Quasi-Monte Carlo methods for two-stage stochastic programs

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Introduction

- Computational methods for solving stochastic programs require (first) a discretization of the underlying probability distribution induced by a numerical integration scheme for the approximate computation of expectations and (second) an efficient solver for a (large scale) finite-dimensional program.
- Discretization means scenario or sample generation.
- Standard approach: Variants of Monte Carlo (MC) methods.
- Recent alternative approaches to scenario generation:
  - (a) Optimal quantization of probability distributions

    (Pflug-Pichler 11).
  - (b) Quasi-Monte Carlo (QMC) methods

    (Koivu-Pennanen 05, Pennanen 09, Homem-de-Mello 08, Heitsch-Leövey-Römisch 12).
  - (c) Sparse grid quadrature rules

    (Chen-Mehrotra 08).

- Known convergence rates in terms of scenario or sample size n:
  MC: $\hat{e}_n(f) = O(n^{-\frac{1}{2}})$ if $f \in L_2$,
  (a): $e_n(f) = O(n^{-\frac{1}{d}})$ if $f \in \mathrm{Lip}$,
  (b): classical: $e_n(f) = O(n^{-1}(\log n)^d)$ if $f \in \mathrm{BV}$,
  recently: $\hat{e}_n(f) \leq C(\delta) n^{-1+\delta}$ $(\delta \in (0, \frac{1}{2}])$ if $f \in W^{(1,\dots,1)}$,
  where $C(\delta)$ does not depend on d,
  (c): $e_n(f) = O(n^{-r}(\log n)^{(d-1)(r+1)})$ if $f \in W^{(r,\dots,r)}$,
  where d is the dimension of the random vector and $e_n(f)$ the quadrature error for integrand f and sample size n, i.e.,

$$e_n(f) = \Big| \int_{[0,1]^d} f(\xi) d\xi - \frac{1}{n} \sum_{i=1}^{n} f(x^i) \Big|$$

  and $\hat{e}_n(f)$ denotes mean (square) quadrature error.

- Monte Carlo methods and (a) may be justified by available stability results for stochastic programs, but there is almost no reasonable justification for (b) and (c) in many cases.

- In applications of stochastic programming dimension d is often large.

Contents

Quasi-Monte Carlo methods

We consider the approximate computation of

$$I_d(f) = \int_{[0,1]^d} f(x)dx$$

by a QMC algorithm

$$Q_{n,d}(f) = \frac{1}{n}\sum_{i=1}^{n} f(x^i)$$

with (non-random) points x^i, $i = 1, \ldots, n$, from $[0,1]^d$.

We assume that f belongs to a linear normed space $\mathbb{F}_d$ of functions on $[0,1]^d$ with norm $\|\cdot\|_d$ and unit ball $\mathbb{B}_d$.

Worst-case error of $Q_{n,d}$ over $\mathbb{B}_d$:

$$e(Q_{n,d}) = \sup_{f\in\mathbb{B}_d} \left| I_d(f) - Q_{n,d}(f) \right|$$

Classical convergence result:

Theorem: (Koksma-Hlawka 61)

If $V_{\mathrm{HK}}(f)$ is the variation of f in the sense of Hardy and Krause, it holds

$$|I_d(f) - Q_{n,d}(f)| \leq V_{\mathrm{HK}}(f) D_n^*(x^1, \ldots, x^n)$$

for any $n \in \mathbb{N}$ and any $x^1, \ldots, x^n \in [0,1]^d$, where

$$D_n^*(x^1, \ldots, x^n) := \|\mathrm{disc}(x)\|_\infty, \quad \mathrm{disc}(x) = \lambda^d([0,x)) - \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{[0,x)}(x^i),$$

is the star-discrepancy of $x^1, \ldots, x^n$ (λ^d denotes Lebesgue's measure on $\mathbb{R}^d$).

Extended Koksma-Hlawka inequality:

$$|I_d(f) - Q_{n,d}(f)| \leq \|\mathrm{disc}(\cdot)\|_{p,p'} \|f\|_{q,q'} \,,$$

where $1 \leq p, p', q, q' \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p'} + \frac{1}{q'} = 1$, and

$$\|\mathrm{disc}(\cdot)\|_{p,p'} = \left(\sum_{u \subseteq D} \left(\int_{[0,1]^{|u|}} |\mathrm{disc}(x_u, 1)|^{p'} \, dx_u \right)^{\frac{p}{p'}} \right)^{\frac{1}{p}}$$

and

$$\|f\|_{q,q'} = \left(\sum_{u \subseteq D} \left(\int_{[0,1]^{|u|}} \left| \frac{\partial^{|u|} f}{\partial x_u}(x_u, 1) \right|^{q'} dx_u \right)^{\frac{q}{q'}} \right)^{\frac{1}{q}}$$

with the obvious modifications if one or more of p, p', q, q' are infinite.

In particular, the classical Koksma-Hlawka inequality essentially corresponds to $p = p' = \infty$ if f belongs to the tensor product Sobolev space $\mathcal{W}_{2,\mathrm{mix}}^{(1,\ldots,1)}([0,1]^d)$ which is defined next.

By $(x_u, 1)$ we mean the d-dimensional vector with the same components as x for indices in u and the rest of the components replaced by 1.

The case of kernel reproducing Hilbert spaces

We assume that $\mathbb{F}_d$ is a kernel reproducing Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and kernel $K : [0,1]^d \times [0,1]^d \to \mathbb{R}$, i.e.,

$$K(\cdot, y) \in \mathbb{F}_d \text{ and } \langle f(\cdot), K(\cdot, y) \rangle = f(y) \quad (\forall y \in [0,1]^d, f \in \mathbb{F}_d).$$

If I_d is a linear bounded functional on $\mathbb{F}_d$, the quadrature error $e_n(Q_{n,d})$ allows the representation

$$e_n(Q_{n,d}) = \sup_{f \in \mathbb{B}_d} \left| I_d(f) - Q_{n,d}(f) \right| = \sup_{f \in \mathbb{B}_d} |\langle f, h_n \rangle| = \|h_n\|_d$$

according to Riesz' theorem for linear bounded functionals.

