Scenario reduction in mixed-integer stochastic programming

R. Henrion, C. Küchler, W. Römisch

WIAS Berlin and Humboldt-University Berlin

http://www.math.hu-berlin.de/~romisch

COPI'08, EDF, Clamart, November 26-28, 2008

DFG Research Center MATHEON
Mathematics for key technologies

Bundesministerium
für Bildung
und Forschung

Introduction

Most approaches for solving stochastic programs of the form

$$\min \left\{ \int_{\Xi} f_0(x,\xi) P(d\xi) : x \in X \right\}$$

with a probability measure P on Ξ and a (normal) integrand f_0, require numerical integration techniques, i.e., replacing the integral by some quadrature formula

$$\int_{\Xi} f_0(x,\xi) P(d\xi) \approx \sum_{i=1}^{n} p_i f_0(x,\xi_i),$$

where $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$ and $\xi_i \in \Xi$, $i = 1, \ldots, n$.
Since f_0 is often expensive to compute, the number n should be as small as possible.

Aim: Given pairs (ξ_i, p_i), $i = 1, \ldots, N$, where N is too large. Find a subset $\{\xi_{i_1}, \ldots, \xi_{i_n}\}$ with $n < N$ and the corresponding probabilities q_j, $j = 1, \ldots, n$, such that the approximation is still reasonable.

Mixed-integer two-stage stochastic programs

We consider

$$\min \left\{ \langle c, x \rangle + \int_{\Xi} \Phi(q(\xi), h(\xi) - T(\xi)x) P(d\xi) : x \in X \right\},$$

where Φ is given by

$$\Phi(u,t) := \inf \left\{ \langle u_1, y_1 \rangle + \langle u_2, y_2 \rangle \,\middle|\, \begin{array}{l} W_1 y_1 + W_2 y_2 \le t \\ y_1 \in \mathbb{R}_+^{m_1}, y_2 \in \mathbb{Z}_+^{m_2} \end{array} \right\}$$

for all pairs $(u,t) \in \mathbb{R}^{m_1+m_2} \times \mathbb{R}^d$, and $c \in \mathbb{R}^m$, X is a closed subset of $\mathbb{R}^m$, Ξ a polyhedron in $\mathbb{R}^s$, $T \in \mathbb{R}^{d\times m}$, $W_1 \in \mathbb{R}^{d\times m_1}$, $W_2 \in \mathbb{R}^{d\times m_2}$, and $q(\xi) \in \mathbb{R}^{m_1+m_2}$ and $h(\xi) \in \mathbb{R}^d$ are affine functions of ξ, and P is a Borel probability measure.

Assumptions:

(C1) The matrices W_1 and W_2 have rational elements.

(C2) For each pair $(x,\xi) \in X \times \Xi$ it holds that $h(\xi) - T(\xi)x \in \mathcal{T}$ (relatively complete recourse), where

$$\mathcal{T} := \{t \in \mathbb{R}^d | \exists y = (y_1, y_2) \in \mathbb{R}^{m_1} \times \mathbb{Z}^{m_2} \text{ with } W_1 y_1 + W_2 y_2 \leq t\}.$$

(C3) For each $\xi \in \Xi$ the recourse cost $q(\xi)$ belongs to the dual feasible set (dual feasibility)

$$\mathcal{U} := \{u = (u_1, u_2) \in \mathbb{R}^{m_1+m_2} | \exists z \in \mathbb{R}^d_- \text{ with } W_j^\top z = u_j, j = 1,2\}.$$

(C4) $P \in \mathcal{P}_r(\Xi)$, i.e., $\int_\Xi \|\xi\|^r P(d\xi) < +\infty$, $r \in \{1,2\}$.

Condition (C2) means that a feasible second stage decision always exists. Both (C2) and (C3) imply $\Phi(u,t)$ to be finite for all $(u,t) \in \mathcal{U} \times \mathcal{T}$. Clearly, it holds $(0,0) \in \mathcal{U} \times \mathcal{T}$ and $\Phi(0,t) = 0$ for every $t \in \mathcal{T}$.

$r = 1$ holds if either $q(\xi)$ is the only quantity depending on ξ or $q(\xi)$ does not depend on ξ. Otherwise, we set $r = 2$.

