.

(i) In [KS99] (see also [KPS07]) there is another way of parametrising all self-adjoint vertex conditions, namely for bounded operators A, B on $\mathscr{G}^{\max}$,

$$\operatorname{dom}\Delta_{(A,B)} = \{\, f \in \mathsf{H}^2_{\max}(X) \mid A\underline{f} + B\widehat{\underline{f}}' = 0 \,\}$$

is the domain of a self-adjoint operator $\Delta_{(A,B)}$ iff
(a) $A \mathbin{\hat{\oplus}} B\colon \mathscr{G}^{\max} \oplus \mathscr{G}^{\max} \longrightarrow \mathscr{G}^{\max}$, $F \oplus \widehat{F} \mapsto AF + B\widehat{F}$, is surjective
(b) AB^* is self-adjoint, i.e., $AB^* = BA^*$.
Given a vertex space $\mathscr{G} \leq \mathscr{G}^{\max}$ and a bounded operator L on $\mathscr{G}$, we have $\Delta_{(A,B)} = \Delta_{(X,\mathscr{G},L)}$ if we choose

$$A \cong \begin{pmatrix} L & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad \text{and} \quad B = P \cong \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathscr{G}^{\max} = \mathscr{G} \oplus \mathscr{G}^\perp$. The associated scattering matrix with spectral parameter $\mu = \sqrt{\lambda}$ is

$$S(\mu) = -(A + \mathrm{i}\mu B)^{-1}(A - \mathrm{i}\mu B) \cong \begin{pmatrix} -(L + \mathrm{i}\mu\mathbb{1})^{-1}(L - \mathrm{i}\mu\mathbb{1}) & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

In particular, $S(\mu)$ is independent of μ iff $L = 0$, and in this case, we have $S(\mu) = \mathbb{1} \oplus -\mathbb{1}$ for all μ.

(ii) In [FKW07], there is a useful description of vertex conditions, closely related to ours: Namely, a vertex condition is determined by the *Dirichlet part* $\underline{f} \in \mathscr{G}$, the *Neumann part* $\widehat{\underline{f}}' \in \mathscr{G}'$ and the *Robin part* $P_1\widehat{\underline{f}}' = L_1P_1\underline{f}$. Here, $\mathscr{G}' = \ker L \oplus \mathscr{G}^\perp$, and P_1 denotes the projection onto $\operatorname{ran} L$. Moreover, L_1 is the restriction of L onto $\operatorname{ran} L = (\ker L)^\perp$. In particular, L_1 is invertible.

The aim of the subsequent section is to express $\Delta_{(X,\mathscr{G},L)}$ as d*d or dd*. Of course, to do so, we need $L \geq 0$ (since operators d*d and dd* are always non-negative). Furthermore, for non-trivial $L \neq 0$, we need to enlarge the L_2-spaces by the vertex space $\mathscr{G}$.

5.3. **Differential forms, exterior derivatives and Dirac operators.** The definition of differential forms is a bit technical, since a priori, we want to include the more general metric graph Laplacians $\Delta_{(X,\mathscr{G},L)}$. As a motivation, we refer to the limit operator defined on an "enlarged" metric graph in the smooth space limit (see Examples 6.6–6.7 and [EP05, P09c, P09a]). The limit operator acts on an "enlarged" space which is precisely the 0-enlarged space defined below.

Definition 5.9. For $p \in \{0, 1\}$, let $\mathscr{G}^p$ be a vertex space. We call the space

$$\mathsf{L}_2(\Lambda^p X) := \mathsf{L}_2(X) \oplus \mathscr{G}^p$$

the L_2*-space of p-forms*.

A subspace $\mathsf{H}^1(\Lambda^p X)$ of $\mathsf{L}_2(\Lambda^p X)$ is called an H^1*-space of p-forms* iff
(i) the space $\mathsf{H}^1(\Lambda^p X)$ is dense in $\mathsf{L}_2(\Lambda^p X)$,
(ii) we have $\iota^p(\mathsf{H}^1_{\mathrm{c}}(X)) \subset \mathsf{H}^1(\Lambda^p X)$ and
(iii) we have $(\iota^p)^*(\mathsf{H}^1(\Lambda^p X)) \subset \mathsf{H}^1_{\max}(X)$,
where

$$\iota^p\colon \mathsf{L}_2(X) \hookrightarrow \mathsf{L}_2(\Lambda^p X), \qquad f \mapsto (f, 0), \qquad p \in \{0, 1\},$$

denote the natural embedding operators. We set

$$\mathsf{L}_2(\Lambda X) := \mathsf{L}_2(\Lambda^0 X) \oplus \mathsf{L}_2(\Lambda^1 X) \quad \text{and} \quad \mathsf{H}^1(\Lambda X) := \mathsf{H}^1(\Lambda^0 X) \oplus \mathsf{H}^1(\Lambda^1 X).$$

In order to avoid an upper bound on ℓ_e, we replace the edge e by the shortened edge $\widetilde{e}_v$ of length $\widetilde{\ell}_e = \max\{\ell_e, 1\}$ starting at v. Then

$$|f_e(v)|^2 \le 2\widetilde{\ell}_e \|f'\|^2_{\widetilde{e}_v} + \frac{2}{\widetilde{\ell}_e}\|f\|^2_{\widetilde{e}_v} \le 2\max\Big\{1, \frac{1}{\ell_0}\Big\}\|f\|^2_{\mathsf{H}^1(e)}$$

using Cauchy-Schwarz. Summing the contributions over $e \in E_v$ and $v \in V$ and using (2.1) we are done. The same arguments hold for $\widehat{\underline{g}}$. □

For a general vertex space $\mathscr{G}$, i.e., a closed subspace of $\mathscr{G}^{\max} := \bigoplus_{v\in V} \mathbb{C}^{E_v}$, we set

$$\mathsf{H}^1_{\mathscr{G}}(X) := \big\{\, f \in \mathsf{H}^1_{\max}(X) \,\big|\, \underline{f} \in \mathscr{G} \,\big\} = (\underline{\cdot})^{-}\mathscr{G},$$

i.e., the preimage of $\mathscr{G}$ under the (unoriented) evaluation map, and similarly,

$$\mathsf{H}^1_{\widehat{\mathscr{G}}}(X) := \big\{\, g \in \mathsf{H}^1_{\max}(X) \,\big|\, \widehat{\underline{g}} \in \mathscr{G} \,\big\} = (\widehat{\cdot})^{-}\mathscr{G}$$

the preimage of $\mathscr{G}$ under the (oriented) evaluation map. In particular, both spaces are closed in $\mathsf{H}^1_{\max}(X)$.

The reason for two different vertex evaluations becomes clear through the following lemma:

Lemma 5.3. *For $f, g \in \mathsf{H}^1_{\max}(X)$, we have*

$$\langle f', g\rangle_X = \langle f, -g'\rangle_X + \langle \underline{f}, \widehat{\underline{g}}\rangle_{\mathscr{G}^{\max}}.$$

Proof. We have

$$\begin{aligned}\langle f', g\rangle_X + \langle f, g'\rangle_X &= \sum_{e\in E}\big(\langle f', g\rangle_e + \langle f, g'\rangle_e\big)\\ &= \sum_{e\in E}\big[\overline{f}g\big]_{\partial e} = \sum_{v\in V}\sum_{e\in E_v}\overline{f}_e(v)\widehat{g}_e(v) = \sum_{v\in V}\langle \underline{f}(v), \widehat{\underline{g}}(v)\rangle_{\mathbb{C}^{E_v}} = \langle \underline{f}, \widehat{\underline{g}}\rangle_{\mathscr{G}^{\max}}\end{aligned}$$

and the latter expression is defined due to Proposition 5.2. □

5.2. **Quantum graphs.**

Definition 5.4. A *Laplacian* on a (continuous) metric graph $X = (V, E, \partial, \ell)$ is an operator Δ_X acting as $(\Delta_X f)_e = -f_e''$ on each edge $e \in E$.

We have the following characterisation from [Ku04, Thm. 17]:

Theorem 5.5. *Assume the condition* (3.1) *on the edge lengths, namely $\ell_e \ge \ell_0 > 0$. Let $\mathscr{G} \le \mathscr{G}^{\max}$ be a (closed) vertex space with orthogonal projection P, and let L be a self-adjoint, bounded operator on $\mathscr{G}$. Then the Laplacian $\Delta_{(X,\mathscr{G},L)}$ with domain*

$$\mathsf{H}^2(X, \mathscr{G}, L) := \big\{\, f \in \mathsf{H}^2_{\max}(X) \,\big|\, \underline{f} \in \mathscr{G}, \quad P\widehat{\underline{f}}' + L\underline{f} = 0 \,\big\}$$

is self-adjoint with associated quadratic form

$$\mathfrak{d}_{(X,\mathscr{G},L)}(f) := \|f'\|^2_X + \langle f, Lf\rangle_{\mathscr{G}} = \sum_{e\in E}\|f_e'\|^2_e + \sum_{v\in V}\langle \underline{f}(v), L(v)\underline{f}(v)\rangle_{\mathscr{G}_v}$$

and domain $\operatorname{dom}\mathfrak{d}_{(X,\mathscr{G},L)} = \mathsf{H}^1_{\mathscr{G}}(X) = \{\, f \in \mathsf{H}^1_{\max}(X) \,|\, \underline{f} \in \mathscr{G} \,\}$.