The representer $h_n \in \mathbb{F}_d$ of the quadrature error is of the form

$$h_n(x) = \int_{[0,1]^d} K(x,y)dy - \tfrac{1}{n} \sum_{i=1}^{n} K(x, x^i) \quad (\forall x \in [0,1]^d),$$

and it holds

$$e_n^2(Q_{n,d}) = \int_{[0,1]^{2d}} K(x,y)dx\,dy - \tfrac{2}{n} \sum_{i=1}^{n} \int_{[0,1]^d} K(x^i, y)dy + \tfrac{1}{n^2} \sum_{i,j=1}^{n} K(x^i, x^j)$$

(Hickernell 98)

Example: Tensor product Sobolev space

$$\mathbb{F}_d = \mathcal{W}_{2,\mathrm{mix}}^{(1,\ldots,1)}([0,1]^d) = \bigotimes_{i=1}^{d} W_2^1([0,1])$$

equipped with the weighted norm $\|f\|_\gamma^2 = \langle f, f\rangle_\gamma$ and inner product

$$\langle f, g\rangle_\gamma = \sum_{u\subseteq\{1,\ldots,d\}} \gamma_u^{-1} \int_{[0,1]^{|u|}} \Big(\int_{[0,1]^{d-|u|}} \frac{\partial^{|u|}}{\partial x^u} f(x) dx^{-u} \Big) \Big(\int_{[0,1]^{d-|u|}} \frac{\partial^{|u|}}{\partial x^u} g(x) dx^{-u} \Big) dx^u$$

where the sequence (γ_i) is positive and nonincreasing, and γ_u is given by

$$\gamma_u = \prod_{i\in u} \gamma_i$$

for $u \subseteq \{1, \ldots, d\}$, is a reproducing kernel Hilbert space with the kernel

$$K_{d,\gamma}(x, y) = \prod_{i=1}^{d} \big(1 + \gamma_i (0.5 B_2(|x_i - y_i|) + B_1(x_i) B_1(y_i))\big) \quad (x, y \in [0,1]^d),$$

where $B_1(x) = x - \frac{1}{2}$ and $B_2(x) = x^2 - x + \frac{1}{6}$ are the Bernoulli polynomials of order 1 and 2, respectively.

Theorem: (Sloan-Woźniakowski 98)
Let $\mathbb{F}_d = \mathcal{W}^{(1,\dots,1)}_{2,\mathrm{mix}}([0,1]^d)$. Then the worst-case error

$$e^2(Q_{n,d}) = \sup_{\|f\|_\gamma \le 1} |I_d(f) - Q_{n,d}(f)| = \sum_{\emptyset \neq u \subseteq D} \prod_{j \in u} \gamma_j \int_{[0,1]^{|u|}} \mathrm{disc}^2(x_u, 1) dx_u$$

is called weighted L_2-discrepancy of $\xi^1, \dots, \xi^n$.

Note that any $f \in \mathbb{F}_d$ is of bounded variation $V_{\mathrm{HK}}(f)$ in the sense of Hardy and Krause and it holds

$$V(f) = \sum_{\emptyset \neq u \subseteq D} \int_{[0,1]^{|u|}} \left| \frac{\partial^{|u|} f}{\partial x_u}(x_u, 1) \right| dx_u \, .$$

Problem: Integrands in two-stage stochastic programming do not belong to F_d
(piecewise linear functions are not of bounded variation (Owen 05)).

First general QMC construction: Digital nets (Sobol 69, Niederreiter 87)

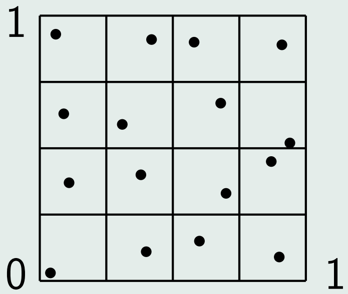
Elementary subintervals E in base b:

$$E = \prod_{j=1}^{d} \left[\frac{a_j}{b^{d_j}}, \frac{a_j + 1}{b^{d_j}} \right),$$

where $a_i, d_i \in \mathbb{Z}_+, 0 \le a_i < b^{d_i}, i = 1, \ldots, d$.

Let $m, t \in \mathbb{Z}_+$, $m > t$. A set of b^m points in $[0,1)^d$ is a (t, m, d)-net in base b if every elementary subinterval E in base b with $\lambda^d(E) = b^{t-m}$ contains b^t points.

Illustration of a $(0, 4, 2)$-net with $b = 2$



A sequence (ξ^i) in $[0,1)^d$ is a (t, d)-sequence in base b if, for all integers $k \in \mathbb{Z}_+$ and $m > t$, the set

$$\{\xi^i : kb^m \le i < (k+1)b^m\}$$

is a (t, m, d)-net in base b.

There exist (t,d)-sequences (ξ^i) in $[0,1]^d$ such that

$$D_n^*(\xi^1,\ldots,\xi^n) = O(n^{-1}(\log n)^{d-1}) \le C(\delta,d)\, n^{-1+\delta} \quad (\forall \delta > 0).$$

Specific sequences: Faure, Sobol', Niederreiter and Niederreiter-Xing sequences (Lemieux 09, Dick-Pillichshammer 10).

Recent development: Scrambled (t,m,d)-nets, where the digits are randomly permuted (Owen 95).

Second general QMC construction: Lattices (Korobov 59, Sloan-Joe 94)

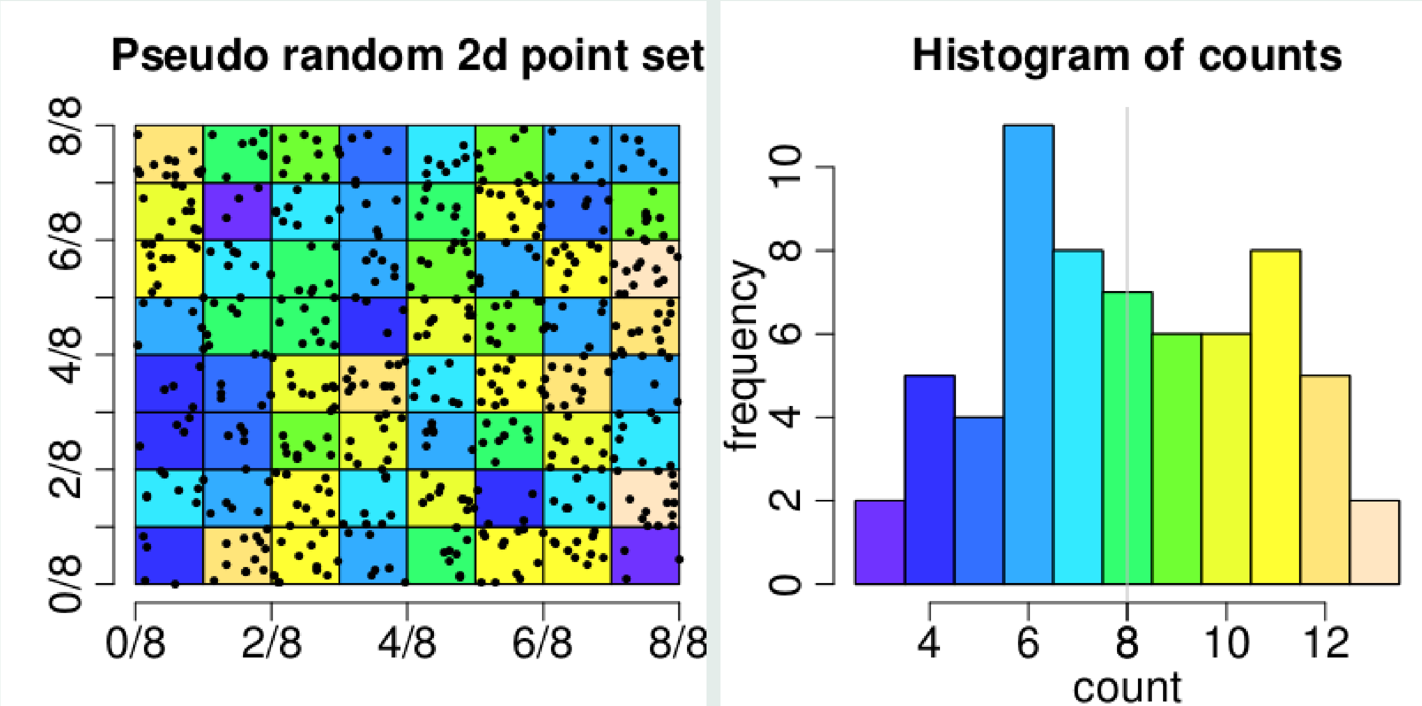
Lattice rules: Let $g \in \mathbb{Z}^d$ and consider the lattice points