With the convex polyhedral cone

$$\mathcal{K} := \{t \in \mathbb{R}^d \mid \exists y_1 \in \mathbb{R}^{m_1} \text{ such that } t \geq W_1 y_1\} = W_1(\mathbb{R}^{m_1}) + \mathbb{R}^d_+$$

one obtains the representation

$$\mathcal{T} = \bigcup_{z \in \mathbb{Z}^{m_2}} (W_2 z + \mathcal{K}).$$

The set $\mathcal{T}$ is always connected (i.e., there exists a polygon connecting two arbitrary points of $\mathcal{T}$) and condition (C1) implies that $\mathcal{T}$ is closed. If, for each $t \in \mathcal{T}$, $Z(t)$ denotes the set

$$Z(t) := \{z \in \mathbb{Z}^{m_2} \mid \exists y_1 \in \mathbb{R}^{m_1} \text{ such that } W_1 y_1 + W_2 z \leq t\},$$

the representation of $\mathcal{T}$ implies that it is decomposable into subsets of the form

$$\begin{aligned} \mathcal{T}(t_0) &:= \{t \in \mathcal{T} \mid Z(t) = Z(t_0)\} \\ &= \bigcap_{z \in Z(t_0)} (W_2 z + \mathcal{K}) \setminus \bigcup_{z \in \mathbb{Z}^{m_2} \setminus Z(t_0)} (W_2 z + \mathcal{K}) \end{aligned}$$

for every $t_0 \in \mathcal{T}$.

In general, the set $Z(t_0)$ is finite or countable, but condition (C1) implies that there exist countably many elements $t_i \in \mathcal{T}$ and $z_{ij} \in \mathbb{Z}^{m_2}$ for j belonging to a finite subset N_i of $\mathbb{N}$, $i \in \mathbb{N}$, such that

$$\mathcal{T} = \bigcup_{i\in\mathbb{N}} \mathcal{T}(t_i) \quad \text{with} \quad \mathcal{T}(t_i) = (t_i + \mathcal{K}) \setminus \bigcup_{j\in N_i} (W_2 z_{ij} + \mathcal{K}).$$

The sets $\mathcal{T}(t_i)$, $i \in \mathbb{N}$, are nonempty and star-shaped, but nonconvex in general.

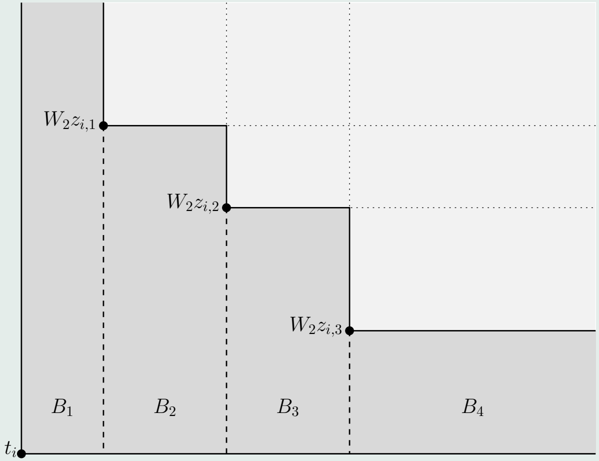


Illustration of $\mathcal{T}(t_i)$ for $W_1 = 0$ and $d = 2$, i.e., $\mathcal{K} = \mathbb{R}^2_+$, with $N_i = \{1, 2, 3\}$ and its decomposition into the sets B_j, $j = 1, 2, 3, 4$, whose closures are rectangular.

If for some $i \in \mathbb{N}$ the set $\mathcal{T}(t_i)$ is nonconvex, it can be decomposed into a finite number of subsets.

This leads to a countable number of subsets B_j, $j \in \mathbb{N}$, of $\mathcal{T}$ whose closures are convex polyhedra with facets parallel to $W_1(\mathbb{R}^{m_1})$ or to suitable facets of $\mathbb{R}^r_+$ and form a partition of $\mathcal{T}$.

Since the sets $Z(t)$ of feasible integer decisions do not change if t varies in some B_j, the function $(u,t) \mapsto \Phi(u,t)$ from $\mathcal{U} \times \mathcal{T}$ to $\mathbb{R}$ has the (local) Lipschitz continuity regions $\mathcal{U} \times B_j$, $j \in \mathbb{N}$ and the estimate

$$|\Phi(u,t) - \Phi(\tilde{u},\tilde{t})| \leq L(\max\{1, \|t\|, \|\tilde{t}\|\}\|u - \tilde{u}\| + \max\{1, \|u\|, \|\tilde{u}\|\}\|t - \tilde{t}\|)$$

holds for all pairs $(u,t), (\tilde{u},\tilde{t}) \in \mathcal{U} \times B_j$ and some (uniform) constant $L > 0$.

