Stochastic Programming:
A Variational Analysis Perspective

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Introduction

What is Stochastic Programming ?

- Mathematics for Decision Making under Uncertainty
- subfield of Mathematical Programming (MSC 90C15)

Stochastic programs are **optimization models**

- having special properties and structures,
- depending on the underlying probability distribution,
- requiring specific approximation and numerical approaches,
- having close relations to practical applications.

Selected recent monographs:

P. Kall/S.W. Wallace 1994, A. Prekopa 1995, J.R. Birge/F. Louveaux 1997, J. Mayer/P. Kall 2005

A. Ruszczynski/A. Shapiro (eds.), Stochastic Programming, Handbook, Elsevier, 2003

S.W. Wallace/W.T. Ziemba (eds.), Applications of Stochastic Programming, MPS-SIAM Series on Optimization, 2005.

Contents:

Part I

Chance Constraints and Nonsmooth Analysis

(R. Henrion (Berlin))

Optimization models under stochastic uncertainty

Let us consider the optimization model

$$\min\{f(\xi, x) \,:\, x \in X,\, g(\xi, x) \leq 0\}\,,$$

where $\xi : \Omega \to \Xi$ is a random vector defined on a probability space $(\Omega, \mathcal{F}, I\!P)$, Ξ and X are closed subsets of $I\!R^s$ and $I\!R^m$, respectively, $f : \Xi \times X \to I\!R$ and $g : \Xi \times X \to I\!R^d$ are lower semicontinuous.

Aim: Finding optimal decisions before knowing the random outcome of ξ (here-and-now decision).

Main approaches:

- Replace the objective by $I\!E[f(\xi, x)]$ or by $I\!F[f(\xi, x)]$, where $I\!E$ denotes expectation (w.r.t. $I\!P$) and $I\!F$ some functional on the space of real random variables (e.g., playing the role of a *risk functional*).

- Replace the random constraints by the constraint

$$\mathbb{P}(\{\omega \in \Omega : g(\xi(\omega), x) \leq 0\}) = \mathbb{P}(g(\xi, x) \leq 0) \geq p$$

  where $p \in [0, 1]$ denotes a probability level, or go back to the *modeling stage* and introduce a recourse action to compensate violations of the constraint.

The first variant leads to stochastic programs with probabilistic or chance constraints:

$$\min\{\mathbb{E}[f(\xi, x)] \, : \, x \in X, \, \mathbb{P}(g(\xi, x) \leq 0) \geq p\}$$

Problem:
If the original optimization problem is smooth, convex or even linear, the probabilistic constraint function

$$G(x) := \mathbb{P}(g(\xi, x) \leq 0)$$

may be non-differentiable, non-Lipschitzian and non-convex.

Properties of chance constraints

Special forms of chance constraints:

- $g(\xi, x) := \xi - h(x)$, where $h : \mathbb{R}^m \to \mathbb{R}^s$, i.e.,

  $$G(x) = \mathbb{P}(\xi \leq h(x)) = F_\mu(h(x)) \geq p\,,$$

  where $F_\mu(y) := \mathbb{P}(\{\omega \in \Omega : \xi(\omega) \leq y\}) = \mu(\{\xi \in \Xi : \xi \leq y\})$ $(y \in \mathbb{R}^s)$ denotes the (multivariate) probability distribution function of ξ and $\mu := \mathbb{P} \cdot \xi^{-1}$ its probability distribution.

- $g(\xi, x) := b(\xi) - A(\xi)x$, where the matrix $A(\cdot)$ and the vector $b(\cdot)$ are affine functions of ξ, i.e.,

  $$G(x) := \mu(\{\xi \in \Xi : A(\xi)x \geq b(\xi)\}) \geq p.$$

Proposition: (Prekopa)
If $H : I\!R^m \to I\!R^s$ is a set-valued mapping with closed graph, the function $G : I\!R^m \to I\!R$ defined by $G(x) := \mu(H(x))$ $(x \in I\!R^m)$ is upper semicontinuous for every probability distribution μ on $I\!R^s$. Hence, the feasible set

$$\mathcal{X}_p(\mu) = \{x \in X : G(x) = \mu(H(x)) \geq p\}$$

is closed.

What about continuity and differentiability properties of G or convexity of $\mathcal{X}_p(\mu)$?

