

# SPINORIAL FIELD EQUATIONS IN SUPERGRAVITY WITH FLUXES

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In the seventies of the last century A. Gray started to consider non-integrable special Riemannian geometries of small dimensions ( $n \leq 8$ ). Some years later – in the second half of the eighties – these structures played a crucial role in the investigations of T. Friedrich et al. concerning eigenvalue estimates for the Dirac operator on a Riemannian manifold.

The interest in non-integrable geometries emerged once again from the developments of theoretical physics in the common sector of type II string theory. Around the end of the nineties T. Friedrich and collaborators developed a new and systematic approach to non-integrable geometries which takes up certain aspects of string theory straightforwardly. This approach leads naturally to the notion of characteristic connection  $\nabla^c$ , an affine connection with totally skew-symmetric torsion  $T^c$ , i.e. a 3-form, which can be associated to certain G-structures in a well-defined way. This point of view grew to show that structures with parallel torsion ( $\nabla^c T^c = 0$ ) are of particular interest. This restricted type clearly carries more information about the properties of these structures and provides a basis to solve Strominger’s equations in a natural manner (see [Str86] and [FI02]). We recommend the article [Agr06] as a compendium of these developments.

Since the turn of the century a more general system of spinorial field equations than those described by Strominger has become a matter of particular interest in type II string theory. These models of supergravity – the so-called supergravity models with fluxes – can be described geometrically by a tuple  $(M^n, g, T, F, \Psi)$  consisting of a Riemannian spin manifold  $(M^n, g)$ , a 3-form  $T$ , a 4-form  $F$  and a spinor field  $\Psi$ . The link between these objects are the equations (see [Duf02])

$$(N) \quad \nabla_X^g \Psi + \frac{1}{4} (X \lrcorner T) \cdot \Psi + p (X \lrcorner F) \cdot \Psi + q (X \wedge F) \cdot \Psi = 0,$$

$$\text{Ric}_{ij}^g - \frac{1}{4} T_{imn} T_{jmn} = 0, \quad \delta(T) = 0$$

where  $\nabla^g$  denotes the Levi-Civita connection of  $(M^n, g)$  and  $p, q \in \mathbb{R}$  are real parameters. The first of these three equations – the so-called Killing equation – should be satisfied for any vector field  $X \in TM$ . If one introduces the new covariant spinorial derivative

$$\nabla_X \Psi := \nabla_X^g \Psi + \frac{1}{4} (X \lrcorner T) \cdot \Psi + p (X \lrcorner F) \cdot \Psi + q (X \wedge F) \cdot \Psi,$$

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then the Killing equation takes the particularly simple form  $\nabla\Psi = 0$ . Considering the Kaluza-Klein reduction of  $\mathcal{M}$ -theory (see [WNW85], [Ali01], [BDS02] and [BJ03]) the relevant dimension for  $M^n$  lies between 4 and 8 ( $4 \leq n \leq 8$ ).

Moreover, additional algebraic constraints occur, for example the algebraic coupling between the 3-form or the 4-form and the spinor field  $\Psi$

$$(\aleph\aleph) \quad T \cdot \Psi = \lambda \cdot \Psi, \quad F \cdot \Psi = \kappa \cdot \Psi, \quad \lambda, \kappa \in \mathbb{C}.$$

Obviously, the system  $(\aleph)$ ,  $(\aleph\aleph)$  generalizes Strominger's model by introducing the new degree of freedom given by a 4-form  $F$ , usually called flux form.

The motivation of the present work is the construction of solutions to the system  $(\aleph)$ ,  $(\aleph\aleph)$ . At first sight, some aspects to that aim are unclear:

- Which is the correct way to obtain solutions to the entire system?
- How can the high degree of freedom in the choice of the differential forms  $T$  and  $F$  be controlled?

The starting point in order to answer the first question is to consider the Killing equation,  $\nabla\Psi = 0$ . Solving this equation is the central task of this work and will be treated in the following chapters in a systematic way. At the end of this summary we will review the solutions obtained in the light of the entire system  $(\aleph)$ ,  $(\aleph\aleph)$ .