(Blair-Jeroslow 77, Bank-Guddat-Kummer-Klatte-Tammer 1982)

The integrand

$$f_0(x,\xi) = \langle c, x\rangle + \Phi(q(\xi), h(\xi) - T(\xi)x) \quad ((x,\xi) \in X \times \Xi)$$

has the property that, for every $x \in X$, and

$$\Xi_{x,j} = \{\xi \in \Xi \,|\, h(\xi) - T(\xi)x \in B_j\} \quad (j \in \mathbb{N})$$

it holds

$$|f_0(x,\xi) - f_0(x,\tilde{\xi})| \leq \hat{L} \max\{1, \|\xi\|^{r-1}, \|\tilde{\xi}\|^{r-1}\} \|\xi - \tilde{\xi}\| \; (\xi, \tilde{\xi} \in \Xi_{x,j})$$
$$|f_0(x,\xi)| \leq C \max\{1, \|x\|\} \max\{1, \|\xi\|^r\} \; (\xi \in \Xi)$$

for all $x \in X$ with some constants $\hat{L}$ and C.

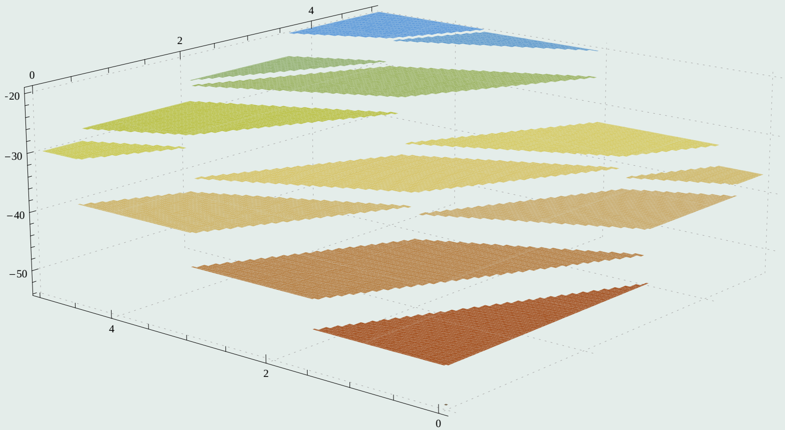
Since the objective function is lower semicontinuous on X if the conditions (C1)–(C4) are satisfied, solutions exist if X is compact. If the probability distribution P has a density, the objective function is continuous, but nonconvex in general. If the support of P is finite, the objective function is piecewise continuous with a finite number of continuity regions, whose closures are polyhedral.

Example: (Schultz-Stougie-van der Vlerk 98)

$m = d = s = 2$, $m_1 = 0$, $m_2 = 4$, $c = (0, 0)$, $X = [0, 5]^2$, $h(\xi) = \xi$, $q(\xi) \equiv q = (-16, -19, -23, -28)$, $y_i \in \{0, 1\}$, $i = 1, 2, 3, 4$, $P \sim \mathcal{U}(5, 10, 15\}$ (discrete)

Second stage problem: MILP with 1764 binary variables and 882 constraints.

$$T = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \qquad W = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 6 & 1 & 3 & 2 \end{pmatrix}$$



Stability

We consider the class of functions

$$\mathcal{F}_{r,\mathcal{B}}(\Xi) := \{f\mathbf{1}_B : f \in \mathcal{F}_r(\Xi), B \in \mathcal{B}\},$$

where $\mathbf{1}_B$ denotes the characteristic function of the set B and the class $\mathcal{F}_r(\Xi)$ consists of all continuous functions $f : \Xi \to \mathbb{R}$ such that the estimates

$$|f(\xi)| \leq \max\{1, \|\xi\|^r\}$$

$$f(\xi) - f(\tilde{\xi}) \leq \max\{1, \|\xi\|^{r-1}, \|\tilde{\xi}\|^{r-1}\}\|\xi - \tilde{\xi}\|$$

hold true for all $\xi, \tilde{\xi} \in \Xi$ and $\mathcal{B}$ is the set of convex polyhedra in Ξ that contains

$$\{\xi \in \Xi : h(\xi) - T(\xi)x \in B\}$$

for all $x \in X$ and all polyhedra B in $\mathbb{R}^d$ with facets, i.e., $(d-1)$-dimensional faces, that are parallel to $W_1(\mathbb{R}^{m_1})$ or parallel to suitable facets of $\mathbb{R}^d_+$.