Examples:

(i) Let $H(x) = x + \mathbb{R}^s_-$ $(\forall x \in \mathbb{R}^s)$ and μ be discrete with finite support, i.e., $\mu = \sum_{i=1}^n p_i \delta_{\xi_i}$, where δ_ξ denotes the Dirac measure placing unit mass at ξ and $p_i > 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n p_i = 1$.
Then $\mathcal{X}_p(\mu) = X \cap (\cup_{i \in I}(\xi_i + \mathbb{R}^s_+))$ holds with some index set $I \subset \{1, \ldots, n\}$ and, hence, it is non-convex in general.
Moreover, $G = F_\mu$ is discontinuous with jumps at $\mathrm{bd}(\xi_i + \mathbb{R}^s_-)$.

(ii) Let $H(x) = x + \mathbb{R}^s_-$ $(\forall x \in \mathbb{R}^s)$ and μ have a density f_μ with respect to the Lebesgue measure on $\mathbb{R}^s$, i.e.,

$$G(x) = F_\mu(x) = \int_{-\infty}^{x} f_\mu(y)dy = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_s} f_\mu(y_1, \ldots, y_s)dy_s \cdots dy_1.$$

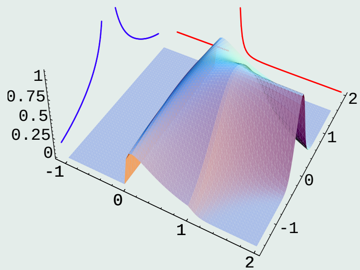
Conjecture: $G = F_\mu$ is Lipschitz continuous if the density f_μ is continuous and bounded.

Answer: The conjecture is true for $s = 1$, but holds no longer for $s > 1$ in general.

Example: (A. Wakolbinger)

$$f_\mu(x_1, x_2) = \begin{cases} 0 & x_1 < 0 \\ c x_1^{1/4} e^{-x_1 x_2^2} & x_1 \in [0, 1] \\ c e^{-x_1^4 x_2^2} & x_1 > 1, \end{cases}$$

where c is chosen such that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_\mu(x_1, x_2) dx_1 dx_2 = 1$.



The density f_μ is continuous and bounded. However, F_μ is not locally Lipschitz continuous (as the marginal density functions are not bounded).

Proposition:

A probability distribution function F_μ with density f_μ is locally Lipschitz continuous if its (one-dimensional) marginal density functions f_μ^i, $i = 1, \ldots, s$, are locally bounded.

F_μ is (globally) Lipschitz continuous iff its marginal density functions are bounded.

$$f_\mu^i(x_i) := \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_\mu(x_1, \ldots, x_s) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_s$$

Is there a reasonable class of probability distributions to which the proposition applies ?

Definition:

A probability measure $\mu \in \mathcal{P}(\mathbb{R}^s)$ is called quasi-concave whenever

$$\mu(\lambda B + (1-\lambda)\tilde{B}) \geq \min\{\mu(B), \mu(\tilde{B})\}$$

holds true for all Borel measurable convex subsets B, $\tilde{B} \subseteq \mathbb{R}^s$ and all $\lambda \in [0,1]$ such that $\lambda B + (1-\lambda)\tilde{B}$ is Borel measurable.

Proposition: (Prekopa)

If $H : \mathbb{R}^m \rightarrow \mathbb{R}^s$ is a set-valued mapping with closed convex graph and $\mu \in \mathcal{P}(\mathbb{R}^s)$ is quasi-concave, the function $G(x) := \mu(H(x))$ ($x \in \mathbb{R}^m$) is quasi-concave. Hence, if X is closed and convex, the feasible set

$$\mathcal{X}_p(\mu) = \{x \in X : G(x) = \mu(H(x)) \geq p\}$$

is closed and convex.

Proof: Let $x, \tilde{x} \in \mathbb{R}^m$, $\lambda \in [0,1]$.

$$\begin{aligned} G(\lambda x + (1-\lambda)\tilde{x}) &= \mu(H(\lambda x + (1-\lambda)\tilde{x})) \geq \mu(\lambda H(x) + (1-\lambda)H(\tilde{x})) \\ &\geq \min\{\mu(H(x)), \mu(H(\tilde{x}))\} = \min\{G(x), G(\tilde{x})\}. \end{aligned}$$

Theorem: (Borell 75)
If $\mu \in \mathcal{P}(\mathbb{R}^s)$ is quasi-concave and has a density f_μ, the function $f_\mu^{-\frac{1}{s}} : \mathbb{R}^s \to \overline{\mathbb{R}}$ is convex.