Remark 5.6.

(i) At least for finite graphs, the converse statement is true, i.e., if Δ is a self-adjoint Laplacian in the sense of Definition 5.4 then $\Delta = \Delta_{(X,\mathscr{G},L)}$ for some vertex space $\mathscr{G}$ and a bounded operator L. For infinite graphs, the operator L may become unbounded but we do not consider this case here.

(ii) Note that $\Delta_{(X,\mathscr{G},L)} \ge 0$ iff $L \ge 0$.

We slightly restrict ourselves and consider only those self-adjoint Laplacians on X that are obtained as in the above theorem:

Proof. We have

$$\begin{aligned}\|\mathsf{d}F\|^2_{\ell_2(E)} &= \sum_{e\in E}\frac{1}{\ell_e}\big|F_e(\partial_+ e) - F_e(\partial_- e)\big|^2\\ &\le 2\sum_{v\in V}\Big(\sum_{e\in E_v^+}\frac{1}{\ell_e}\big|F_e(v)\big|^2 + \sum_{e\in E_v^-}\frac{1}{\ell_e}\big|F_e(v)\big|^2\Big)\\ &\le 2\sum_{v\in V}\sum_{e\in E_v}\frac{1}{\ell_e}\big|F_e(v)\big|^2\\ &\le 2\sum_{v\in V}\big\langle [\ell^{-1}]_v\underline{F}(v), \underline{F}(v)\big\rangle\\ &= 2\sum_{v\in V}\big\langle [\ell^{-1}]_v\underline{F}(v), P_v\underline{F}(v)\big\rangle\\ &\le 2\kappa\|F\|^2_{\mathscr{G}}\end{aligned}$$

using Eq. (2.1) and the fact that $\underline{F}(v) \in \mathscr{G}_v$. For the second assertion, we calculate

$$\begin{aligned}\langle \mathsf{d}F, \eta\rangle &= \sum_{e\in E}\frac{1}{\ell_e}\big(\overline{F}_e(\partial_+ e) - \overline{F}_e(\partial_- e)\big)\eta_e\\ &= \sum_{v\in V}\Big(\sum_{e\in E_v^+}\frac{1}{\ell_e}\overline{F}_e(v)\,\eta_e - \sum_{e\in E_v^-}\frac{1}{\ell_e}\overline{F}_e(v)\,\eta_e\Big)\\ &= \sum_{v\in V}\Big\langle P_vF, \Big\{\frac{1}{\ell_e}\widehat{\eta}_e(v)\Big\}_{e\in E_v}\Big\rangle_{\mathscr{G}_v^{\max}} = \langle F, \mathsf{d}^*\eta\rangle\end{aligned}$$

since $\underline{F}(v) \in \mathscr{G}_v$, i.e., $P_v\underline{F}(v) = \underline{F}(v)$. □

Example 3.4.

(i) For the minimal vertex space, we have $\mathsf{d} = 0$ and $\mathsf{d}^* = 0$. Obviously, these operators are decoupled, i.e., they do not feel any connection information of the graph.

(ii) For the maximal vertex space, we have (denoting $\mathsf{d} = \mathsf{d}^{\max}$)

$$(\mathsf{d}^*\eta)_e(v) = \frac{1}{\ell_e}\,\widehat{\eta}_e(v).$$

The operator $\mathsf{d} = \mathsf{d}^{\max}$ decomposes as $\bigoplus_e \mathsf{d}_e$ with respect to the decomposition of $\mathscr{G}^{\max}$ in Eq. (2.3) and $\ell_2(E)$ in Eq. (2.4). Here,

$$\big(\mathsf{d}_e\colon \mathbb{C}^2 \longrightarrow \mathbb{C}\big) \cong \begin{pmatrix}1 & -1\end{pmatrix} \quad \text{and} \quad \big(\mathsf{d}_e^*\colon \mathbb{C} \longrightarrow \mathbb{C}^2\big) \cong \frac{1}{\ell_e}\begin{pmatrix}1\\ -1\end{pmatrix}$$

where $F_e = (F_e(\partial_+ e), F_e(\partial_- e)) \in \mathbb{C}^2$. Again, the operators are decoupled, since any connection information of the graph is lost.

We can now calculate the exterior derivative and its adjoint in several general cases. The proofs are straightforward. We start with the relation to the dual vertex space:

Lemma 3.5. *Let $\mathscr{G}$ be a vertex space with exterior derivative $\mathsf{d} = \mathsf{d}_{\mathscr{G}}$, then*

$$\begin{aligned}&\mathsf{d}_{\mathscr{G}} \overset{\curlywedge}{\oplus} \mathsf{d}_{\mathscr{G}^\perp} = \mathsf{d}^{\max}\colon \mathscr{G}^{\max} \longrightarrow \ell_2(E), && F \oplus F^\perp \mapsto \mathsf{d}_{\mathscr{G}}F + \mathsf{d}_{\mathscr{G}^\perp}F^\perp\\ &\mathsf{d}^*_{\mathscr{G}} \overset{\curlyvee}{\oplus} \mathsf{d}^*_{\mathscr{G}^\perp} = (\mathsf{d}^{\max})^*\colon \ell_2(E) \longrightarrow \mathscr{G}^{\max}, && \eta \mapsto \mathsf{d}^*_{\mathscr{G}}\eta \oplus \mathsf{d}^*_{\mathscr{G}^\perp}\eta.\end{aligned}$$

In particular,

$$((\mathsf{d}^*_{\mathscr{G}^\perp}\eta)_e(v) = \frac{1}{\ell_e}\widehat{\eta}_e(v) - (\mathsf{d}^*_{\mathscr{G}}\eta)_e(v).$$

For a continuous vertex space, it is convenient to use the unitary transformation from $\mathscr{G}$ onto $\ell_2(V)$ (see Lemma 2.12 (iii)):

Lemma 3.6. *For a continuous vertex space, the exterior derivative* $\widetilde{\mathsf{d}} := \mathsf{d}\circ U^{-1}$ *transformed back to* $\ell_2(V)$ *is given by*

$$(\widetilde{\mathsf{d}}\widetilde{F})_e = p_e(\partial_+e)\widetilde{F}(\partial_+e) - p_e(\partial_-e)\widetilde{F}(\partial_-e)$$

and its adjoint $\widetilde{\mathsf{d}}^* = U\circ\mathsf{d}^*$ *by*

$$(\widetilde{\mathsf{d}}^*\eta)(v) = \frac{1}{\deg v}\sum_{e\in E_v}\frac{\overline{p_e(v)}}{\ell_e}\widehat{\eta}_e(v).$$

Switching the orientation on or off leads to another class of examples:

Lemma 3.7. *If* $\hat{\mathscr{G}}$ *is a vertex space with projection* $\hat{P}$ *and if we define the "unoriented" exterior derivative* $\hat{\mathsf{d}}$ *via*

$$\hat{\mathsf{d}}\colon \hat{\mathscr{G}} \longrightarrow \ell_2(E), \qquad (\hat{\mathsf{d}}F)_e := F_e(\partial_+e) + F_e(\partial_-e),$$

then its adjoint is given by

$$(\hat{\mathsf{d}}^*\eta)(v) = \hat{P}_v\Big(\Big\{\frac{1}{\ell}_e\,\eta_e(v)\Big\}\Big).$$

In addition, if $\mathscr{G} = \tau\hat{\mathscr{G}}$ *is the vertex space with switched orientation, then* $\mathsf{d} = \hat{\mathsf{d}}\circ\tau$ *and* $\mathsf{d}^* = \tau\circ\hat{\mathsf{d}}^*$*, i.e., the above "unoriented" exterior derivative* $\hat{\mathsf{d}}$ *occurs as an exterior derivative in the sense of Definition 3.1 for the vertex space* $\tau\mathscr{G}$ *with switched orientation.*

We give now some examples of exterior derivatives on continuous vertex spaces and their duals:

Example 3.8.