Metric on $\mathcal{P}_r(\Xi)$:

$$\zeta_{r,\mathcal{B}}(P,Q) := \sup\left\{\left|\int_\Xi f(\xi)(P-Q)(d\xi)\right| : f \in \mathcal{F}_{r,\mathcal{B}}(\Xi)\right\}$$

Let $v(P)$ denote the optimal value of the stochastic program, i.e.,

$$v(P) := \inf\left\{\int_\Xi f_0(x,\xi)P(d\xi) : x \in X\right\}.$$

Proposition: (Rö-Vigerske 08)

Assume (C1)–(C4) and let X be compact. Then the estimate

$$|v(P) - v(Q)| \leq L\varphi_P(\zeta_{r,\mathcal{B}}(P,Q))$$

holds for every $Q \in \mathcal{P}_r(\Xi)$, where the function φ_P is defined by $\varphi_P(0) = 0$ and

$$\varphi_P(t) := \inf_{R \geq 1}\left\{R^{d+1}t + \int_{\{\xi\in\Xi \,|\, \|\xi\|>R\}} \|\xi\|^r P(d\xi)\right\} \quad (t > 0).$$

The function characterizes the tail behavior of P and is continuous at $t = 0$. If P has a finite pth moment, i.e., if $\int_\Xi \|\xi\|^p P(d\xi) < +\infty$, for some $p > r$, the estimate

$$\varphi_P(t) \leq C t^{\frac{p-r}{p+d-1}} \quad (t \geq 0)$$

is valid for some constant $C > 0$. If Ξ is bounded, we have $\varphi_P(t) \leq Ct$.

The metric $\zeta_{r,\mathcal{B}}$ is difficult to handle, but it holds:

Proposition:

Convergence with respect to the metric $\zeta_{r,\mathcal{B}}$ is equivalent to convergence with respect to ζ_r (Fortet-Mourier metric of order r) and with respect to $\alpha_{\mathcal{B}}$ ($\mathcal{B}$-discrepancy), where

$$\zeta_r(P,Q) := \sup\left\{\left|\int_\Xi f(\xi)(P-Q)(d\xi)\right| : f \in \mathcal{F}_r(\Xi)\right\},$$

$$\alpha_{\mathcal{B}}(P,Q) := \sup_{B\in\mathcal{B}} |P(B) - Q(B)|$$

If the set Ξ is bounded, it even holds

$$\alpha_{\mathcal{B}}(P,Q) \le \zeta_{r,\mathcal{B}}(P,Q) \le C\alpha_{\mathcal{B}}(P,Q)^{\frac{1}{s+1}}$$

with some constant C depending on Ξ.

Since the class $\mathcal{B}$ strongly depends on the structure of the underlying mixed-integer stochastic program, we sometimes consider the rectangular discrepancy with $\mathcal{B} = \mathcal{B}_{\text{rect}}$

$$\mathcal{B}_{\text{rect}} := \{I_1 \times I_2 \times \cdots \times I_s \mid \emptyset \ne I_j \text{ is a closed interval in } \mathbb{R}\}.$$

The metric Fortet-Mourier metric ζ_r allows the following dual representation as transportation problem: Let

$$c_r(\xi, \tilde{\xi}) := \max\{1, \|\xi\|^{r-1}, \|\tilde{\xi}\|^{r-1}\}\|\xi - \tilde{\xi}\| \quad (\xi, \tilde{\xi} \in \Xi).$$

Proposition: (Rachev/Rüschendorf 98)
Let Ξ be bounded.

$$\zeta_r(P, Q) = \inf\left\{\int_{\Xi\times\Xi} \hat{c}_r(\xi, \tilde{\xi})\Theta(d\xi, d\tilde{\xi}) : \pi_1\Theta = P, \pi_2\Theta = Q\right\}$$

where the reduced cost $\hat{c}$ is of the form

$$\hat{c}_r(\xi, \tilde{\xi}) := \inf\left\{\sum_{i=1}^{n-1} c_r(\xi_{l_i}, \xi_{l_{i+1}}) : n \in \mathbb{N}, \xi_{l_i} \in \Xi, \xi_{l_1} = \xi, \xi_{l_n} = \tilde{\xi}\right\}.$$

is a metric on Ξ with $\hat{c}_r \le c_r$.