Theorem: (Henrion/Römisch 05)
The probability distribution function F_μ of a quasi-concave probability measure $\mu \in \mathcal{P}(\mathbb{R}^s)$ is Lipschitz continuous iff $\operatorname{supp} \mu$ is not contained in a $(s-1)$-dimensional hyperplane.

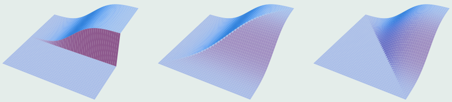
Question: Are distribution functions of quasi-concave measures differentiable, too ?

Examples: (of quasi-concave probability measures)
Multivariate normal distributions $N(m, C)$ (with mean $m \in \mathbb{R}^s$ and $s \times s$ symmetric, positive semidefinite covariance matrix C; nondegenerate or singular), uniform distributions on convex compact subsets of $\mathbb{R}^s$, Dirichlet-, Pareto-, Gamma-distributions etc.

Example: (singular normal distributions)
The probability distribution functions F_μ of 2-dimensional normal distributions $N(0, C)$ with

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

are not differentiable on $\mathbb{R}^2$.



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Theorem: (Henrion/Römisch 05)
Let ξ be an s-dimensional normal random vector whose covariance matrix is nonsingular. Let F_η denote the probability distribution function of the random vector $\eta = A\xi + b$ where A is an $m \times s$-matrix and $b \in I\!R^m$.
Then F_η is infinitely many times differentiable at any $\bar{x} \in I\!R^m$ for which the system $(A, \bar{x} - b)$ satisfies the *Linear Independence Constraint Qualification* (LICQ), i.e., the rows a_i, $i = 1, \ldots, m$, of A satisfy the condition $\operatorname{rank}\{a_i : i \in I\} = \#I$ for every index set $I \in \{1, \ldots, m\}$ such that there exists $z \in I\!R^s$ with

$$a_i^T z = \bar{x}_i - b_i \quad (i \in I), \quad a_i^T z < \bar{x}_i - b_i \quad (i \in \{1, \ldots, m\} \setminus I).$$

Example:
Our second example of singular normal distributions corresponds to the probability distribution function F_η of

$$\eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xi, \quad \xi \sim N(0,1).$$

The result implies the C^∞-property of F_η on $R^2 \setminus \{(x,x) : x \in I\!R\}$.

Example: (Henrion)

Let $\mu \in \mathcal{P}(I\!R)$ be the standard normal $(N(0,1))$ distribution with probability distribution function

$$\Phi(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{x} \exp(-\frac{\xi^2}{2}) d\xi,$$

$A = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $b(\xi) = \begin{pmatrix} \xi \\ \xi \end{pmatrix}$ for each $\xi \in I\!R$. Then we have

$$\begin{aligned} G(x) &= \mu(\{\xi \in I\!R : Ax \geq b(\xi)\}) \\ &= \mu(\{\xi \in I\!R : x \geq \xi, -x \geq \xi\}) = \Phi(\min\{-x, x\}). \end{aligned}$$

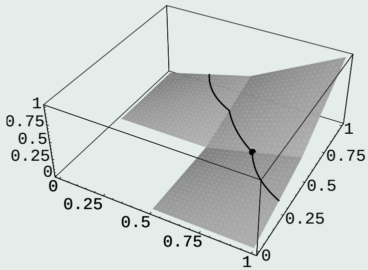
Hence, although Φ is in $C^\infty(I\!R)$, G is non-differentiable.

Example: (Henrion/Römisch 99)

Let $m = s = 2$, $X = [0,2] \times [0,2]$, $A := I$, $p = 1/6$ and μ be the uniform distribution on $\Xi := ([0,1] \times [0,1]) \setminus ([0,1/2] \times [0,1/2])$. Around the feasible point $\bar{x} = (3/4, 1/2)$ (the probability level is binding at $\bar{x}$) the constraint function is of the form

$$G(x) := F_\mu(x) = 4/3 \max\{x_2(x_1 - 1/2), x_1(x_2 - 1/2), x_1 x_2 - 1/4\}$$

and is non-differentiable at $\bar{x}$, although $\bar{x}$ lies in the interior of the support of the underlying constant density. Note that μ is not quasi-concave since the support of μ is non-convex.