$$\left\{\xi^i = \left\{\tfrac{i}{n}\, g\right\} : i = 0,\ldots,n-1\right\},$$

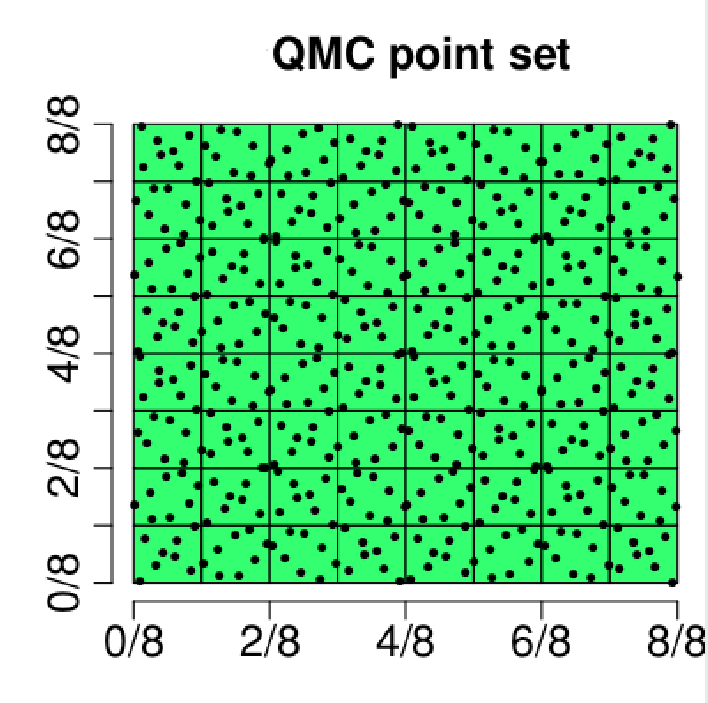
where $\{z\}$ is defined as *componentwise fractional part* of $z \in \mathbb{R}_+$, i.e., $\{z\} = z - \lfloor z \rfloor \in [0,1)$.

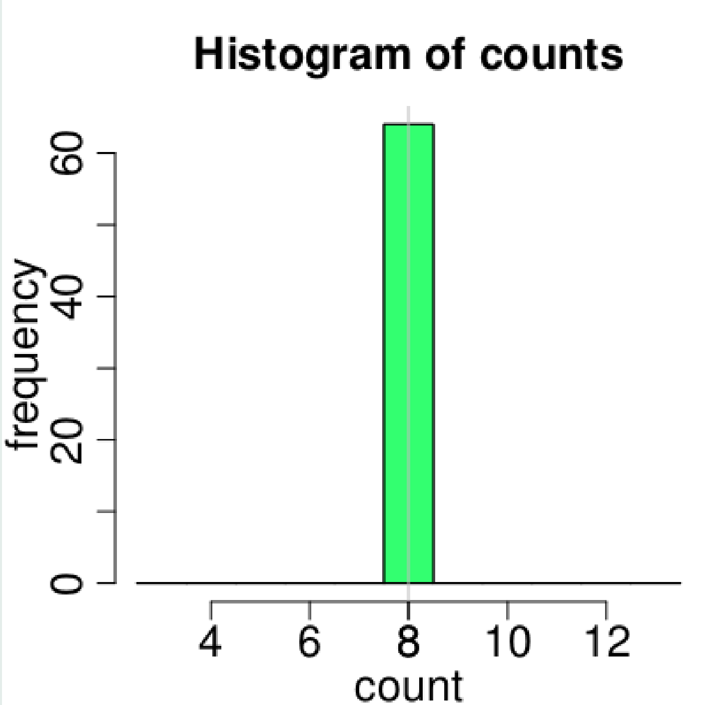
The generator g is chosen such that the lattice rule has good convergence properties. Such lattice rules may achieve better convergence rates $O(n^{-k+\delta})$, $k \in \mathbb{N}$, for integrands in C^k.

$n = 2^9$ pseudo random numbers in $[0,1]^2$ generated by the Mersenne Twister



Sobol point set with $n = 2^9$ in $[0, 1]^2$





Recent development: Randomized lattice rules.

Randomly shifted lattice points:
If $\triangle$ is a sample from uniform distribution in $[0,1]^d$ put

$$Q_{n,d}(f) = \frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n}\, g + \triangle\right),$$

where $g \in \mathbb{Z}^d$ is the generator of the lattice.

Theorem:
Let n be prime, $\mathbb{F}_d = \mathcal{W}_{2,\mathrm{mix}}^{(1,\dots,1)}([0,1]^d)$ and $g \in \mathbb{Z}^d$ be constructed componentwise. Then there exists for any $\delta \in (0, \frac{1}{2}]$ a constant $C(\delta) > 0$ such that the mean quadrature error attains the optimal convergence rate

$$\hat{e}_n(Q_{n,d}) = \left(\mathbb{E}_\Delta \left| I_d(f) - Q_{n,d}(f)\right|^2\right)^{\frac{1}{2}} \le C(\delta) n^{-1+\delta},$$

where the constant $C(\delta)$ grows when δ decreases, but does not depend on the dimension d if the sequence (γ_j) satisfies the condition

$$\sum_{j=1}^{\infty} \gamma_j^{\frac{1}{2(1-\delta)}} < \infty \qquad \text{(e.g. } \gamma_j = \tfrac{1}{j^3}\text{)}.$$

(Sloan/Wožniakowski 98, Sloan/Kuo/Joe 02, Kuo 03)

ANOVA decomposition of multivariate functions

Idea: Decompositions of f may be used, where most of the terms are smooth, but hopefully only some of them relevant.