Scenario reduction

Let P be a probability measure with finite support $\{\xi^1, \ldots, \xi^N\}$ and set $p_i := P(\{\xi^i\}) > 0$ for $i = 1, \ldots, N$. Denoting by δ_ξ the Dirac measure placing mass one at the point ξ, P has the form

$$P = \sum_{i=1}^{N} p_i \delta_{\xi^i}.$$

The problem of optimal scenario reduction consists in determining a discrete probability measure Q on $\mathbb{R}^s$ supported by a subset of $\{\xi^1, \ldots, \xi^N\}$ and being close to P with respect to

$$d_\lambda := \lambda \alpha_{\mathcal{B}} + (1-\lambda)\zeta_r \quad (\lambda \in [0,1]).$$

It can be written as

$$\min \left\{ d_\lambda \left(\sum_{i=1}^{N} p_i \delta_{\xi^i}, \sum_{j=1}^{n} q_j \delta_{\eta^j} \right) \middle| \begin{array}{l} \{\eta^1, \ldots, \eta^n\} \subset \{\xi^1, \ldots, \xi^N\} \\ q_j \geq 0 \; j = 1, \ldots, n, \sum_{j=1}^{n} q_j = 1 \end{array} \right\}.$$

This optimization problem may be decomposed into an outer problem for determining $\mathrm{supp}\,(Q) = \eta$ and an inner problem for choosing the probabilities q_j, $j = 1, \ldots, n$.

To this end, we denote

$$d_\lambda(P,(\eta,q)) \;:=\; d_\lambda\left(\sum_{i=1}^{N} p_i\delta_{\xi^i}, \sum_{j=1}^{n} q_j\delta_{\eta^j}\right)$$

$$S_n \;:=\; \{q \in \mathbb{R}^n_+ : \sum\nolimits_{j=1}^{n} q_j = 1\}.$$

Then the scenario reduction problem may be rewritten as

$$\min_{\eta}\{\min_{q\in S_n} d_\lambda(P,(\eta,q)) : \eta \subset \{\xi^1,\ldots,\xi^N\}, |\eta| = n\}$$

with the inner problem (optimal redistribution)

$$\min\{d_\lambda(P,(\eta,q)) : q \in S_n\}$$

for the fixed support η. The outer problem is a combinatorial optimization problem (NP hard) while the inner problem may be reformulated as a linear program.

We assume for the sake of notational simplicity, that $\eta = \{\xi^1, \ldots, \xi^n\}$. Then the inner problem is of the form:

$$\min\{d_\lambda(P, (\{\xi^1, \ldots, \xi^n\}, q)) : q \in S_n\}$$

The finiteness of the support of P allows to define for $B \in \mathcal{B}$ the critical index set $I(B)$ by

$$I(B) := \{i \in \{1, \ldots, N\} : \ \xi^i \in B\}$$

and to write

$$|P(B) - Q(B)| = \left| \sum_{i \in I(B)} p_i - \sum_{j \in I(B) \cap \{1,\ldots,n\}} q_j \right|.$$

Furthermore, we define the system of critical index sets of $\mathcal{B}$ as

$$\mathcal{I}_\mathcal{B} := \{I(B) : B \in \mathcal{B}\}.$$

Thus, the $\mathcal{B}$-discrepancy between P and Q may be reformulated as follows:

$$\alpha_\mathcal{B}(P, Q) = \max_{I \in \mathcal{I}_\mathcal{B}} \left| \sum_{i \in I} p_i - \sum_{j \in I \cap \{1,\ldots,n\}} q_j \right|.$$

This allows to compute $\alpha_{\mathcal{B}}$ by means of the following linear program:

$$\min \left\{ t \;\middle|\; \begin{array}{l} -\sum_{j\in I\cap\{1,\ldots,n\}} q_j \leq t - \sum_{i\in I} p_i \\ \sum_{j\in I\cap\{1,\ldots,n\}} q_j \leq t + \sum_{i\in I} p_i,\, I \in \mathcal{I}_{\mathcal{B}} \end{array} \right\}$$

Since $|\mathcal{I}_{\mathcal{B}}| \leq 2^N$, the number of inequalities is too large to solve this LP numerically.