Metric regularity of chance constraints

Let $H : \mathbb{R}^m \to \mathbb{R}^s$ be a set-valued mapping with closed graph, $X \subseteq \mathbb{R}^m$ be closed and $\mu \in \mathcal{P}(\mathbb{R}^s)$. We consider the set-valued mapping (from $\mathbb{R}$ to $\mathbb{R}^m$)

$$y \mapsto \mathcal{X}_y(\mu) = \{x \in X : \mu(H(x)) \geq y\}.$$

Definition:
The chance constraint function $\mu(H(\cdot)) - p$ is metrically regular with respect to X at $\bar{x} \in \mathcal{X}_p(\mu)$ if there exist positive constants a and ε such that

$$d(x, \mathcal{X}_y(\mu)) \leq a \max\{0, y - \mu(H(x))\}$$

holds for all $x \in X \cap \mathbb{B}(\bar{x}, \varepsilon)$ and $|p - y| \leq \varepsilon$.

Motivation: Continuity properties of the feasible set $\mathcal{X}_p(\mu)$ with respect to perturbations of $\mu \in \mathcal{P}(\mathbb{R}^s)$ measured in terms of a suitable distance on $\mathcal{P}(\mathbb{R}^s)$, e.g., the $\mathcal{B}$-discrepancy
$\alpha_{\mathcal{B}}(\mu, \nu) := \sup_{B \in \mathcal{B}} |\mu(B) - \nu(B)|$ with $\mathcal{B} := \{H(x) : x \in X\}$.

The convex case

Proposition: (Römisch/Schultz 91)
Let the set-valued mapping H have closed and convex graph, X be closed and convex, $p \in (0,1)$ and $\mu \in \mathcal{P}(\mathbb{R}^s)$ be r-concave for some $r \in (-\infty, +\infty]$. Suppose there exists a Slater point $\bar{x} \in X$ such that $\mu(H(\bar{x})) > p$.
Then $\mu(H(\cdot)) - p$ is metrically regular with respect to X at each $x \in \mathcal{X}_p(\mu)$.

The proof is based on the Robinson-Ursescu theorem applied to the set-valued mapping $\Gamma(x) := \{v \in \mathbb{R} : x \in X, p^r - (\mu(H(x)))^r \geq v\}$ for some $r < 0$ (w.l.o.g.).

The proposition applies to $H(x) = \{\xi \in \mathbb{R}^s : h(x) \geq \xi\}$, i.e., $\mu(H(x)) = F_\mu(h(x))$, where h has concave components. However, even for $h(x) = Ax$ the matrix A has to be non-stochastic.
For stochastic A there exist only specific results (Henrion/Strugarek 06).
Metric regularity results for the general case are an open problem.

Definition:

A probability measure $\mu \in \mathcal{P}(\mathbb{R}^s)$ is called r- concave for some $r \in [-\infty, +\infty]$ if the inequality

$$\mu(\lambda B + (1-\lambda)\tilde{B}) \geq m_r(\mu(B), \mu(\tilde{B}); \lambda)$$

holds for all $\lambda \in [0,1]$ and all convex Borel subsets $B, \tilde{B}$ of $\mathbb{R}^s$ such that $\lambda B + (1-\lambda)\tilde{B}$ is Borel.

Here, the generalized mean function m_r on $\mathbb{R}_+ \times \mathbb{R}_+ \times [0,1]$ for $r \in [-\infty, \infty]$ is given by

$$m_r(a,b;\lambda) := \begin{cases} (\lambda a^r + (1-\lambda)b^r)^{1/r} & , \; r > 0 \text{ or } r < 0, ab > 0, \\ 0 & , \; ab = 0, r < 0, \\ a^\lambda b^{1-\lambda} & , \; r = 0, \\ \max\{a,b\} & , \; r = \infty, \\ \min\{a,b\} & , \; r = -\infty. \end{cases}$$

Notice that $r = -\infty$ corresponds to quasi-concavity.