Let $D = \{1, \ldots, d\}$ and $f \in L_{1,\rho}(\mathbb{R}^d)$ with $\rho(\xi) = \prod_{j=1}^{d} \rho_j(\xi_j)$, where

$$f \in L_{p,\rho}(\mathbb{R}^d) \quad \text{iff} \quad \int_{\mathbb{R}^d} |f(\xi)|^p \rho(\xi) d\xi < \infty \quad (p \geq 1).$$

Let the projection P_k, $k \in D$, be defined by

$$(P_k f)(\xi) := \int_{-\infty}^{\infty} f(\xi_1, \ldots, \xi_{k-1}, s, \xi_{k+1}, \ldots, \xi_d) \rho_k(s) ds \quad (\xi \in \mathbb{R}^d).$$

Clearly, $P_k f$ is constant with respect to ξ_k. For $u \subseteq D$ we write

$$P_u f = \Big(\prod_{k \in u} P_k \Big)(f),$$

where the product means composition, and note that the ordering within the product is not important because of Fubini's theorem. The function $P_u f$ is constant with respect to all x_k, $k \in u$.

ANOVA-decomposition of f:

$$f = \sum_{u \subseteq D} f_u \,,$$

where $f_\emptyset = I_d(f) = P_D(f)$ and recursively (Kuo-Sloan-Wasilkowski-Woźniakowski 10)

$$f_u = \sum_{v \subseteq u} (-1)^{|u|-|v|} P_{-v} f = P_{-u}(f) + \sum_{v \subset u} (-1)^{|u|-|v|} P_{u-v}(P_{-u}(f)),$$

where P_{-u} and P_{u-v} mean integration with respect to ξ_j, $j \in D \backslash u$ and $j \in u \backslash v$, respectively. The second representation motivates that f_u is essentially as smooth as $P_{-u}(f)$.

If f belongs to $L_{2,\rho}(\mathbb{R}^d)$, its ANOVA terms $\{f_u\}_{u \subseteq D}$ are orthogonal in $L_{2,\rho}(\mathbb{R}^d)$.

We set $\sigma^2(f) = \|f - I_d(f)\|_{L_2}^2$ and $\sigma_u^2(f) = \|f_u\|_{L_2}^2$, and have

$$\sigma^2(f) = \|f\|_{L_2}^2 - (I_d(f))^2 = \sum_{\emptyset \neq u \subseteq D} \sigma_u^2(f) \,.$$

The normalized ratios $\frac{\sigma_u^2(f)}{\sigma^2(f)}$ serve as indicators for the importance of ξ^u in f.

Owen's superposition (truncation) dimension distribution of f: Probability measure ν_S (ν_T) defined on the power set of D

$$\nu_S(s) := \sum_{|u|=s} \frac{\sigma_u^2(f)}{\sigma^2(f)} \qquad \left(\nu_T(s) = \sum_{\max\{j:j\in u\}=s} \frac{\sigma_u^2(f)}{\sigma^2(f)}\right) \ (s \in D).$$

Effective superposition (truncation) dimension $d_S(\varepsilon)$ ($d_T(\varepsilon)$) of f is the $(1-\varepsilon)$-quantile of ν_S (ν_T):

$$d_S(\varepsilon) \ = \ \min\left\{s \in D : \sum_{|u|\le s} \sigma_u^2(f) \ge (1-\varepsilon)\sigma^2(f)\right\} \le d_T(\varepsilon)$$

$$d_T(\varepsilon) \ = \ \min\left\{s \in D : \sum_{u\subseteq\{1,\ldots,s\}} \sigma_u^2(f) \ge (1-\varepsilon)\sigma^2(f)\right\}$$

It holds

$$\max\left\{\Big\|f - \sum_{|u|\le d_S(\varepsilon)} f_u\Big\|_{2,\rho}, \Big\|f - \sum_{u\subseteq\{1,\ldots,d_T(\varepsilon)\}} f_u\Big\|_{2,\rho}\right\} \le \sqrt{\varepsilon}\sigma(f).$$

(Caflisch-Morokoff-Owen 97, Owen 03, Wang-Fang 03)

Two-stage linear stochastic programs

We consider the linear two-stage stochastic program

$$\min\Big\{\int_\Xi f(x,\xi)P(d\xi) : x\in X\Big\},$$

where f is extended real-valued defined on $\mathbb{R}^m\times\mathbb{R}^d$ given by

$$f(x,\xi)=\langle c,x\rangle+\Phi(q(\xi),h(\xi)-T(\xi)x),\ (x,\xi)\in X\times\Xi,$$

$c\in\mathbb{R}^m$, $X\subseteq\mathbb{R}^m$ and $\Xi\subseteq\mathbb{R}^d$ are convex polyhedral, W is an $(r,\overline{m})$-matrix, P is a Borel probability measure on Ξ, and the vectors $q(\xi)\in\mathbb{R}^{\overline{m}}$, $h(\xi)\in\mathbb{R}^r$ and the (r,m)-matrix $T(\xi)$ are affine functions of ξ, Φ is the second-stage optimal value function

$$\Phi(u,t)=\inf\{\langle u,y\rangle : Wy=t, y\geq 0\}\quad ((u,t)\in\mathbb{R}^{\overline{m}}\times\mathbb{R}^r),$$

Let $\mathrm{pos}\,W=W(\mathbb{R}^{\overline{m}}_+)$, $\mathcal{D}=\{u\in\mathbb{R}^{\overline{m}}:\{z\in\mathbb{R}^r : W^\top z\leq u\}\neq\emptyset\}$.

Assumptions:

(A1) $h(\xi)-T(\xi)x\in\mathrm{pos}\,W$ and $q(\xi)\in\mathcal{D}$ for all $(x,\xi)\in X\times\Xi$.

(A2) $\int_\Xi\|\xi\|^2P(d\xi)<\infty$.

Proposition:
(A1) and (A2) imply that the two-stage stochastic program represents a convex minimization problem with respect to the first stage decision x with polyhedral constraints.