A direct construction of solutions to  $\nabla\Psi = 0$  on an arbitrary Riemannian spin manifold is not to be expected, as suggested by the discussion of the case  $F = 0$  of [AFNP05]. Instead we study a certain class of non-integrable G-structures with characteristic connection  $\nabla^c$  and parallel torsion  $T^c$ . We will introduce these structures and their properties in the first section of chapter 1. This approach has the advantage that many geometric properties are known already due to the parallelism of  $T^c$ . In this case a natural ansatz for the 3-form  $T$  is to require that it is fixed up to a real parameter:  $T \sim T^c$ . Unfortunately no analogue approach to geometric structures is known that determines a 4-form intrinsically, to the effect that – at least in principle –  $F$  is completely arbitrary. To overcome this problem we shall make a special ansatz, and furthermore demand  $F$  to be parallel with respect to  $\nabla^c$ .

We thus take a class  $(M^n, g, \nabla^c T^c = 0)$  of non-integrable geometric structures with parallel torsion  $T^c$ , fix  $\nabla^c$ -parallel 4-forms  $F_i$  and make the following ansatz for  $T$  and  $F$ :

$$F = \sum_i A_i \cdot F_i, \quad T = B \cdot T^c, \quad A_i, B \in \mathbb{R}.$$

With these we try to solve  $\nabla\Psi = 0$  on the corresponding geometric structure. Which are then the classes of geometric structures dealt with in this work?

Our treatise includes the dimensions 5, 6 and 7 in chapters 2, 3 and 4, respectively. In dimension 5 we investigate quasi-Sasakian structures. As far as dimension 6 is concerned we choose almost Hermitian structures with parallel torsion,  $\nabla^c T^c = 0$ . These were classified by N. Schömann (see [Sch06]). We say that six-dimensional almost Hermitian structures with parallel torsion and  $\text{Hol}(\nabla^c) \subset \text{G} \subset \text{Iso}(T^c)$  belong to the class  $\mathcal{C}[\text{G}]$ . Here  $\text{Hol}(\nabla^c)$  is the holonomy group of the characteristic connection and  $\text{Iso}(T^c)$  the connected component of the identity of the isotropy group of  $T^c$ . In dimension 7 we solve  $\nabla\Psi = 0$  both on  $\alpha$ -Sasakian structures and cocalibrated  $\text{G}_2$ -structures with parallel torsion, the latter fully described in the work of Friedrich (see [Fri06]).

The cocalibrated, non-nearly parallel  $G_2$ -structures with parallel torsion, for which the holonomy algebra  $\mathfrak{hol}(\nabla^c)$  of the characteristic connection equals  $\mathfrak{g}$ , are said to belong to the class  $\mathcal{C}[\mathfrak{g}]$ .

To solve  $\nabla\Psi = 0$  in terms of the previous set-up we proceed in two steps:

- (1) We ‘classify’ solutions, i.e. determine necessary conditions for the solvability of  $\nabla\Psi = 0$ .
- (2) We construct solutions, with the help of (1).

Step one is described in detail in the second section of chapter 1, where we discuss how the kernel of the endomorphism

$$\sum_k e_k \cdot R^\nabla(e_k, X)$$

can be used to classify solutions. We will refer to this endomorphism of spinors as the ‘first contraction’ of the curvature tensor  $R^\nabla$  with respect to  $\nabla$ . It can be computed algebraically, provided we know the Ricci tensor  $\text{Ric}^c$  of the characteristic connection, and if  $T$  and  $F$  are parallel with respect to  $\nabla^c$ .