Therefore, we consider the following reduced system of critical index sets

$$\mathcal{I}^*_{\mathcal{B}} := \{I(B) \cap \{1,\ldots,n\} : B \in \mathcal{B}\}.$$

Thereby, every member $J \in \mathcal{I}^*_{\mathcal{B}}$ of the reduced system is associated with a family $\varphi(J) \subset \mathcal{I}_{\mathcal{B}}$ of critical index sets, all of which share the same intersection with $\{1,\ldots,n\}$:

$$\varphi(J) := \{I \in \mathcal{I}_{\mathcal{B}} : J = I \cap \{1,\ldots,n\}\} \quad (J \in \mathcal{I}^*_{\mathcal{B}}).$$

Finally, we consider the quantities

$$\gamma^J := \max_{I \in \varphi(J)} \sum_{i \in I} p_i \quad \text{and} \quad \gamma_J := \min_{I \in \varphi(J)} \sum_{i \in I} p_i \quad (J \in \mathcal{I}^*_{\mathcal{B}}),$$

and write the inner problem as

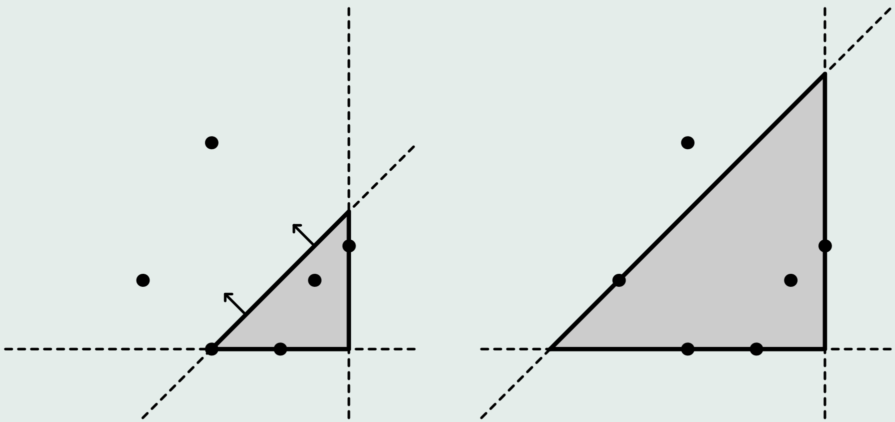
$$\min \left\{ \lambda t_\alpha + (1-\lambda) t_\zeta \;\middle|\; \begin{array}{l} t_\alpha, t_\zeta \geq 0,\ q_j \geq 0,\ \sum_{j=1}^n q_j = 1, \\ \eta_{ij} \geq 0, i = 1, \ldots, N,\ j = 1, \ldots, n, \\ t_\zeta \geq \sum_{i=1}^N \sum_{j=1}^n \hat{c}_r(\xi^i, \xi^j) \eta_{ij}, \\ \sum_{j=1}^n \eta_{ij} = p_i,\ i = 1, \ldots, N, \\ \sum_{i=1}^N \eta_{ij} = q_j,\ j = 1, \ldots, n, \\ -\sum_{j \in I^*} q_j \leq t_\alpha - \gamma^{I^*},\ I^* \in \mathcal{I}^*_{\mathcal{B}} \\ \quad \sum_{j \in I^*} q_j \leq t_\alpha + \gamma_{I^*},\ I^* \in \mathcal{I}^*_{\mathcal{B}} \end{array} \right\}$$

Now we have $|\mathcal{I}^*_{\mathcal{B}}| \leq 2^n$ and, hence, drastically reduced the maximum number of inequalities. This can make the LP solvable at least for moderate values of n.

How to determine $\mathcal{I}^*_{\mathcal{B}}$, γ_J and γ^J?

Observation:

$\mathcal{I}^*_{\mathcal{B}}$, γ_J and γ^J are determined by those polyhedra (belonging to $\mathcal{P}$), each of whose facets contains an element of $\{\xi^1, \ldots, \xi^n\}$, such that it can not be enlarged without changing its interior's intersection with $\{\xi^1, \ldots, \xi^n\}$. The polyhedra in $\mathcal{P}$ are called supporting.