Lemma: (Walkup-Wets 69, Nožička-Guddat-Hollatz-Bank 74)
Φ is finite, polyhedral and continuous on the $(\overline{m}+r)$-dimensional convex polyhedral cone $\mathcal{D} \times \mathrm{pos}\, W$ and there exist $(r, \overline{m})$-matrices C_j and $(\overline{m}+r)$-dimensional convex polyhedral cones $\mathcal{K}_j$, $j = 1, ..., \ell$, such that

$$\bigcup_{j=1}^{\ell} \mathcal{K}_j \;=\; \mathcal{D} \times \mathrm{pos}\, W \quad \text{and} \quad \mathrm{int}\, \mathcal{K}_i \cap \mathrm{int}\, \mathcal{K}_j = \emptyset\,,\; i \neq j,$$

$$\Phi(u,t) \;=\; \langle C_j u, t\rangle, \quad \text{for each} \;\; (u,t) \in \mathcal{K}_j,\; j = 1, ..., \ell.$$

The function $\Phi(u,\cdot)$ is convex on $\mathrm{pos}\, W$ for each $u \in \mathcal{D}$, and $\Phi(\cdot, t)$ is concave on $\mathcal{D}$ for each $t \in \mathrm{pos}\, W$. The intersection $\mathcal{K}_i \cap \mathcal{K}_j$, $i \neq j$, is either equal to $\{0\}$ or contained in a $(\overline{m}+r-1)$-dimensional subspace of $\mathbb{R}^{\overline{m}+r}$ if the two cones are adjacent.

Error estimates for optimal values and solution sets

With $v(P)$ and $S(P)$ denoting the optimal value and solution set of

$$\min\Big\{\int_\Xi f(x,\xi)P(d\xi) : x\in X\Big\},$$

it holds

$$|v(P)-v(Q)| \;\leq\; L\sup_{x\in X}\Big|\int_\Xi f(x,\xi)P(d\xi)-\int_\Xi f(x,\xi)Q(d\xi)\Big|$$

$$\emptyset\neq S(Q) \;\subseteq\; S(P)+\Psi_P\Big(L\sup_{x\in X}\Big|\int_\Xi f(x,\xi)(P-Q)(d\xi)\Big|\Big),$$

where $L>0$ is some constant, P the original probability distribution and Q its perburbation, and Ψ_P the conditioning function given by

$$\Psi_P(\eta):=\eta+\psi_P^{-1}(2\eta)\quad(\eta\in\mathbb{R}_+),$$

where the growth function ψ_P is

$$\psi_P(\tau):=\min\Big\{\int_\Xi f_0(x,\xi)P(d\xi)-v(P) : d(x,S(P))\geq\tau, x\in X\Big\}$$

with inverse $\psi_P^{-1}(t):=\sup\{\tau\in\mathbb{R}_+ : \psi_P(\tau)\leq t\}$. (Römisch 03)

ANOVA decomposition of two-stage integrands

Assumptions: (A1), **(A2)** and

(A3) P has a density of the form $\rho(\xi) = \prod_{j=1}^{d} \rho_j(\xi_j)$ ($\xi \in \mathbb{R}^d$) with continuous marginal densities ρ_j, $j \in D$.

(A4) All common faces of adjacent convex polyhedral sets

$$\Xi_j(x) = \{\xi \in \Xi : (q(\xi), h(\xi) - T(\xi)x) \in \mathcal{K}_j\} \quad (j = 1, \ldots, \ell)$$

do not parallel any coordinate axis for all $x \in X$ (*geometric condition*).

Proposition:

(A1) implies that the function $f(x, \cdot)$, where

$$f_x(\xi) := f(x, \xi) = \langle c, x\rangle + \Phi(q(\xi), h(\xi) - T(\xi)x) \quad (x \in X, \xi \in \Xi)$$

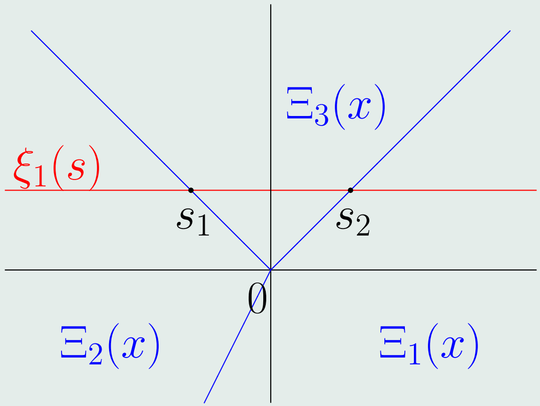
is the two-stage integrand, is continuous and piecewise linear-quadratic.

For each $x \in X$, $f(x, \cdot)$ is linear-quadratic on each convex polyhedral set $\Xi_j(x)$, $j = 1, \ldots, \ell$. It holds $\operatorname{int} \Xi_j(x) \neq \emptyset$, $\operatorname{int} \Xi_j(x) \cap \operatorname{int} \Xi_i(x) = \emptyset$, $i \neq j$, and the sets $\Xi_j(x)$, $j = 1, \ldots, \ell$, decompose Ξ. Furthermore, the intersection of two adjacent sets $\Xi_i(x)$ and $\Xi_j(x)$, $i \neq j$, is contained in some $(d-1)$-dimensional affine subspace.

To compute projections $P_k f$ for $k \in D$, let $\xi_i \in \mathbb{R}$, $i = 1, \ldots, d$, $i \neq k$, be given. We set $\xi^k = (\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_d)$ and

$$\xi_k(s) = (\xi_1, \ldots, \xi_{k-1}, s, \xi_{k+1}, \ldots, \xi_d) \in \mathbb{R}^d \quad (s \in \mathbb{R}).$$

We fix $x \in X$ and consider the one-dimensional affine subspace $\{\xi_k(s) : s \in \mathbb{R}\}$:



Example with $d = 2 = p$, where the polyhedral sets are cones

It meets the nontrivial intersections of two adjacent polyhedral sets $\Xi_i(x)$ and $\Xi_j(x)$, $i \neq j$, at finitely many points s_i, $i = 1, \ldots, p$ if all $(d-1)$-dimensional subspaces containing the intersections do not parallel the kth coordinate axis.

The $s_i = s_i(\xi^k)$, $i = 1, \ldots, p$, are affine functions of ξ^k. It holds

$$s_i = - \sum_{l=1, l\neq k}^{p} \frac{g_{il}}{g_{ik}} \xi_l + a_i \quad (i = 1, \ldots, p)$$

for some $a_i \in \mathbb{R}$ and $g_i \in \mathbb{R}^d$ belonging to an intersection of polyhedral sets.