(iii) For the standard vertex space $\mathscr{G}^{\mathrm{std}}$, the exterior derivative and its adjoint are unitarily equivalent to

$$\widetilde{\mathsf{d}}\colon \ell_2(V) \longrightarrow \ell_2(E), \qquad (\widetilde{\mathsf{d}}F)_e = F(\partial_+e) - F(\partial_-e)$$

and

$$(\widetilde{\mathsf{d}}^*\eta)(v) = \frac{1}{\deg v}\sum_{e\in E_v}\frac{1}{\ell_e}\widehat{\eta}_e(v),$$

i.e., $\widetilde{\mathsf{d}}$ is the classical coboundary operator and $\widetilde{\mathsf{d}}^*$ its adjoint.

(iv) If $\widehat{\mathscr{G}^{\mathrm{std}}} = \tau\mathscr{G}^{\mathrm{std}}$ is the oriented standard vertex space, then the exterior derivative d is unitarily equivalent to

$$\widetilde{\mathsf{d}}\colon \ell_2(V) \longrightarrow \ell_2(E), \qquad (\widetilde{\mathsf{d}}F)_e = F(\partial_+e) + F(\partial_-e)$$

and

$$(\widetilde{\mathsf{d}}^*\eta)(v) = \frac{1}{\deg v}\sum_{e\in E_v}\frac{1}{\ell_e}\eta_e(v).$$

(v) For the (unoriented) sum vertex space $\mathscr{G}^{\Sigma} = (\mathscr{G}^{\mathrm{std}})^{\perp}$, we have

$$(\mathsf{d}^*\eta)_e(v) = \frac{1}{\ell}_e\,\widehat{\eta}_e(v) - \frac{1}{\deg v}\sum_{e'\in E_v}\frac{1}{\ell}_{e'}\,\widehat{\eta}_{e'}(v)$$

(vi) For the (oriented) sum vertex space $\mathscr{G}^{\widehat{\Sigma}} = (\widehat{\mathscr{G}^{\mathrm{std}}})^{\perp}$, we have

$$(\mathsf{d}^*\eta)_e(v) = \pm\Big(\frac{1}{\ell}_e\,\eta_e(v) - \frac{1}{\deg v}\sum_{e'\in E_v}\frac{1}{\ell}_{e'}\eta_{e'}\Big)$$

if $v = \partial_\pm e$.

(vii) For the magnetic vertex space $\mathscr{G}^{\mathrm{mag},\alpha}$, we have

$$\widetilde{\mathsf{d}}\colon \ell_2(V) \longrightarrow \ell_2(E), \qquad (\widetilde{\mathsf{d}}F)_e = \mathrm{e}^{-\mathrm{i}\alpha_e/2}F(\partial_+e) - \mathrm{e}^{\mathrm{i}\alpha_e/2}F(\partial_-e)$$

and

$$\widetilde{\mathsf{d}}^*\colon \ell_2(E) \longrightarrow \ell_2(V), \qquad (\widetilde{\mathsf{d}}^*\eta)(v) = \frac{1}{\deg v}\sum_{e\in E_v}\frac{1}{\ell_e}\mathrm{e}^{\mathrm{i}\widehat{\alpha}_e/2}\widehat{\eta}_e(v).$$

5.1. **Continuous metric graphs.** A (continuous) metric graph $X = (V, E, \partial, \ell)$ is formally given by the same data as a discrete (edge-)weighted graph. The difference is the interpretation of the space X: We define X as

$$X := \bigcup_{e\in E}^{\cdot}[0,\ell_e]/\sim_\psi$$

where we identify $x\sim_\psi y$ iff $\psi(x) = \psi(y)$ with

$$\psi\colon \bigcup_{e\in E}^{\cdot}\{0,\ell_e\} \longrightarrow V, \qquad 0_e\mapsto\partial_-e, \quad \ell_e\mapsto\partial_+e.$$

In the sequel, we often identify the edge e with the interval $(0,\ell_e)$ and use $x = x_e$ as coordinate. In addition, we denote $\mathrm{d}x = \mathrm{d}x_e$ the Lebesgue measure on e inducing a natural measure on X. The space X becomes a metric space by defining the distance of two points to be the length of the shortest path in X joining these points.

We first define several Hilbert spaces associated with X. Our basic Hilbert space is

$$\mathsf{L}_2(X) := \bigoplus_{e\in E}\mathsf{L}_2(e) \tag{5.1}$$

where again e is identified with $(0,\ell_e)$.

Remark 5.1. The interpretation of an edge e as a "continuous" *interval* is in contrast with the *discrete* case where e is considered as a single point, e.g. in (2.4). Another point of view is that we use different types of measures, in the discrete case a point measure and in the metric case the Lebesgue measure. This fact of choosing two different types of measures (or even combinations of them) is pointed out in the works of Friedman and Tillich (cf. [FT04a, FT04b]) and also in [BF06, BR07].

More generally, we define the *decoupled* Sobolev space of order k by

$$\mathsf{H}^k_{\max}(X) := \bigoplus_{e\in E}\mathsf{H}^k(e).$$

Obviously, for $k = 0$, there is no difference between $\mathsf{L}_2(X)$ and the decoupled space. Namely, evaluation of a function at a point only makes sense if $k\geq 1$ due to the next lemma. We need the following notation: For $f\in\mathsf{H}^1_{\max}(X)$, we denote

$$\underline{f} = \{\underline{f}(v)\}_{v\in V}, \qquad \underline{f}(v) = \{f_e(v)\}_{e\in E_v}, \qquad f_e(v) := \begin{cases} f_e(0), & v = \partial_-e\\ f_e(\ell_e), & v = \partial_+e\end{cases}$$

the *unoriented* evaluation at the vertex v. Similarly, for $g\in\mathsf{H}^1_{\max}(X)$, we denote

$$\widehat{\underline{g}} = \{\widehat{\underline{g}}(v)\}_{v\in V}, \qquad \widehat{\underline{g}}(v) = \{\widehat{g}_e(v)\}_{e\in E_v}, \qquad \widehat{g}_e(v) := \begin{cases} -g_e(0), & v = \partial_-e\\ g_e(\ell_e), & v = \partial_+e\end{cases} \tag{5.2}$$

the *oriented* evaluation at the vertex v.

The following proposition is a simple consequence of a standard estimate for Sobolev spaces:

Proposition 5.2. *Assume the condition* (3.1) *on the edge lengths, i.e., there is* $\ell_0\leq 1$ *such that* $\ell_e\geq\ell_0 > 0$ *for all* $e\in E$*. Then the evaluation maps*

$$\underline{(\cdot)}\colon \mathsf{H}^1_{\max}(X) \longrightarrow \mathscr{G}^{\max}, \quad f\mapsto\underline{f} \quad \textit{and} \quad \widehat{\underline{(\cdot)}}\colon \mathsf{H}^1_{\max}(X) \longrightarrow \mathscr{G}^{\max}, \quad g\mapsto\widehat{\underline{g}},$$

are bounded by $2/\sqrt{\ell_0}$.

Proof. By density, we can assume that f is smooth on each edge. For $e\in E_v$, let $\chi_{v,e}$ be the affine linear function with value 1 at v and 0 at the other vertex v_e. Then

$$f_e(v) = \int_e (f_e\chi_{v,e})'(x)\,\mathrm{d}x\,\widehat{\mathbb{1}}_e(v) = \int_e (f_e'\chi_{v,e})(x)\,\mathrm{d}x\,\widehat{\mathbb{1}}_e(v) + \frac{1}{\ell_e}\int_e f_e(x)\,\mathrm{d}x. \tag{5.3}$$

(iv) Of course, the theorem is a consequence of the rank theorem for matrices (the dimension of the kernel minus the dimension of the cokernel equals the dimension of the initial space minus the dimension of the target space). Nevertheless, we would like to give a prove which is more related to the geometry of the problem.

Before proving our index theorem, we use a deformation argument in order to calculate the index:

Lemma 4.8. *Let $\mathscr{G}_0$ and $\mathscr{G}_1$ be two vertex spaces with $\dim\mathscr{G}_0 = \dim\mathscr{G}_1 = n$, then $\operatorname{ind}\mathsf{D}_{\mathscr{G}_0} = \operatorname{ind}\mathsf{D}_{\mathscr{G}_1}$ for the Dirac operators associated to the vertex spaces.*

Proof. Denote P_t the associated orthogonal projections, $t \in \{0,1\}$. Note that P_0 and P_1 can be connected by a (norm-)continuous path P_t inside the space of orthogonal projections of rank n: This can be seen as follows: Let $\{\varphi_{0,k}\}_k$ and $\{\varphi_{1,k}\}_k$ be two orthonormal bases such that the first n vectors span the range of P_0 and P_1, respectively. Let U_1 be the unitary operator mapping $\varphi_{0,k}$ onto $\varphi_{1,k}$. Since the space of unitary operators is connected, we can find a (norm)-continuous path $t \mapsto U_t$ from the identity operator $U_0 := \mathbb{1}$ to U_1 such that all operators U_t are unitary. Define $P_t := U_t^* P_0 U_t$, then $t \mapsto P_t$ is a continuous path from P_0 to P_1.