Non supporting polyhedron (left) and supporting polyhedron (right). The dots represent the remaining scenarios $\xi^1, \ldots, \xi^n$

Proposition:

$$
\begin{aligned}
\mathcal{I}^*_{\mathcal{B}} &= \{J \subseteq \{1,\ldots,n\} : \exists B \in \mathcal{P}, \cup_{j\in J}\{\xi^j\} = \{\xi^1,\ldots,\xi^n\} \cap \operatorname{int} B\} \\
\gamma^J &= \max\{P(\operatorname{int} B) : B \in \mathcal{P}, \cup_{j\in J}\{\xi^j\} = \{\xi^1,\ldots,\xi^n\} \cap \operatorname{int} B\} \\
\gamma_J &= \sum_{i\in I} p_i \quad \text{with} \quad I \subseteq \{1,\ldots,N\} \quad \text{defined by} \\
I &:= \left\{ i : \min_{j\in J} \langle m^l, \xi^j\rangle \le \langle m^l, \xi^i\rangle \le \max_{j\in J} \langle m^l, \xi^j\rangle, l = 1,\ldots,k \right\},
\end{aligned}
$$

where m^j, $j = 1,\ldots,k$, are the rows of a matrix $M \in \mathbb{R}^{k\times s}$ having the property that every polyhedron $B \in \mathcal{B}$ can be written as

$$B = \{\xi \in \mathbb{R}^s : \underline{a}^B \le M\xi \le \bar{a}^B\}$$

for some $\underline{a}^B$ and $\bar{a}^B$ in $\overline{\mathbb{R}}^k$.

Note that $|\mathcal{P}| \le \binom{n+2}{2}^k$!

For $n = 5$, $k = 3$ and $n = 20$, $k = 12$, the latter is equal to 3375 and $7.36 \cdot 10^{27}$, respectively.

Numerical results

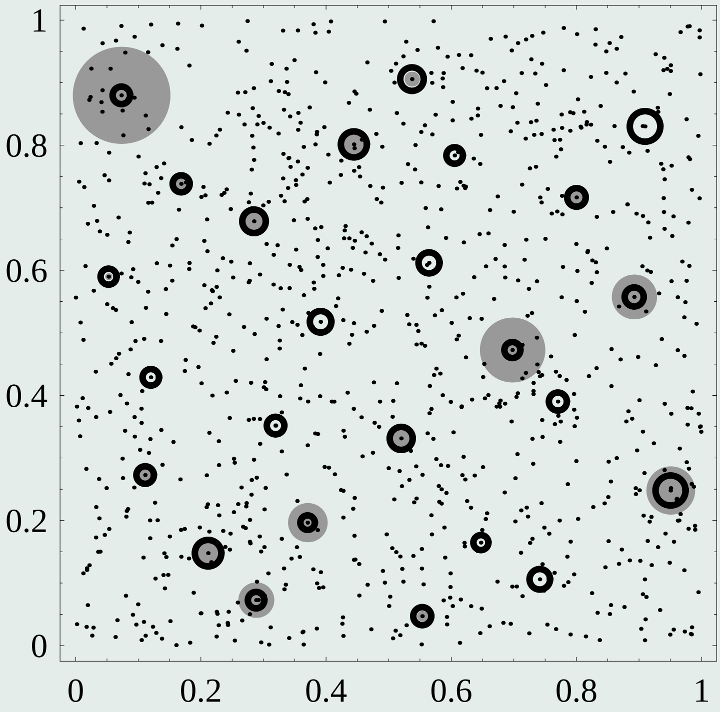
Optimal redistribution w.r.t. the polyhedral discrepancy $\alpha_{\mathcal{B}}$:

	k	n=5	n=10	n=15	n=20
	cell	0.01	0.01	0.01	0.05
$\mathbb{R}^3$	3	0.01	0.04	0.56	6.02
N=100	6	0.03	1.03	14.18	157.51
	9	0.15	7.36	94.49	948.17
	cell	0.01	0.01	0.05	0.30
$\mathbb{R}^4$	4	0.01	0.19	1.83	17.22
N=100	8	0.11	5.66	59.28	521.31
	12	0.67	39.86	374.15	3509.34
	cell	0.01	0.01	0.01	0.07
$\mathbb{R}^3$	3	0.01	0.05	0.53	4.28
N=200	6	0.03	0.76	11.80	132.21
	9	0.12	4.22	78.49	815.79
	cell	0.01	0.01	0.06	0.29
$\mathbb{R}^4$	4	0.01	0.20	2.56	41.73
N=200	8	0.11	4.44	73.70	1042.78
	12	0.74	28.29	473.72	6337.68

Running times [sec] of the optimal redistribution algorithm

Example 2:

We consider $\Xi = [0,1]^2$, $N = 1000$ samples from the uniform distribution on Ξ, $n = 25$. Consider $d_\lambda = \lambda \alpha_{\mathcal{B}_{\mathrm{rect}}} + (1-\lambda)\zeta_2$.