Proposition:
Let $k \in D$, $x \in X$ and assume (A1)–(A4).
Then the kth projection $P_k f$ has the explicit representation

$$P_k f(\xi^k) = \sum_{i=1}^{p+1} \sum_{j=0}^{2} p_{ij}(\xi^k; x) \int_{s_{i-1}}^{s_i} s^j \rho_k(s) ds,$$

where $s_0 = -\infty$, $s_{p+1} = +\infty$ and $p_{ij}(\cdot; x)$ are polynomials in ξ^k of degree $2 - j$, $j = 0, 1, 2$, with coefficients depending on x, and is continuously differentiable on $\mathbb{R}^d$. $P_k f$ is s-times continuously differentiable almost everywhere on $\mathbb{R}^d$ if the marginal density ρ_k belongs to $C^{s-1}(\mathbb{R})$.

Theorem:
Let $x \in X$, assume (A1)–(A4) and $f = f(x, \cdot)$ be the two-stage integrand.
Then the second order ANOVA approximation of f

$$f^{(2)} := \sum_{|u| \leq 2} f_u \qquad \text{where} \qquad f = f^{(2)} + \sum_{|u|=3}^{d} f_u$$

belongs to $W^{(1,\ldots,1)}_{2,\rho,\mathrm{mix}}(\mathbb{R}^d)$ if all marginal densities ρ_k, $k \in D$, belong to $C^1(\mathbb{R})$.

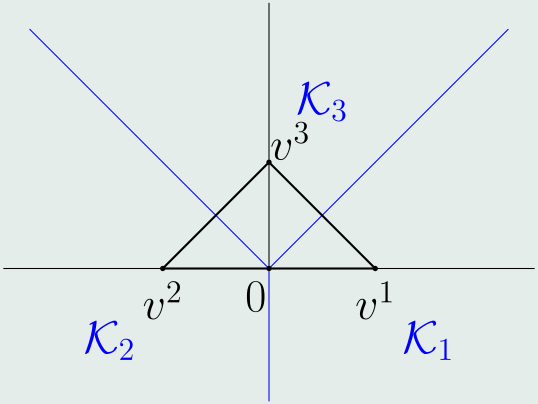
Remark:
The second order ANOVA approximation $f^{(2)}$ is a good approximation of f if the effective superposition dimension $d_S(\varepsilon)$ is at most 2. Then

$$\Big\| \sum_{|u|=3}^{d} f_u \Big\|^2_{2,\rho} = \sum_{|u|=3}^{d} \|f_u\|^2_{2,\rho} \leq \varepsilon \sigma^2(f)$$

and f belongs essentially to the tensor product Sobolev space $\mathcal{W}^{(1,\ldots,1)}_{2,\mathrm{mix}}(\mathbb{R}^d)$.
Hence, a favorable behavior of randomly shifted lattice rules may be expected.

Example: Let $\bar{m} = 3$, $d = 2$, P satisfy (A2) and (A3), $h(\xi) = \xi$, q and T be fixed and W be given such that (A1) is satisfied and the dual feasible set is

$$\{z \in \mathbb{R}^2 : -z_1 + z_2 \leq 1, z_1 + z_2 \leq 1, -z_2 \leq 0\}.$$



Dual feasible set, its vertices v^j and the normal cones $\mathcal{K}_j$ to its vertices

The function Φ and the integrand are of the form

$$\Phi(t) = \max_{i=1,2,3} \langle v^i, t \rangle = \max\{t_1, -t_1, t_2\} = \max\{|t_1|, t_2\}$$

$$f(\xi) = \langle c, x \rangle + \Phi(\xi - Tx) = \langle c, x \rangle + \max\{|\xi_1 - [Tx]_1|, \xi_2 - [Tx]_2\}$$

and the convex polyhedral sets are $\Xi_j(x) = Tx + \mathcal{K}_j$, $j = 1, 2, 3$.

The ANOVA projection $P_1 f$ is in C^1, but $P_2 f$ is not differentiable.

QMC quadrature error estimates

If the assumptions of the theorem are satisfied, one may argue

$$
\begin{aligned}
\Big| \int_{\mathbb{R}^d} f(\xi)\rho(\xi)d\xi - n^{-1}\sum_{j=1}^{n} f(\xi^j)\Big| &= \Big| \int_{[0,1]^d} g(x)dx - n^{-1}\sum_{j=1}^{n} g(x^j)\Big| \\
&\le \sum_{0<|u|\le d} \Big| \int_{[0,1]^{|u|}} g_u(x^u)dx^u - n^{-1}\sum_{j=1}^{n} g_u(x^j)\Big| \\
&\le \sum_{|u|=1}^{2} \mathrm{Disc}_{n,u}(x^1,\ldots,x^n)\|g_u\|_\gamma + \\
&\quad \sum_{|u|=3}^{d} \Big| \int_{[0,1]^d} g_u(x)dx - n^{-1}\sum_{j=1}^{n} g_u(x^j)\Big| \\
&\le C(\delta)n^{-1+\delta} + O(\sqrt{\varepsilon})
\end{aligned}
$$

if effective superposition dimension satisfies $d_S(\varepsilon) \le 2$ and g_u, $|u| = 1, 2$, belongs to the tensor product Sobolev space on $[0,1]^d$ with weighted norm $\|\cdot\|_\gamma$.

The function g is defined by

$$g(x) = \begin{cases} (f \circ \varphi^{-1})(x) & \text{if } x \in (0,1)^d, \\ 0 & \text{if } x \in [0,1]^d \setminus (0,1)^d \end{cases}$$

where

$$\varphi := (\varphi_1, \ldots, \varphi_d), \quad \varphi_i(t) := \int_{-\infty}^{t} \rho_i(s) ds \quad (i \in D).$$

Since f_u, $|u| = 1, 2$, is first and second order partially differentiable in the sense of Sobolev under certain assumptions and φ^{-1} can be assumed to be smooth, g_u, $|u| = 1, 2$, is also first and second order partially differentiable in the sense of Sobolev.

However, the derivatives of g_u are in general not quadratically integrable. Hence, the Sobolev spaces have to be modified by introducing weight functions.

(Kuo-Sloan-Wasilkowski-Waterhouse 10).

Question: How restrictive is the geometric condition (A4) ?

Partial answer: If P is normal with nonsingular covariance matrix, (A4) is a generic property. Namely, it holds

Proposition: Let $x \in X$, (A1), (A2) be satisfied and P be a normal distribution with nonsingular covariance matrix Σ.
Then for almost all covariance matrices Σ the second order ANOVA approximation $f^{(2)}$ of f belongs to the mixed Sobolev space $\mathcal{W}_{2,\rho,\mathrm{mix}}^{(1,\ldots,1)}(\mathbb{R}^d)$.

Question: For which two-stage stochastic programs is the effective superposition dimension $d_S(\varepsilon)$ of f is less than or equal to 2?