Let $\varphi_{t,k} := U_t\varphi_{0,k}$, then $t \mapsto \{\varphi_{t,k}\}_k$ is a continuous family of orthogonal bases. Let D_t be the Dirac operator defined with respect to $\mathscr{G}_t = \operatorname{ran} P_t$. Passing to the basis $\{\varphi_{t,k}\}_k$ in $\mathscr{G}_t$, we may assume that the family $t \mapsto \mathsf{D}_t$ is defined on $\mathbb{C}^n \oplus \ell_2(E)$. Moreover, the family $t \mapsto \mathsf{D}_t$ is continuous, since $t \mapsto U_t$ is. The index formula follows from Proposition 1.5. □

Proof of Theorem 4.6. For each dimension $n = \dim\mathscr{G}$, we use a simple vertex space $\widetilde{\mathscr{G}}$ of dimension n. In particular, we choose the space $\widetilde{\mathscr{G}}_v := \mathbb{C}^{d_v} \oplus 0 \subset \mathbb{C}^{\deg v} = \mathscr{G}_v^{\max}$ where $d_v := \dim\mathscr{G}_v$ is the dimension of the original vertex space at v. This vertex space corresponds to d_v "Neumann" boundary conditions at the first d_v edges, and $\deg v - d_v$ "Dirichlet" boundary conditions at the remaining edges in E_v. Due to the stability of the index shown in Lemma 4.8, it suffices to calculate the index of the model vertex space $\widetilde{\mathscr{G}}$.

It is easily seen as in Example 4.5 (i)–(ii) that for a graph consisting of a single edge with two adjacent vertices, we have

$$\operatorname{ind}\mathsf{D}^{\mathrm{DD}} = -1, \qquad \operatorname{ind}\mathsf{D}^{\mathrm{DN}} = 0 \quad \text{and} \quad \operatorname{ind}\mathsf{D}^{\mathrm{NN}} = 1.$$

For example, for the mixed case, $\mathsf{d}F = F(1)$ if the Neumann space is at the vertex 1. In particular, d and d^* are both injective, so $\ker\mathsf{d} = 0$ and $\ker\mathsf{d}^* = 0$.

Due to the additivity of the index with respect to orthogonal sums, we therefore have

$$\operatorname{ind}\widetilde{\mathsf{D}} = \#\{\text{edges with N-N}\} - \#\{\text{edges with D-D}\}.$$

for the Dirac operator associated with the vertex space $\widetilde{\mathscr{G}}$. It remains to show that

$$\#\{\text{edges with N-N}\} - \#\{\text{edges with D-D}\} = \#\{\text{all N}\} - |E|. \tag{4.3}$$

In order to show this last equality, we argue by induction over the dimension n of $\widetilde{\mathscr{G}}$, i.e., the total number of Neumann conditions in $\widetilde{\mathscr{G}}$. For $n = 0$, the vertex space is the minimal or Dirichlet vertex space, for which the index formula is correct by Example 4.5 (i). For the induction step $n \to n+1$ we have to distinguish two cases:

Case A. In an existing edge with two Dirichlet or one Dirichlet and one Neumann vertex space, we replace one Dirichlet space by a Neumann one. This increases the LHS of Eq. (4.3) as well as the RHS by 1.

Case B. We add an edge with Dirichlet and Neumann vertex space to the graph. The LHS is unchanged, and in the RHS, we increase the number of Neumann conditions by 1, but subtract also one additional edge. □

5. Exterior derivatives on quantum graphs

In this section, we develop the notion of exterior derivatives on metric graphs. We first start with the definition of a metric graph, and some general results needed later.

3.2. **Discrete Dirac operators and Laplacians.** Let $\mathsf{D} = \mathsf{D}_{\mathscr{G}}$ be the Dirac operator associated to the exterior derivative $\mathsf{d} = \mathsf{d}_{\mathscr{G}}$ on the vertex space $\mathscr{G}$, i.e.,

$$\mathsf{D} = \begin{pmatrix} 0 & \mathsf{d}^* \\ \mathsf{d} & 0 \end{pmatrix} \quad \text{with respect to} \quad \ell_2(\Lambda X) := \ell_2(\Lambda^0 X) \oplus \ell_2(\Lambda^1 X) = \mathscr{G} \oplus \ell_2(E)$$

(cf. Definition 1.1).

Definition 3.9. We define as in the abstract supersymmetric setting the *Laplacians* associated to a vertex space $\mathscr{G}$ by

$$\Delta_{\Lambda X} := \Delta_{\mathscr{G}} := \mathsf{D}_{\mathscr{G}}^2, \qquad \Delta_{\Lambda^0 X} := \Delta^0_{\mathscr{G}} := \mathsf{d}^*_{\mathscr{G}}\mathsf{d}_{\mathscr{G}} \quad \text{and} \quad \Delta_{\Lambda^1 X} := \Delta^1_{\mathscr{G}} := \mathsf{d}_{\mathscr{G}}\mathsf{d}^*_{\mathscr{G}}.$$

In particular, we have

$$(\Delta^0_{\mathscr{G}} F)(v) = P_v\Big(\Big\{\frac{1}{\ell_e}\big(F_e(v) - F_e(v_e)\big)\Big\}\Big) \tag{3.3a}$$

$$(\Delta^1_{\mathscr{G}}\eta)_e = \Big(P_{\partial_+ e}\Big(\Big\{\frac{1}{\ell_{e'}}\widehat{\eta}_{e'}(\partial_+ e)\Big\}_{e'}\Big) - P_{\partial_- e}\Big(\Big\{\frac{1}{\ell_{e'}}\widehat{\eta}_{e'}(\partial_- e)\Big\}_{e'}\Big)\Big)_e \tag{3.3b}$$

where v_e denotes the vertex on $e \in E_v$ opposite to v. Here, we see that the orientation plays no role for the 0-form Laplacian.

We have a sort of Hodge decomposition (see Proposition 1.3):

Proposition 3.10. *Assume that D has a spectral gap at 0, i.e., that $\operatorname{dist}(0, \sigma(\mathsf{D}) \setminus \{0\}) > 0$ (e.g., X finite is sufficient). Then*

$$\ell_2(\Lambda X) = \ker\mathsf{D} \oplus \operatorname{ran}\mathsf{d}^* \oplus \operatorname{ran}\mathsf{d}, \qquad \textit{i.e.,}$$
$$\ell_2(\Lambda^0 X) = \mathscr{G} = \ker\mathsf{d} \oplus \operatorname{ran}\mathsf{d}^* \quad \textit{and} \quad \ell_2(\Lambda^1 X) = \ell_2(E) = \ker\mathsf{d}^* \oplus \operatorname{ran}\mathsf{d}.$$

Let us start with the Laplacians acting on the trivial vertex spaces:

Example 3.11.

(i) For the minimal vertex space, we have $\Delta^p_{\mathscr{G}_0} = 0$ for $p \in \{0,1\}$.

(ii) For the maximal vertex space, we have ($\Delta^p_{\max} := \Delta^p_{\mathscr{G}^{\max}}$)

$$(\Delta^0_{\max} F)_e(v) = \Big\{\frac{1}{\ell_e}\big(F_e(v) - F_e(v_e)\big)\Big\}_{e\in E_v}.$$

The operator $\Delta^0_{\max}$ decomposes as $\bigoplus_e (\Delta^0_{\max})_e$ with respect to the decomposition of $\mathscr{G}^{\max}$ in Eq. (2.3), where

$$\big((\Delta^0_{\max})_e \colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2\big) \cong \frac{1}{\ell_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Similarly,

$$(\Delta^1_{\max}\eta)_e = \frac{2}{\ell_e}\eta_e,$$

i.e., $\Delta^1_{\max} = 2[\ell^{-1}]$ is a multiplication operator on $\ell_2(E)$.

Proposition 3.12.

(i) *The Laplacian $\Delta^p_{\mathscr{G}}$ on p-forms associated to the vertex space $\mathscr{G}$ is a bounded operator with norm bounded by 2κ.*

(ii) *On 1-forms, we have $\Delta^1_{\max} = \Delta^1_{\mathscr{G}} + \Delta^1_{\mathscr{G}^\perp}$ or $\Delta^1_{\mathscr{G}^\perp} = 2\big[\ell^{-1}\big] - \Delta^1_{\mathscr{G}}$ on $\ell_2(E)$. In particular, if all length $\ell_e = 1$, then*

$$\Delta^1_{\mathscr{G}^\perp} = 2 - \Delta^1_{\mathscr{G}} \quad \textit{and} \quad \sigma(\Delta^1_{\mathscr{G}^\perp}) = 2 - \sigma(\Delta^1_{\mathscr{G}}),$$

i.e., $\lambda \in \sigma(\Delta^1_{\mathscr{G}^\perp})$ iff $2 - \lambda \in \sigma(\Delta^1_{\mathscr{G}})$.