As typical examples of the results obtained via the above two-step procedure, we mention the following, taken from chapters 3 and 4:

**Theorem 1.** *Let  $G$  be a connected, non-abelian subgroup of  $U(3)$  that stabilizes a non-trivial 3-form  $T^c \in \Lambda^3(\mathbb{R}^6)$  and  $(M^6, g, J)$  a six-dimensional almost Hermitian spin manifold of class  $\mathcal{C}[G]$  with characteristic connection  $\nabla^c$ , characteristic torsion  $T^c$ , 4-form  $F = A \cdot *\Omega \neq 0$ , 3-form  $T = B \cdot T^c$  and covariant spinorial derivative*

$$\nabla_X^0 \Psi = \nabla_X^g \Psi + \frac{1}{4} (X \lrcorner T) \cdot \Psi + \frac{1}{2} (X \lrcorner F) \cdot \Psi + (X \wedge F) \cdot \Psi.$$

*Then there exists one  $\nabla^0$ -parallel spinor field  $\Psi_0$  if and only if the following conditions are satisfied:*

- (1) *The spinor field  $\Psi_0$  is parallel with respect to the characteristic connection  $\nabla^c$ .*
- (2) *The spinor field  $\Psi_0$  satisfies  $*\Omega \cdot \Psi = -3 \cdot \Psi$ .*
- (3) *The 3-form  $T$  coincides with the characteristic torsion,  $T = T^c$ .*

**Theorem 2.** *Let  $\mathfrak{g}$  be a proper, non-abelian subalgebra of  $\mathfrak{g}_2$  and  $(M^7, g, \omega^3)$  a seven-dimensional, cocalibrated  $G_2$ -manifold of class  $\mathcal{C}[\mathfrak{g}]$  with characteristic connection  $\nabla^c$ , characteristic torsion  $T^c$ , 4-form  $F = A \cdot *\omega^3 \neq 0$ , 3-form  $T = B \cdot T^c$  and covariant spinorial derivative*

$$\nabla_X^0 \Psi = \nabla_X^g \Psi + \frac{1}{4} (X \lrcorner T) \cdot \Psi + \frac{3}{4} (X \lrcorner F) \cdot \Psi + (X \wedge F) \cdot \Psi.$$

*In case  $B \neq -7$ , there exists one  $\nabla^0$ -parallel spinor field  $\Psi_0$  if and only if the following conditions are satisfied:*

- (1) *The spinor field  $\Psi_0$  is parallel with respect to the characteristic connection  $\nabla^c$ .*
- (2) *The spinor field  $\Psi_0$  satisfies  $*\omega^3 \cdot \Psi = -7 \cdot \Psi$ .*
- (3) *The 3-form  $T$  coincides with the characteristic torsion,  $T = T^c$ .*

These theorems have two common aspects. Firstly, both refer to a particular ratio between the parameters  $p$  and  $q$

$$4p = (n - 4)q.$$

This is special in several ways, so we will call the corresponding equation ‘of special type’. The role of this ratio was mentioned in [AF03] in relation to the fact that the Dirac operator coming

TABLE 1. Existence of solutions to  $\nabla\Psi = 0$ .  $\mathcal{N}$  denotes the maximum number of constructed, distinct spinor fields which are parallel with respect to a certain family of covariant spinorial derivatives  $\nabla$ . The superscript  $c$  refers to the characteristic connection  $\nabla^c$ . In the second last column we determine whether all constructed solutions are eigenspinors of  $T$ .

Dim.	Structure		$F \neq 0$				$F = 0$	
			$\mathcal{N}$	$\mathcal{N}(\text{Ric}^T = 0)$		$T \cdot \Psi = \lambda \cdot \Psi$	$\mathcal{N}^c$	
				$T \neq T^c$	$T = T^c$			
$n = 5$	$\alpha$ -Sasakian structure		1	1	1	✓	2	
	quasi-Sasakian $M^5(a, b, c, d)$		1	–	–	✓	2	
$n = 6$	almost Hermitian structure of class $\mathcal{C}[G]$	SU(3)	2	1	–	✓	2	
		SO(3)	2	1	2	✓	2	
		SU(2)	2	–	–	✗	4	
		U(2) <sub>0</sub>	2	–	–	✗	4	
		U(2) <sub>1</sub>	no solutions constructable					
		U(2) <sub>-1</sub>	2	1	–	✓	2	
$n = 7$	nearly parallel $G_2$ -structure		2	2	–	✓	1	
	cocalibrated $G_2$ -structure	$\mathfrak{su}(3)$	I	2	–	–	✓	2
			II	2	2	–	✓	2
	of class	$\mathfrak{so}(3)$	I	2	–	–	✓	2
			II	2	2	1	✓	2
	$\mathcal{C}[\mathfrak{g}]$	$\mathfrak{su}(2)$	I	4	4	3	✓	4
			II	4	–	–	✓	4
			III	3	–	–	✓	4
		u(2)	I	2	2	2	✓	2
			II	2	2	–	✓	2
		$\mathfrak{su}_c(2)$ rel.		1	1	1	✓	1
	$\alpha$ -Sasakian structure			2	2	2	✓	2