Partial answer: In case of a (log)normal probability distribution P the effective dimension depends on the mode of decomposition of the covariance matrix in order to transform the random vector to one with independent components.

Dimension reduction in case of (log)normal distributions

Let P be the normal distribution with mean μ and nonsingular covariance matrix Σ. Let A be a matrix satisfying $\Sigma = A\,A^\top$. Then η defined by $\xi = A\eta + \mu$ is standard normal.

The (lower triangular) standard Cholesky matrix $A = L_C$ performing the factorization $\Sigma = L_C L_C^\top$ seems to assign the same importance to every variable and, hence, is not suitable to reduce the effective dimension.

A universal principle is principal component analysis (PCA). Here, one uses $A = (\sqrt{\lambda_1}u_1, \ldots, \sqrt{\lambda_d}u_d)$, where $\lambda_1 \geq \cdots \geq \lambda_d > 0$ are the eigenvalues of Σ in decreasing order and the corresponding orthonormal eigenvectors u_i, $i = 1, \ldots, d$. Wang-Fang 03, Wang-Sloan 05 report an enormous reduction of the effective truncation dimension in financial models if PCA is used. Our numerical results confirm this observation.

However, there is no consistent dimension reduction effect for any such matrix A (Papageorgiou 02, Wang-Sloan 11).

Some computational experience

We consider a stochastic production planning problem which consists in minimizing the expected costs of a company during a certain time horizon. The model contains stochastic demands ξ_δ and prices ξ_c as components of

$$\xi = (\xi_{\delta,1}, \ldots, \xi_{\delta,T}, \xi_{c,1}, \ldots, \xi_{c,T})^\top.$$

The company aims to satisfy stochastic demands $\xi_{\delta,t}$ in a time horizon $\{1, \ldots, T\}$, but its production capacity based on their own units does eventually not suffice to cover the demand. The model is of the form

$$\max \Big\{ \sum_{t=1}^{T} \Big(c_t^\top x_t + \int_{\mathbb{R}^T} q_t(\xi)^\top y_t P(d\xi) \Big) : Wy + Vx = h(\xi), y \geq 0, x \in X \Big\}$$

We assume that the stochastic demands and prices $\xi_{\delta,t}, \xi_{c,t}$ may be modeled as a multivariate ARMA(1,1) process, i.e.,

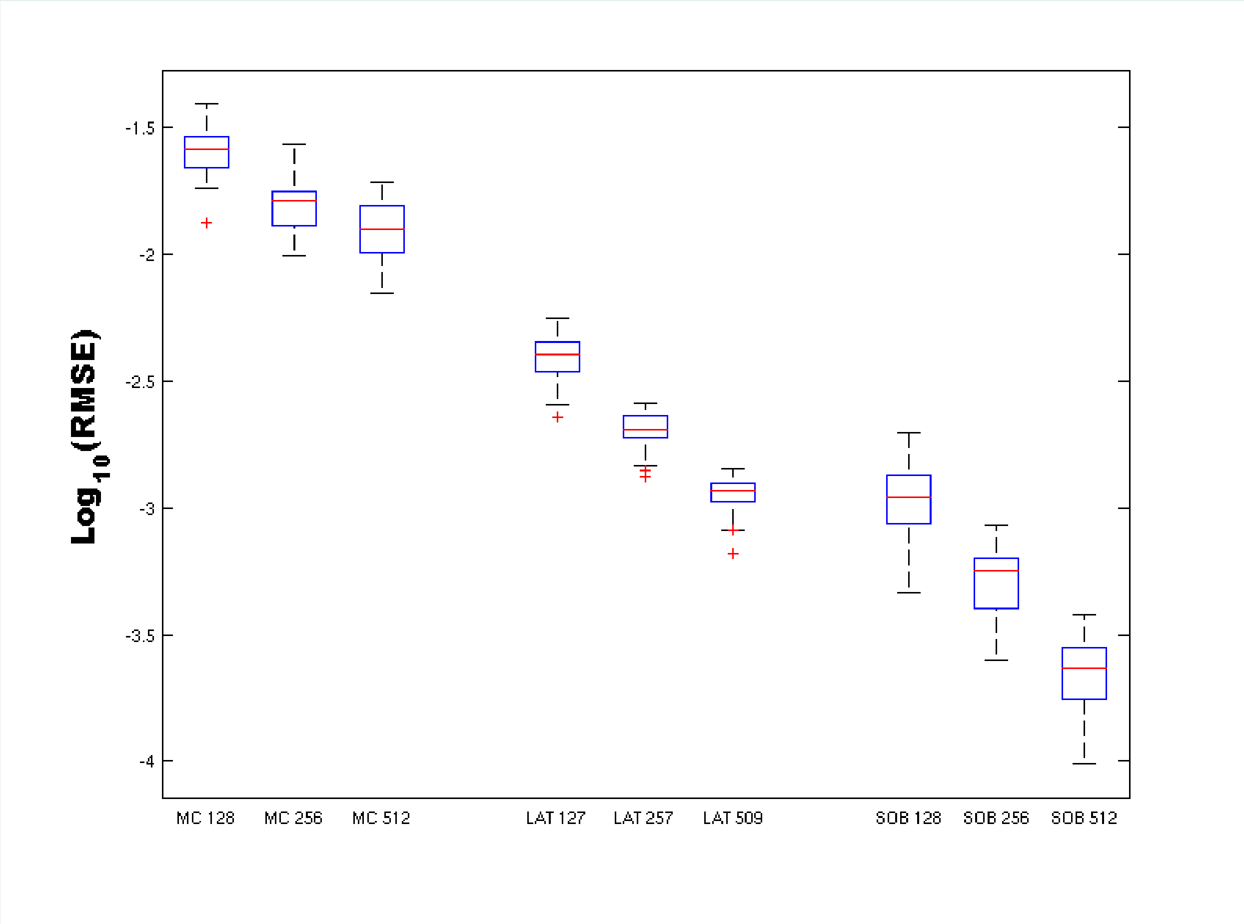
$$\begin{pmatrix} \xi_{\delta,t} \\ \xi_{c,t} \end{pmatrix} = \begin{pmatrix} \bar{\xi}_{\delta,t} \\ \bar{\xi}_{c,t} \end{pmatrix} + \begin{pmatrix} E_{1,t} \\ E_{2,t} \end{pmatrix}, \quad \text{for } t = 1, \ldots, T, \text{ and}$$

$$\begin{pmatrix} \bar{\xi}_{\delta,1} \\ \bar{\xi}_{c,1} \end{pmatrix} = B_1 \begin{pmatrix} \gamma_{1,1} \\ \gamma_{2,1} \end{pmatrix}, \quad \begin{pmatrix} \bar{\xi}_{\delta,t} \\ \bar{\xi}_{c,t} \end{pmatrix} = A \begin{pmatrix} \bar{\xi}_{\delta,t-1} \\ \bar{\xi}_{c,t-1} \end{pmatrix} + B_1 \begin{pmatrix} \gamma_{1,t} \\ \gamma_{2,t} \end{pmatrix} + B_2 \begin{pmatrix} \gamma_{1,t-1} \\ \gamma_{2,t-1} \end{pmatrix}$$