(iii) *Assume that $\ell_e = 1$ then we have the spectral relation*

$$\sigma(\Delta^0_{\mathscr{G}^\perp}) \setminus \{0,2\} = 2 - (\sigma(\Delta^0_{\mathscr{G}}) \setminus \{0,2\})$$

on 0-forms, i.e., if $\lambda \neq 0, 2$, then $\lambda \in \sigma(\Delta^0_{\mathscr{G}^\perp})$ iff $2 - \lambda \in \sigma(\Delta^0_{\mathscr{G}})$.

Proof. The first assertion follows immediately from Lemma 3.3. The second is a consequence of Lemma 3.5. The last spectral equality follows from the spectral equality for 1-forms and supersymmetry to pass from 1-forms to 0-forms (cf. Proposition 1.2). □

In Proposition 4.4 we will prove a relation between the kernels, namely $\ker \Delta^0_{\mathscr{G}^\perp} \cong \ker \Delta^1_{\widehat{\mathscr{G}}}$.

Proposition 3.13. *Let $\mathscr{G}$ be a continuous vertex space, $\widetilde{\mathsf{d}}\colon \ell_2(V) \longrightarrow \ell_2(E)$ the unitarily equivalent exterior derivative as defined in Lemma 3.6 and $\widetilde{\mathsf{d}}^*$ its adjoint, then $\widetilde{\Delta}^0_{\mathscr{G}} := \widetilde{\mathsf{d}}^*\widetilde{\mathsf{d}}$ and $\Delta^1_{\mathscr{G}}$ are given by*

$$(\widetilde{\Delta}^0_{\mathscr{G}} F)(v) = \frac{1}{\deg v} \sum_{e\in E_v} \frac{\overline{p_e(v)}}{\ell_e}\big(p_e(v)F(v) - p_e(v_e)F(v_e)\big)$$

$$(\Delta^1_{\mathscr{G}}\widehat{\eta})_e = -\sum_{e\sim e'} \frac{(\overline{p}_{e'}p_e)(e\cap e')}{\ell_{e'}\deg(e\cap e')}\widehat{\eta}_{e'}(e) + \Big(\frac{|p_e(\partial_+ e)|^2}{\deg \partial_+ e} + \frac{|p_e(\partial_- e)|^2}{\deg \partial_- e}\Big)\frac{1}{\ell_e}\eta_e,$$

where $e'\sim e$ means that $e'\neq e$ and e', e have the vertex $e'\cap e$ in common. Furthermore, $\widehat{\eta}_{e'}(e) = \eta_{e'}$ if the orientation of e, e' gives an orientation of the path formed by e, e', and $\widehat{\eta}_{e'}(e) = -\eta_{e'}$ otherwise.

We have several important special cases of continuous vertex spaces and their duals:

Example 3.14.

(iii) For the standard vertex space $\mathscr{G}^{\mathrm{std}}$, we have the standard (weighted) Laplacian Δ^0_{std} transformed to $\widetilde{\Delta}^0_{\mathrm{std}} = \Delta^0_X = \Delta^0_{(X,\ell^{-1})}$ on $\ell_2(V)$ and Δ^1_{std} on $\ell_2(E)$, where

$$(\widetilde{\Delta}^0_{\mathrm{std}} F)(v) = \frac{1}{\deg v}\sum_{e\in E_v}\frac{1}{\ell_e}\big(F(v) - F(v_e)\big) \tag{3.4}$$

$$(\Delta^1_{\mathrm{std}}\eta)_e = -\sum_{e'\sim e}\frac{1}{\ell_{e'}\deg(e'\cap e)}\widehat{\eta}_{e'}(e) + \Big(\frac{1}{\deg\partial_+ e} + \frac{1}{\deg\partial_- e}\Big)\frac{1}{\ell_e}\eta_e.$$

(iv) For the oriented standard space $\mathscr{G}^{\widehat{\mathrm{std}}}$, we have

$$(\widetilde{\Delta}^0_{\widehat{\mathrm{std}}} F)(v) = \frac{1}{\deg v}\sum_{e\in E_v}\frac{1}{\ell_e}\big(F(v) + F(v_e)\big)$$

$$(\Delta^1_{\widehat{\mathrm{std}}}\eta)_e = \sum_{e'\sim e}\frac{1}{\ell_{e'}\deg(e'\cap e)}\eta_{e'} + \Big(\frac{1}{\deg\partial_+ e} + \frac{1}{\deg\partial_- e}\Big)\frac{1}{\ell_e}\eta_e.$$

Note that

$$\widetilde{\Delta}^0_{\widehat{\mathrm{std}}} = 2[L^\Sigma] - \widetilde{\Delta}^0_{\mathrm{std}}, \tag{3.5}$$

where $\widetilde{\Delta}^0_{\mathrm{std}}$ is the standard Laplacian of Example (iii) and $[L^\Sigma]$ is the multiplication operator with

$$L^\Sigma(v) := \frac{1}{\deg v}\sum_{e\in E_v}\frac{1}{\ell_e}.$$

(v) For the (unoriented) sum vertex space $\mathscr{G}^\Sigma$, the dual of $\mathscr{G}^{\mathrm{std}}$, we have

$$(\Delta^0_\Sigma F)_e(v) = \frac{1}{\ell_e}\big(F_e(v) - F_e(v_e)\big) - \frac{1}{\deg v}\sum_{e'\in E_v}\frac{1}{\ell_{e'}}\big(F_{e'}(v) - F_{e'}(v_{e'})\big)$$

$$(\Delta^1_\Sigma\eta)_e = \sum_{e'\sim e}\frac{1}{\ell_{e'}\deg(e'\cap e)}\widehat{\eta}_{e'}(e) - \Big(\frac{1}{\deg\partial_+ e} + \frac{1}{\deg\partial_- e} - 2\Big)\frac{1}{\ell_e}\eta_e.$$

(v) For the (unoriented) sum vertex space $\mathscr{G}^\Sigma$ we can apply Proposition 4.4. In particular,

$$\ker \mathsf{d}_{\mathscr{G}^\Sigma} = \Big\{ F\in \mathscr{G}^{\max} \,\Big|\, F_e(\partial_+ e) = F_e(\partial_- e) =: F_e,\ \sum_{e\in E_v} F_e = 0\Big\}$$

is isomorphic to $\ker \mathsf{d}_{\mathscr{G}^{\widehat{\mathrm{std}}}}$, i.e., $b^0_\Sigma = b^1_{\widehat{\mathrm{std}}} = |E|-|V|+1$ iff X is bipartite and $b^0_\Sigma = |E|-|V|$ otherwise. Furthermore,

$$\ker \mathsf{d}^*_{\mathscr{G}^\Sigma} = \Big\{ \eta\in\ell_2(E)\,\Big|\, \frac{1}{\ell_e}\widehat{\eta}_e(v) \text{ is independent of } e\in E_v \text{ for } v\in V\Big\}$$

is a sort of "oriented" continuity condition. In particular, $b^1_\Sigma = b^0_{\widehat{\mathrm{std}}} = 1$ iff X is bipartite and $b^1_\Sigma = 0$ otherwise. Finally,

$$\operatorname{ind}\mathsf{D}_{\mathscr{G}^\Sigma} = -\chi_{\mathrm{std}}(X) = |E| - |V|.$$

(vi) The oriented sum vertex space $\mathscr{G}^{\widehat{\Sigma}}$ is related to the standard vertex space, i.e, $b^0_{\widehat{\Sigma}} = |E| - |V| + 1$ and $b^1_{\widehat{\Sigma}} = 1$ by Proposition 4.4. In particular,

$$\operatorname{ind}\mathsf{D}_{\mathscr{G}^{\widehat{\Sigma}}} = -\chi_{\mathrm{std}}(X) = |E| - |V|.$$