A non-convex and non-differentiable situation

Proposition: (Henrion/Römisch 99)
Let $\mu \in \mathcal{P}(\mathbb{R}^s)$ have a density f_μ on $\mathbb{R}^s$, $p \in (0,1)$ and $h : \mathbb{R}^m \to \mathbb{R}^s$ be locally Lipschitz continuous.
Then $F_\mu(h(\cdot)) - p$ is metrically regular with respect to X at $\bar{x} \in \mathcal{X}_p(\mu)$ if the following conditions are satisfied:

(i) $(h(\bar{x}) + \mathrm{bd}\, \mathbb{R}^s_-) \cap \mathcal{D}^+ \neq \emptyset$ if $F_\mu(h(\bar{x})) = p$, where
$\mathcal{D}^+ := \{\xi \in \mathbb{R}^s : \exists \varepsilon > 0$ such that $f_\mu(z) \geq \varepsilon, \forall z \in \mathbb{B}(\xi, \varepsilon)\}$.

(ii) $\partial_a \langle y^*, h \rangle(\bar{x}) \cap (-N_a(X; \bar{x})) = \emptyset$, $\forall y^* \in \mathbb{R}^s_- \setminus \{0\}$, where
N_a and ∂_a denote the approximate normal cone and subdifferential (of Mordukhovich), respectively.

(decomposition into growth condition and constraint qualification)

Corollary:
If X is closed and convex, and all components of h are concave, condition (ii) is satisfied if there exists $\hat{x} \in X$ with $h(\hat{x}) > h(\bar{x})$.

Illustration of quantitative stability results

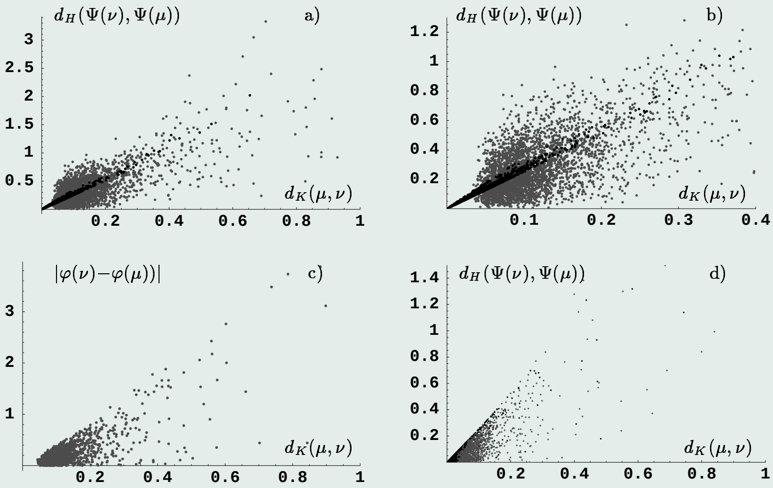
We consider the 2-dimensional example

$$\min\{x_1 + x_2 | I\!P(\xi_1 \le x_1, \xi_2 \le x_2) \ge 1/2\},$$

where ξ is assumed to have a distribution μ which is normal with independent $N(0,1)$ components. The solution set consists of a singleton $\Psi(\mu) = \{(q,q)\}$, where $q \approx 0.55$ is the $1/\sqrt{2}$-quantile of the $N(0,1)$ distribution.

We consider two specific approximations of μ:
(i) The empirical measure $\nu = N^{-1} \sum_{i=1}^{N} \delta_{\xi_i}$, where ξ_i, $i = 1, \ldots, N$ are i.i.d. observations of ξ.
(ii) A parametric estimate for the mean m and the covariance matrix C in $\nu \sim N(m, C)$.

The figure shows $d_H(\Psi(\mu), \Psi(\nu))$ relative to the Kolmogorov distance $d_K(\mu, \nu) = \sup_{\xi \in I\!R^2} |F_\mu(\xi) - F_\nu(\xi)|$. The grey dots correspond to the empirical estimates and the black dots to the parametric estimates.