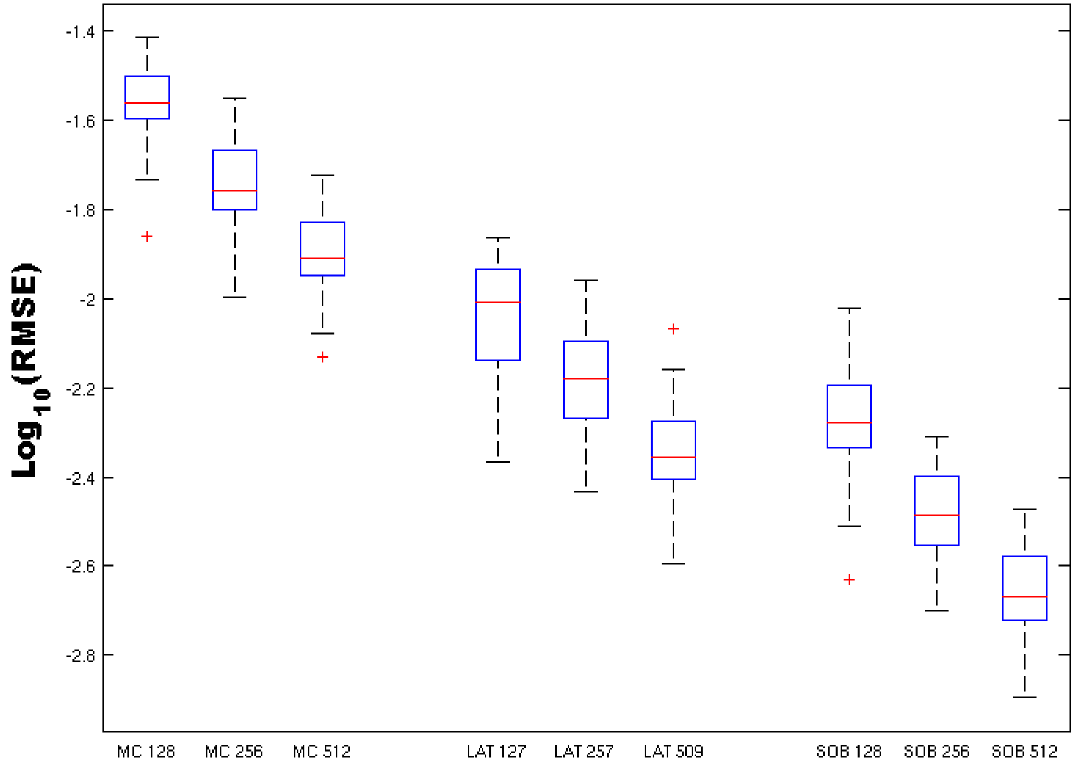
for $t = 2, \ldots, T$, where $\gamma_{1,t}, \gamma_{2,t} \sim$ N(0,1) and i.i.d. and $T = 100$.

We used PCA and CH for decomposing the covariance matrix of ξ. PCA has led to effective truncation dimension $d_T(0.01) = 2$ while for CH $d_T(0.01) = 200$. As QMC methods we used a randomly scrambled Sobol sequence (SSobol) and a randomly shifted lattice rule (Sloan-Kuo-Joe) with weights $\gamma_j = \frac{1}{j^3}$ and for MC the Mersenne-Twister.
We used $n = 128, 256, 512$ for the Mersenne Twister and for Sobol' points. For randomly shifted lattices we used $n = 127, 257, 509$. The random shifts were generated using the Mersenne Twister. We estimated the relative root mean square errors (RMSE) of the optimal costs by taking 10 runs for each experiment, and repeat the process 30 times for the box plots in the figures.
The average of the estimated rates of convergence under PCA was approximately -0.9 for randomly shifted lattice rules, and -1.0 for the randomly scrambled Sobol' points. This is clearly superior compared to the MC rate -0.5.

The box-plots show the first quartile as lower bound of the box, the third quartile as upper bound and the median as line between the bounds, Outliers are marked as plus signs and the rest of the results lie between the brackets.



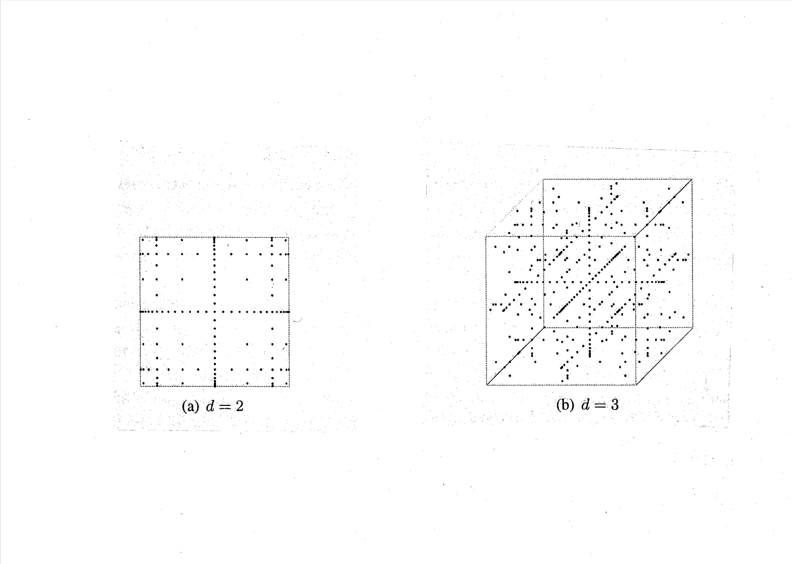
$\log_{10}$ of the relative errors of MC, SLA (randomly shifted lattice rule) and SSOB (scrambled Sobol' points) using PCA



$\log_{10}$ of the relative errors of MC, SLA (randomly shifted lattice rule) and SSOB (scrambled Sobol' points) using Cholesky

Conclusions

- Our analysis provides a theoretical basis for applying QMC methods accompanied by dimension reduction techniques to two-stage stochastic programs.
- The analysis also applies to sparse grid quadrature techniques.



Sparse grids in the unit cube $[0, 1]^d$

- The results seem to be extendable to mixed-integer two-stage models, to multi-stage situations, and to models with chance and dominance constraints.

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