(vii) Assume that X is finite. For the magnetic vertex space $\mathscr{G}^{\mathrm{mag},\alpha}$, we need to define the flux through a circuit c: If $c = \sum_{i=1}^n p_i e_i \in H_1(X,\mathbb{Z})$ is a cycle represented in the homology group, then we define the *flux* of α through c as

$$c\cdot\alpha := \sum_{i=1}^n p_i\widehat{\alpha}_e(e_i),$$

where $\widehat{\alpha}_e(e_i)$ is defined in Proposition 3.13. Now, $b^0_{\mathrm{mag},\alpha} = 1$ iff $c\cdot\alpha\in 2\pi\mathbb{Z}$ for all cycles $c\in H_1(X,\mathbb{Z})$ and $b^0_{\mathrm{mag},\alpha} = 0$ otherwise. In order to calculate $b^1_{\mathrm{mag},\alpha}$, we note that the linear system $(\widetilde{\mathsf{d}}^*_{\mathrm{mag},\alpha}\eta)(v) = 0$ for all $v\in V$ consists of $|E|$ variables η_e and $|V|$ equations, therefore $b^1_{\mathrm{mag},\alpha}\geq |E|-|V|$. It remains to show that the rank of the coefficient matrix can increase by 1 iff $c\cdot\alpha\in 2\pi\mathbb{Z}$ for all $c\in H_1(X,\mathbb{Z})$, i.e., $b^1_{\mathrm{mag},\alpha} = |E|-|V|+1$ in this case and $b^1_{\mathrm{mag},\alpha} = |E|-|V|$ otherwise. We do not give a formal proof of this fact here, since the result follows by abstract arguments of the next theorem. Indeed, we have

$$\operatorname{ind}\mathsf{D}_{\mathscr{G}^{\mathrm{mag},\alpha}} = \chi_{\mathrm{std}}(X) = |V| - |E|.$$

These examples suggest the following theorem:

Theorem 4.6. *Assume that X is a finite graph, i.e., that $|E|<\infty$, with vertex space $\mathscr{G}$. Then the index of the Dirac operator D associated to the exterior derivative d as defined in Definition 3.1 is given by*

$$\operatorname{ind}\mathsf{D} = \dim\mathscr{G} - |E|.$$

Remark 4.7.

(i) We can interpret the above theorem as a discrete "Gauß-Bonnet"-theorem for general vertex spaces, namely

$$\chi_{\mathscr{G}}(X) = \sum_{v\in V}\kappa_{\mathscr{G}}(v) \tag{4.2}$$

using Eq. (2.2), where $\chi_{\mathscr{G}}(X)$ is defined in Definition 4.1 and $\kappa_{\mathscr{G}}(v)$ in Definition 4.2.

(ii) The index $\operatorname{ind}\mathsf{D}$ gives at least some simple information on the vertex space, namely $\mathscr{G}$ is trivial (i.e., $\mathscr{G}$ is the maximal or minimal vertex space $\mathscr{G}^{\max}$ or $\mathscr{G}^0 = 0$) iff $\operatorname{ind}\mathsf{D} = \pm|E|$. This follows from Eq. (2.2) and Example 4.5 (i)–(ii).

(iii) For continuous vertex spaces $\mathscr{G}$, we obtain the classical case where $\dim\mathscr{G} = |V|$, i.e., the classical discrete Gauß-Bonnet formula Eq. (1.6).

Before calculating the kernel of the Dirac operator in some examples, we establish a general result on the dual $\mathscr{G}^\perp$ of a vertex space $\mathscr{G}$. It shows that actually, $\mathscr{G}^\perp$ and the *oriented* version of $\mathscr{G}$ are related:

Proposition 4.4. *Assume that the global length bound*

$$\ell_0 \le \ell_e \le \ell_+ \qquad \text{for all } e \in E \tag{4.1}$$

holds for some constants $0 < \ell_0 \le \ell_+ < \infty$. Then

$$H^0_{\mathscr{G}^\perp}(X,\mathbb{C}) = \ker \mathsf{d}_{\mathscr{G}^\perp} \cong \ker \mathsf{d}^*_{\widehat{\mathscr{G}}} = H^1_{\widehat{\mathscr{G}}}(X,\mathbb{C})$$

are isomorphic. In particular, if X is finite, then

$$b^0_{\mathscr{G}^\perp}(X) = b^1_{\widehat{\mathscr{G}}}(X), \qquad b^1_{\mathscr{G}^\perp}(X) = b^0_{\widehat{\mathscr{G}}}(X) \quad \text{and} \quad \chi_{\mathscr{G}^\perp}(X) = -\chi_{\widehat{\mathscr{G}}}(X).$$

If in addition, D has a spectral gap at 0, then $\operatorname{ran} \mathsf{d}^*_{\mathscr{G}^\perp} \cong \operatorname{ran} \mathsf{d}_{\widehat{\mathscr{G}^\perp}}$.

Proof. We define $\psi\colon \ell_2(E) \longrightarrow \mathscr{G}^\perp$ via $(\psi\eta)(v) := P^\perp_v\{\frac{1}{\ell_e}\eta\}$, then

$$\|\psi\eta\|^2_{\mathscr{G}^{\max}} = \sum_{v\in V}\sum_{e\in E_v}\frac{1}{\ell_e^2}|\eta_e|^2 = 2\sum_{e\in E}\frac{1}{\ell_e^2}|\eta_e|^2 \le \frac{2}{\ell_0}\|\eta\|^2_{\ell_2(E)},$$

and therefore $\psi\eta \in \mathscr{G}^\perp$. Furthermore, $\psi(\ker \mathsf{d}^*_{\widehat{\mathscr{G}}}) \subset \ker \mathsf{d}_{\mathscr{G}^\perp}$ since $\mathsf{d}^*_{\widehat{\mathscr{G}}}\eta = 0$ implies that $P_v\{\frac{1}{\ell_e}\eta_e\} = 0$ and therefore, $(\psi\eta)_e(v) = \frac{1}{\ell_e}\eta_e$. In particular, the latter expression is independent of $\partial_\pm e$, so that $\mathsf{d}_{\mathscr{G}^\perp}(\psi\eta) = 0$. The other inclusion can be shown similarly: Let $F \in \ker \mathsf{d}_{\mathscr{G}^\perp}$ and set $\eta_e := \ell_e F_e(v)$ independent of $v = \partial_\pm e$. Then $\eta \in \ell_2(E)$ using the global *upper* bound $\ell_e \le \ell_+$. Furthermore, $(\psi\eta)(v) = P^\perp \underline{F}(v)) = \underline{F}(v)$ and $\mathsf{d}^*_{\widehat{\mathscr{G}}}\eta = \widehat{\mathbb{1}}PF = 0$ since $F \in \mathscr{G}^\perp$. The other assertions follow from the definitions and the fact that $\perp$ and $\widehat{\cdot}$ are involutions. The isomorphism of the ranges follows from Proposition 3.10. □

When writing the index of D, we implicitly assume that the graph is finite, i.e., that $|E| < \infty$. We calculate the cohomology for the list of our examples. For simplicity, we assume that X is finite and connected. In particular, the global length bound (4.1), i.e., $0 < \ell_0 \le \ell_e \le \ell_+ < \infty$ is fulfilled.

Example 4.5.

(i) For the minimal vertex space, we have $\ker \mathsf{d}_{\mathscr{G}^0} = 0$ and $\ker \mathsf{d}^*_{\mathscr{G}^0} = \ell_2(E)$. In particular, $\operatorname{ind} \mathsf{D}_{\mathscr{G}^0} = -|E|$.

(ii) For the maximal vertex space, we see from Proposition 4.4 that $\ker \mathsf{d}_{\mathscr{G}^{\max}} \cong \ker \mathsf{d}^*_{\mathscr{G}^0} = \ell_2(E)$ and $\ker \mathsf{d}^*_{\mathscr{G}^{\max}} \cong \ker \mathsf{d}_{\mathscr{G}^0} = 0$. In particular, $\operatorname{ind} \mathsf{D} = |E|$.

(iii) For the standard vertex space $\mathscr{G}^{\mathrm{std}}$ we obtain the classical homology groups $H^p(X,\mathbb{C})$. The 0-th Betti-number counts the number of components, i.e., $b^0_{\mathrm{std}}(X) = 1$, and the 1-st the number of prime cycles. It is a classical fact that $b^1_{\mathrm{std}}(X) = |P(E)| = |E| - |V| + 1$ and therefore

$$\operatorname{ind} \mathsf{D}_{\mathscr{G}^{\mathrm{std}}} = b^0_{\mathrm{std}}(X) - b^1_{\mathrm{std}}(X) = |V| - |E| = \chi_{\mathrm{std}}(X)$$

(iv) For the oriented standard vertex space $\mathscr{G}^{\widehat{\mathrm{std}}}$, the 0-th Betti number counts the number of bipartite components of X, i.e., if X is connected, then $b^0_{\widehat{\mathrm{std}}} = 1$ if X is bipartite and 0 otherwise. This can be seen using the characterisation that $\widetilde{X}$ is bipartite iff X contains no odd cycle. Note that $(\widetilde{\mathsf{d}}\widetilde{F})_e = 0$ for each edge e in an odd cycle c implies that $\widetilde{F}$ vanishes on each vertex in c.