from  $\nabla$  does not depend on  $F$  for  $p/q = (n - 4)/4$  (see chapter 1). Secondly, in either case solving  $\nabla\Psi = 0$  is reduced to the existence of solutions to  $\nabla^c\Psi = 0$ , which were classified in many works relative to Strominger's equations over the last years. In other words we have a good starting point for constructing solutions.

We then proceed to the full system (N), (NN). If we define

$$\text{Ric}^T := \text{Ric}_{ij}^g - \frac{1}{4} T_{imn} T_{jmn},$$

the following relations:

$$\delta T = 0, \quad F \cdot \Psi = \kappa \cdot \Psi, \quad \operatorname{div}^c(\operatorname{Ric}^T) = \operatorname{div}(\operatorname{Ric}^T) = 0$$

hold in all cases discussed here by [AFNP05]. If we replace – as suggested in [AFNP05] – the equation  $\operatorname{Ric}^T = 0$  in (N) by  $\operatorname{div}(\operatorname{Ric}^T) = 0$ , then every solution  $\Psi$  constructed satisfies the entire system (N), (NN), provided furthermore that  $\Psi$  is an eigenspinor of  $T$ .

Table 1 contains a summary of our results regarding  $\nabla\Psi = 0$ . It is to be understood as follows: Cocalibrated  $G_2$ -structures of class  $\mathcal{C}[\mathfrak{su}(2)]$  with torsion type I will for instance admit at most four distinct (i.e. linearly independent) spinor fields parallel with respect to a certain family of covariant spinorial derivatives with non-vanishing 4-form. Should we further consider certain structures of this type and choose  $T = T^c$ , there exists a family  $\tilde{\nabla}$  rendering three distinct spinor fields parallel. At the same time  $\operatorname{Ric}^{T^c} = \operatorname{Ric}^c = 0$  is fulfilled. There exist other structures of the same type and a family  $\tilde{\tilde{\nabla}}$  with now four distinct spinor fields such that  $\tilde{\tilde{\nabla}}\tilde{\Psi} = 0$ ,  $\operatorname{Ric}^T = 0$  and  $T \neq T^c$ . All these spinor fields are eigenspinors relative to the 3-form  $T$ . Finally, there exist at most four distinct  $\nabla^c$ -parallel spinor fields for that type of geometric structures.

To conclude, a few comments on possible generalizations of our spinorial field equations. Solving the following Killing equation is the main concern of supergravity models in type II string theory (see [GLW06]):

$$\nabla_X^g \Psi + \frac{1}{4} (X \lrcorner T) \cdot \Psi + \sum_i p_i (X \lrcorner F^i) \cdot \Psi + \sum_i q_i (X \wedge F^i) \cdot \Psi = 0.$$

The differential forms  $F^i$  are of degree  $2i$  (type IIa) or of degree  $2i + 1$  (type IIb). Due to the complexity of the algebraic systems the approach of this thesis is unlikely to be suitable for this general kind of equation. However,  $\nabla^c$ -parallel spinor fields may represent natural candidates to begin with when constructing solutions. If we start from one of the structures considered in this work and set  $T = B \cdot T^c$ , then the above equation will read

$$\frac{B-1}{4} (X \lrcorner T^c) \cdot \Psi_0 + \sum_i p_i (X \lrcorner F^i) \cdot \Psi_0 + \sum_i q_i (X \wedge F^i) \cdot \Psi_0 = 0$$

for a  $\nabla^c$ -parallel spinor field  $\Psi_0$ . The last column of table 1 tells us how many such spinor fields exist. We conjecture that this purely algebraic equation could be solved with an appropriate ansatz for the differential forms  $F^i$ .

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