25 scenarios chosen by Quasi Monte Carlo out of 1000 samples from the uniform distribution on $[0,1]^2$ and optimal probabilities adjusted w.r.t. d_λ for $\lambda = 1$ (gray balls) and $\lambda = 0.9$ (black circles)

Example 2: (continued)

Solving the outer combinatorial optimization problem by different heuristics:

- **Forward selection**:

Step [0]: $J^{[0]} := \varnothing$.

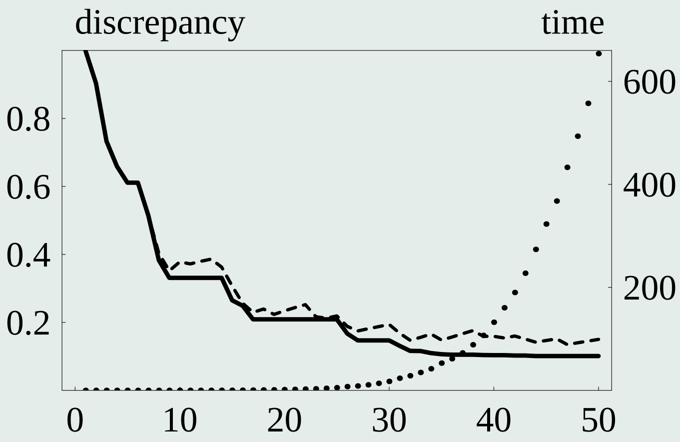
Step [i]: $l_i \in \text{argmin}_{l \notin J^{[i-1]}} \inf_{q \in S_i} d_\lambda(P, (\{\xi^{l_1}, \ldots, \xi^{l_{i-1}}, \xi^{l}\}, q)), J^{[i]} := J^{[i-1]} \cup \{l_i\}$.

Step [n+1]: Minimize $d_\lambda(\{P, (\xi^{l_1}, \ldots, \xi^{l_n}\}, q))$ s.t. $q \in S_n$.

- **(next neighbor) Quasi Monte Carlo (QMC)**: Take the first n points of the Halton sequences with bases 2 and 3 in $[0,1]^2$. The closest scenarios to these points are determined and weight $1/n$ is associated. The resulting distance to the initial measure is computed for $\lambda = 1$.

- **(next neighbor) adjusted QMC**: The probabilities of the closest scenarios to the Halton points are adjusted by optimal redistribution and the distance d_λ is computed for $\lambda = 1$.

Conclusion: Forward selection provides good results, but is very slow due to the optimal redistribution after each step.
Next neighbor adjusted QMC performs significantly better than next neighbor QMC.



Distance $\alpha_{\mathcal{B}_{\text{rect}}}$ between P (with $N = 1000$) and equidistributed QMC-points (dashed), QMC-points, whose probabilities are adjusted (bold), and running times of the QMC-based heuristic (in seconds).

References

Dupačová,J.; Gröwe-Kuska, N.; Römisch, W.: Scenario reduction in stochastic programming: An approach using probability metrics, *Mathematical Programming* 95 (2003), 493–511.

Heitsch, H., Römisch, W.: A note on scenario reduction for two-stage stochastic programs, *Operations Research Letters* 35 (2007), 731–736.

Henrion, R., Küchler, C., Römisch, W.: Scenario reduction in stochastic programming with respect to discrepancy distances, *Computational Optimization and Applications* (to appear).

Henrion, R., Küchler, C., Römisch, W.: Discrepancy distances and scenario reduction in two-stage stochastic mixed-integer programming, *Journal of Industrial and Management Optimization* 4 (2008), 363–384.

Römisch, W., Vigerske, S.: Quantitative stability of fully random mixed-integer two-stage stochastic programs, *Optimization Letters* 2 (2008), 377–388.