The 1-st Betti number counts the number of prime cycles $|P(E)|$, where one has to subtract 1 if there is an odd prime cycle. But the existence of an odd (prime) cycle is equivalent to the fact that X is not bipartite. In particular,$b^1_{\widehat{\mathrm{std}}} = |E| - |V| + 1$ if X is bipartite and $b^1_{\widehat{\mathrm{std}}} = |E| - |V|$ otherwise. Again, we have $\operatorname{ind} \mathsf{D}_{\mathscr{G}^{\widehat{\mathrm{std}}}} = |V| - |E|$.

(vi) For the oriented sum vertex space $\mathscr{G}^{\widehat{\Sigma}}$, we have

$$(\Delta^0_{\widehat{\Sigma}}F)_e(v) = \frac{1}{\ell_e}\big(F_e(v) - F_e(v_e)\big) - \frac{\widehat{\mathbb{1}}_e(v)}{\deg v}\sum_{e'\in E_v}\frac{\widehat{\mathbb{1}}_{e'}(v)}{\ell_{e'}}\big(F_{e'}(v) - F_{e'}(v_{e'})\big)$$

$$(\Delta^1_{\widehat{\Sigma}}\eta)_e = -\sum_{e'\sim e}\frac{1}{\ell_{e'}\deg(e'\cap e)}\eta_{e'} - \Big(\frac{1}{\deg \partial_+ e} + \frac{1}{\deg \partial_- e} - 2\Big)\frac{1}{\ell_e}\eta_e.$$

(vii) For the magnetic vertex space $\mathscr{G}^{\mathrm{mag},\alpha}$, we have

$$(\widetilde{\Delta}^0_{\mathrm{mag},\alpha}F)(v) = \frac{1}{\deg v}\sum_{e\in E_v}\frac{1}{\ell_e}\big(F(v) - \mathrm{e}^{\mathrm{i}\widehat{\alpha}_e(v)}F(v_e)\big)$$

$$(\Delta^1_{\mathrm{mag},\alpha}\eta)_e = -\sum_{e'\sim e}\frac{\mathrm{e}^{\mathrm{i}\widehat{\alpha}_{e',e}}}{\ell_{e'}\deg(e'\cap e)}\widehat{\eta}_{e'}(e) + \Big(\frac{1}{\deg \partial_+ e} + \frac{1}{\deg \partial_- e}\Big)\frac{1}{\ell_e}\eta_e,$$

where $\widehat{\alpha}_{e',e} := (\widehat{\alpha}_{e'} - \widehat{\alpha}_e)(e\cap e')$ denotes the oriented flux along e' and e.

3.3. **Magnetic operators, line graphs and subdivision graphs.** Let us briefly interpret certain above defined abstract Laplacians as well-known objects in other situations.

3.3.1. *Magnetic operators.* For a real-valued 1-form $\beta \in \mathbb{R}^E$ we define unitary operators U_β on $\mathscr{G}^{\max}$ and $\ell_2(E)$ by $(U_\beta F)_e(v) = \mathrm{e}^{\mathrm{i}\beta_e}F_e(v)$ and $(U_\beta\eta)_e = \mathrm{e}^{\mathrm{i}\beta_e}\eta_e$, respectively. Recall that a magnetic vertex space is given by $\mathscr{G}^{\mathrm{mag},\alpha}_v = \mathbb{C}\underline{p}(v)$ with $p_e(v) = \mathrm{e}^{-\mathrm{i}\widehat{\alpha}_e(v)/2}$ for a magnetic vector potential $\alpha \in \mathbb{R}^E$ (see Example 2.10 (vii)). Note that any vertex space $\hat{\mathscr{G}}_v = \underline{\hat{p}}(v)$ with $|\hat{p}_e(v)| = 1$ arises in this form:

Proposition 3.15. *Let $\hat{\mathscr{G}} = \bigoplus_v \mathbb{C}\underline{\hat{p}}(v)$ with $\hat{p}_e(v) = \mathrm{e}^{-\mathrm{i}\hat{A}_e(v)}$ for some $\hat{A} = \{\underline{\hat{A}}(v)\}$ with $\underline{\hat{A}}(v) \in \mathbb{R}^{E_v}$. Then there exists a 1-form $\beta \in \mathbb{R}^E$ given by $\beta_e = -(\hat{A}_e(\partial_+e) + \hat{A}_e(\partial_-e))/2$ such that U_β is a unitary operator from $\hat{\mathscr{G}}$ onto $\mathscr{G}^{\mathrm{mag},\alpha}$ for the vector potential $\alpha := \mathsf{d}\hat{A} \in \mathbb{R}^E$.*

Moreover, U_β intertwines the exterior derivative on $\mathscr{G}^{\mathrm{mag},\alpha}$ and $\hat{\mathscr{G}}$, i.e.,

$$\mathsf{d}_{\mathscr{G}^{\mathrm{mag},\alpha}}U_\beta = U_\beta \mathsf{d}_{\hat{\mathscr{G}}}.$$

In particular, the corresponding Laplacians and Dirac operators for $\mathscr{G}^{\mathrm{mag},\alpha}$ and $\hat{\mathscr{G}}$ are unitary equivalent.

Proof. By definition of $\mathsf{d} = \mathsf{d}^{\max}$, we have

$$\alpha_e = \hat{A}_e(\partial_+e) - \hat{A}_e(\partial_-e),$$

where $\mathsf{d} = \mathsf{d}^{\max}$. Let $A_e(v) := \widehat{\alpha}_e(v)/2$, then $\mathsf{d}A = \mathsf{d}\hat{A}$, i.e., $A - \hat{A} \in \ker \mathsf{d}$. But the kernel of d consists of the values B such that $B_e(\partial_+e) = B_e(\partial_-e) =: \beta_e$ for all $e \in E$ where $\beta \in \mathbb{R}^E$ is given as above; in particular,

$$A_e(v) = \hat{A}_e(v) + \beta_e.$$

It is now easy to see that $F \in \mathscr{G}^{\mathrm{mag},\alpha}$ iff $U_\beta F \in \hat{\mathscr{G}}$. In particular, U_β maps $\hat{\mathscr{G}}$ unitarily onto $\mathscr{G}^{\mathrm{mag},\alpha}$. The intertwining property and therefore the unitary equivalences are obvious. □

Similarly, we can prove the following result on gauge invariance. Let $\Lambda \in \mathbb{R}^V$ be a function and $\alpha \in \mathbb{R}^E$ a vector potential. We define a new vector potential by $\hat{\alpha} := \alpha + \widetilde{\mathsf{d}}\Lambda$.

Proposition 3.16. *There exists a 1-form $\beta \in \mathbb{R}^E$ given by $\beta_e = (\Lambda(\partial_+e) + \Lambda(\partial_-e))/2$ such that U_β is a unitary operator from $\mathscr{G}^{\mathrm{mag},\alpha}$ onto $\mathscr{G}^{\mathrm{mag},\hat{\alpha}}$*

Moreover, U_β intertwines the exterior derivative on $\mathscr{G}^{\mathrm{mag},\alpha}$ and $\mathscr{G}^{\mathrm{mag},\hat{\alpha}}$, i.e.,

$$\mathsf{d}_{\mathscr{G}^{\mathrm{mag},\hat{\alpha}}}U_\beta = U_\beta \mathsf{d}_{\mathscr{G}^{\mathrm{mag},\alpha}}.$$

In particular, the corresponding Laplacians and Dirac operators for $\mathscr{G}^{\mathrm{mag},\alpha}$ and $\mathscr{G}^{\mathrm{mag},\hat{\alpha}}$ are unitary equivalent.

Proof. The only point to be shown here is that U_β maps $\mathscr{G}^{\mathrm{mag},\alpha}$ onto $\mathscr{G}^{\mathrm{mag},\hat{\alpha}}$. But this follows from the equality $\beta_e - \widehat{\alpha}_e/2 = \Lambda(v) - \widehat{\hat{\alpha}}_e(v)/2$. □

3.3.2. *Line graphs.* The 1-form Laplacian of Proposition 3.13 and especially of Example 3.14 (iv) above can be viewed as an operator on the line graph. In order to define the line graph, we assume for simplicity, that X has no self-loops and multiple edges, and that no edge is isolated (i.e., $\deg \partial_+ e$ and $\deg \partial_- e$ are not both equal to 1). Let $L(X)$ be the line graph associated to the graph X, i.e, $V(L(X)) = E(X)$ and two "vertices" in the line graph (i.e., edges in the original graph) e, e' are adjacent iff $e \neq e'$ and $e \cap e' \neq \emptyset$, i.e., if they meet in a common vertex. We have

$$\deg_{L(X)} e = \deg_X \partial_+ e + \deg_X \partial_- e - 2,$$

and in particular, if X is a d-regular graph, then $L(X)$ is $(2d-2)$-regular.