$d_H(\Psi(\nu),\Psi(\mu))$
a)
3
2.5
2
1.5
1
0.5
$d_K(\mu,\nu)$
0.2 0.4 0.6 0.8 1
$d_H(\Psi(\nu),\Psi(\mu))$
b)
1.2
1
0.8
0.6
0.4
0.2
$d_K(\mu,\nu)$
0.1 0.2 0.3 0.4
$|\varphi(\nu)-\varphi(\mu))|$
c)
3
2
1
$d_K(\mu,\nu)$
0.2 0.4 0.6 0.8 1
$d_H(\Psi(\nu),\Psi(\mu))$
d)
1.4
1.2
1
0.8
0.6
0.4
0.2
$d_K(\mu,\nu)$
0.2 0.4 0.6 0.8 1

Part II

Solution Estimates for Two-Stage Models

(S. T. Rachev (Karlsruhe), R. J-B Wets (Davis))

Stochastic programming models with recourse

Consider a linear program with stochastic parameters of the form

$$\min\{\langle c, x\rangle : x \in X,\, T(\xi)x = h(\xi)\},$$

where $\xi : \Omega \to \Xi$ is a random vector defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $c \in \mathbb{R}^m$, Ξ and X are polyhedral subsets of $\mathbb{R}^s$ and $\mathbb{R}^m$, respectively, and the $d \times m$-matrix $T(\cdot)$ and vector $h(\cdot) \in \mathbb{R}^d$ are affine functions of ξ.

Idea:

Introduce a recourse variable $y \in \mathbb{R}^{\overline{m}}$, recourse costs $q(\xi) \in \mathbb{R}^{\overline{m}}$, recourse $d \times \overline{m}$-matrix $W(\xi)$ and a (deterministic) polyhedral cone $Y \subseteq \mathbb{R}^{\overline{m}}$, and solve the second-stage or recourse program

$$\min\{\langle q(\xi), y\rangle : y \in Y, W(\xi)y = h(\xi) - T(\xi)x\}.$$

Add the expected minimal recourse costs $\mathbb{E}[\hat{\Phi}(\xi, x)]$ (depending on the first-stage decision x) to the original objective and solve

$$\min\left\{\langle c, x\rangle + \mathbb{E}[\hat{\Phi}(\xi, x)] : x \in X\right\},$$

where $\hat{\Phi}(\xi, x) := \inf\{\langle q(\xi), y\rangle : y \in Y, W(\xi)y = h(\xi) - T(\xi)x\}$.

Two formulations of two-stage models

Deterministic equivalent of the two-stage model:

$$\min\left\{\langle c,x\rangle + \int_\Xi \hat{\Phi}(\xi,x)P(d\xi) : x \in X\right\},$$

where $P := I\!\!P\xi^{-1} \in \mathcal{P}(\Xi)$ is the probability distribution of the random vector ξ and $\hat{\Phi}(\cdot,\cdot)$ is the infimum function of the second-stage program.

Infinite-dimensional optimization model:

$$\min\left\{\langle c,x\rangle \ + \ \int_\Xi \langle q(\xi), y(\xi)\rangle P(d\xi) : x \in X,\ y \in L_r(\Xi, \mathcal{B}(\Xi), P),\right.$$
$$\left. y(\xi) \in Y,\ W(\xi)y(\xi) = h(\xi) - T(\xi)x\right\},$$

where $r \in [1,+\infty]$ is selected properly.

If the probability distribution P of ξ is assumed to have p-th order moments, i.e., $\int_\Xi \|\xi\|^p P(d\xi) < \infty$, with $p > 1$, r should be chosen such that the constraints of y are consistent with these moment conditions and $I\!\!E[\langle q(\xi), y(\xi)\rangle]$ is finite. For example, if the recourse matrix is fixed (i.e., $W(\xi) \equiv W$), $r = \frac{p}{p-1}$ is consistent.

Structural properties of two-stage models

We consider the infimum function of the parametrized linear (second-stage) program and the dual feasible set of the second-stage program, namely,

$$\Phi(\xi,u,t):=\inf\{\langle u,y\rangle : W(\xi)y=t,\, y\in Y\}\;((\xi,u,t)\in\Xi\times\mathbb{R}^{\overline{m}}\times\mathbb{R}^{d})$$
$$D(\xi):=\{z\in\mathbb{R}^{r} : W(\xi)^{\top}z-q(\xi)\in Y^{*}\}\;(\xi\in\Xi),$$

where $W(\xi)^{\top}$ is the transposed of $W(\xi)$ and Y^{*} the polar cone of Y. Then we have

$$\hat{\Phi}(\xi,x)=\Phi(\xi,q(\xi),h(\xi)-T(\xi)x)=\sup\{\langle h(\xi)-T(\xi)x,z\rangle : z\in D(\xi)\}.$$