The above example of the 1-form Laplacian is a line graph Laplacian up to a multiplication operator with the complex edge "weight"

$$\rho_{e,e'} = \frac{(\overline{p}_{e'} p_e)(e \cap e')\, \widehat{\mathbb{1}}_{e'}(e) \deg_{L(X)} e}{\ell_{e'} \deg(e \cap e')}.$$

We will now show how $\Delta_{L(X)}$ becomes a Laplacian with positive weights.

If $\ell_e = 1$ for all edges and $p_e(v) = 1$, then the 1-form Laplacian is related to the 0-form Laplacian $\Delta^0_{(L(X),\rho)}$ on the line graph with edge weights

$$\rho_{e,e'} = \frac{\deg_{L(X)} e}{\deg_X(e \cap e')} = \frac{\deg_X \partial_+ e + \deg_X \partial_- e - 2}{\deg_X(e \cap e')}$$

via

$$\Delta^1_{\widehat{\text{std}}} = [L] - \Delta^0_{(L(X),\rho)}$$

where $[L]$ is the multiplication operator on $\ell_2(V(L(X)))$ with the function

$$L(e) = \Big(\frac{1}{\deg \partial_+ e} + \frac{1}{\deg \partial_- e}\Big) + \sum_{e' \sim e} \frac{1}{\deg_X(e \cap e')}.$$

In particular, if X is d-regular, then $L = 2$. Moreover, $\rho_{e,e'} = (2d-2)/d$ and

$$\Delta^1_{\widehat{\text{std}}} = 2 - \frac{(2d-2)}{d} \Delta^0_{L(X)}. \tag{3.6}$$

where now, $\Delta^0_{L(X)} = \Delta^0_{(L(X),1)}$ is the line graph Laplacian with edge weights set to 1. In addition, we can recover a result of [Shi00, Ogu02][3], namely a spectral relation for the line graph Laplacian and the Laplacian on the graph itself,

$$\sigma(\Delta^0_{L(X)}) \setminus \Big\{\frac{d}{d-1}\Big\} = \frac{d}{2(d-1)}\big(\sigma(\Delta^0_X) \setminus \{2\}\big),$$

using supersymmetry, (3.5) and (3.6). In particular, the spectrum of the line graph is always contained in the interval $[0, d/(d-1)]$ and is therefore not bipartite (if $d \geq 3$).

3.3.3. *Subdivision graphs.* There is another interesting example which relates a Dirac operator on X to the (standard) Laplacian on the *subdivision graph* $S(X)$ defined as follows (cf. [Shi00, Ogu02]). Again, we assume for simplicity, that X has no self-loops and no double edges and that $\ell_e = 1$. As vertices we set $V(S(X)) = V(X) \cup E(X)$, and the edges are given by $\{v, e\}$ if $v \in \partial e$ in the original graph (we do not care about the orientation here). In other words, $S(X)$ is obtained from X by introducing a new vertex on each edge. The subdivision graph $S(X)$ is always bipartite (choose the above decomposition). If X is d-regular, then $S(X)$ is $(d,2)$-semiregular, i.e., $\deg v = d$ for vertices in $v \in V(X) \subset V(S(X))$ and $\deg e = 2$ for vertices $e \in E(X) \subset V(S(X))$ with respect to the bipartite decomposition.

[3]Shirai and Ogurisu actually showed more: If X is infinite and $d \geq 3$, then $d/(d-1)$ is contained in the spectrum of the line graph, being an eigenvalue of infinite multiplicity. A corresponding eigenfunction lies in $\ker \Delta^1_{\widehat{\text{std}}} = \ker \mathsf{d}^*_{\widehat{\text{std}}}$ by (3.6). For an infinite regular graph, one can see that this space is infinite-dimensional (see also Example 4.5 (iv)).

The standard Laplacian on $S(X)$ is given by

$$(\Delta^0_{S(X)})H(e) = -\frac{1}{2}\big(H(\partial_+ e) + H(\partial_- e)\big) + H(e)$$
$$(\Delta^0_{S(X)})H(v) = -\frac{1}{d}\sum_{e \in E_v} H(e) + H(v)$$

for $H \in \ell_2(V(S(X)))$. In particular,

$$\Delta^0_{S(X)} \cong \begin{pmatrix} 1 & -\widetilde{\mathsf{d}}^*_{\widehat{\text{std}}} \\ -\frac{1}{2}\widetilde{\mathsf{d}}_{\widehat{\text{std}}} & 1 \end{pmatrix} = \mathbb{1} - \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \widetilde{\mathsf{D}}_{\widehat{\text{std}}}$$

where $\widetilde{\mathsf{d}}_{\widehat{\text{std}}}$ is the (transformed) exterior derivative and $\widetilde{\mathsf{D}}_{\widehat{\text{std}}}$ the Dirac operator associated to the oriented standard space (cf. Example 3.8 (iv)). Furthermore,

$$\big(\Delta^0_{S(X)} - 1\big)^2 \cong \frac{1}{2}\begin{pmatrix} \widetilde{\Delta}^0_{\mathscr{G}^{\widehat{\text{std}}}} & 0 \\ 0 & \Delta^1_{\mathscr{G}^{\widehat{\text{std}}}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \widetilde{\Delta}^0_{\mathscr{G}^{\widehat{\text{std}}}} & 0 \\ 0 & \Delta^1_{\mathscr{G}^{\widehat{\text{std}}}} \end{pmatrix}$$

and (for $\lambda \neq 1$) we have $\lambda \in \sigma(\Delta^0_{S(X)})$ iff $2(\lambda-1)^2 \in \sigma(\widetilde{\Delta}^0_{\mathscr{G}^{\widehat{\text{std}}}})$ by supersymmetry. But the latter operator equals $2 - \Delta^0_X$ by Eq. (3.5) since we assumed $\ell_e = 1$. Therefore

$$\sigma(\Delta^0_{S(X)}) \setminus \{1\} = \eta^{-1}\big(\sigma(\Delta^0_X) \setminus \{2\}\big)$$

where $\eta(\lambda) = 2 - 2(\lambda-1)^2 = 2\lambda(2-\lambda)$. One can show that 1 is also an eigenvalue with infinite multiplicity of the subdivision graph Laplacian. In particular, we recover again a result of [Shi00].

4. Indices for Dirac operators on discrete graphs

We start this section with a short excursion into cohomology (for the standard vertex space and its oriented version see also [LP08]). Assume that $\mathsf{d} = \mathsf{d}_{\mathscr{G}} \colon \mathscr{G} \longrightarrow \ell_2(E)$ is an exterior derivative for the vertex space $\mathscr{G}$.

Definition 4.1. We define the *(ℓ_2-)cohomology* of the graph X associated to the vertex space $\mathscr{G}$ as

$$H^0_{\mathscr{G}}(X, \mathbb{C}) := \ker \mathsf{d} \quad \text{and} \quad H^1_{\mathscr{G}}(X, \mathbb{C}) := \ker \mathsf{d}^* = (\operatorname{ran} \mathsf{d})^\perp.$$

We call

$$b^p_{\mathscr{G}}(X) := \dim_{\mathbb{C}} H^p_{\mathscr{G}}(X, \mathbb{C}) \quad \text{and} \quad \chi_{\mathscr{G}}(X) := b^0_{\mathscr{G}}(X) - b^1_{\mathscr{G}}(X)$$

the *p-th Betti-number* and *Euler characteristic* associated to the vertex space $\mathscr{G}$, respectively.

From the definition, it follows that

$$\chi_{\mathscr{G}}(X) = \operatorname{ind} \mathsf{D}_{\mathscr{G}}.$$

In order to derive a sort of "Gauß-Bonnet"-theorem, we need the notion of curvature at a vertex for general vertex spaces:

Definition 4.2. We define the *curvature* of the vertex space $\mathscr{G}_v$ at the vertex $v \in V$ as

$$\kappa_{\mathscr{G}}(v) := \dim \mathscr{G}_v - \frac{1}{2} \deg v.$$

The reason for the name will become clear in Remark 4.7 (i). Note that there are other notions of curvature, especially for tessellations (see e.g. [BP01]).

In order to calculate the Betti-numbers for a vertex space $\mathscr{G}$, we need some more notation. For simplicity, we assume that X is connected. Let X' be a *spanning tree* of X, i.e, X' is simply connected and $V(X') = V(X)$. For each $e \in P(X) := E(X) \setminus E(X')$, there exists a unique cycle c_e (closed path without repetitions) in X containing e.

Definition 4.3. A *prime cycle* is a cycle c_e for some $e \in P(X)$ associated to a spanning tree X' of X as above. A cycle c is said to be *even/odd* if the number of edges in c is even/odd.