Theorem: (Walkup/Wets 69)
For any $\xi\in\Xi$, the function $\Phi(\xi,\cdot,\cdot)$ is finite and continuous on the polyhedral set $D(\xi)\times W(\xi)Y$. Furthermore, the function $\Phi(\xi,u,\cdot)$ is piecewise linear convex on the polyhedral set $W(\xi)Y$ for fixed $u\in D(\xi)$, and $\Phi(\xi,\cdot,t)$ is piecewise linear concave on $D(\xi)$ for fixed $t\in W(\xi)Y$.

Assumptions:

(A1) *relatively complete recourse*: for any $(\xi, x) \in \Xi \times X$, $h(\xi) - T(\xi)x \in W(\xi)Y$;

(A2) *dual feasibility*: $D(\xi) \neq \emptyset$ holds for all $\xi \in \Xi$.

Note that (A1) is satisfied if $W(\xi)Y = \mathbb{R}^d$ (complete recourse). In general, (A1) and (A2) impose a condition on the support of P.

Proposition:

Then the deterministic equivalent of the two-stage model represents a finite convex program (with polyhedral constraints) if the integrals $\int_\Xi \Phi(\xi, q(\xi), h(\xi) - T(\xi)x)P(d\xi)$ are finite for all $x \in X$.
For fixed recourse ($W(\xi) \equiv W$), it suffices to assume

$$\int_\Xi \|\xi\|^2 P(d\xi) < \infty.$$

Convex subdifferentials, optimality conditions, conditions for differentiability, duality results are well known.

(Ruszczyński/Shapiro, Handbook, 2003)

Towards stability

We define the integrand $f_0 : \Xi \times I\!R^m \to \overline{I\!R}$ by

$$f_0(\xi, x) = \begin{cases} \langle c, x\rangle + \Phi(\xi, q(\xi), h(\xi) - T(\xi)x) & \text{if } h(\xi) - T(\xi)x \in W(\xi)Y,\ D(\xi) \neq \emptyset, \\ +\infty & \text{otherwise,} \end{cases}$$

and note that f_0 is a convex random lsc function with $\Xi \times X \subseteq \text{dom } f_0$ if (A1) and (A2) are satisfied.

The two-stage stochastic program can thus be expressed as

$$\min\Big\{ \int_{\Xi} \mathbf{f_0}(\xi, \mathbf{x}) \mathbf{P}(\mathbf{d}\xi) : \ \mathbf{x} \in \mathbf{X} \Big\}.$$

We are interested in studying the behavior of its solutions when perturbing (approximating, estimating) the probability measure P. By $v(P)$, $S(P)$ and $S_\varepsilon(P)$ ($\varepsilon \geq 0$) we denote its optimal value, solution set and set of ε-approximate solutions, i.e.,

$$v(P) := \inf\{\int_\Xi f_0(\xi,x)P(d\xi) : x\in X\}$$
$$S(P) := \text{argmin}_{x\in X}\int_\Xi f_0(\xi,x)P(d\xi) := S_0(P),$$
$$S_\varepsilon(P) := \{x\in X : \int_\Xi f_0(\xi,x)P(d\xi)\le v(P)+\varepsilon\}.$$

We consider classes of relevant functions and probability measures, namely, $\mathcal{F} = \{f_0(\cdot,x) : x\in X\}$ and

$$\mathcal{P}_\mathcal{F} = \{Q\in\mathcal{P}(\Xi) : \int_\Xi \inf_{x\in X\cap\rho\mathbb{B}} f_0(\xi,x)Q(d\xi) > -\infty \text{ , and } \sup_{x\in X\cap\rho\mathbb{B}}\int_\Xi f_0(\xi,x)Q(d\xi)<\infty \text{ , for all } \rho>0\},$$

where $\mathbb{B}$ is the closed unit ball in $\mathbb{R}^m$. We note that the infimum function $\xi\mapsto\inf_{x\in X\cap\rho\mathbb{B}} f_0(\xi,x)$ is measurable for each $\rho>0$ as f_0 is a random lsc function.

For any $\rho > 0$ and probability measures P, $Q \in \mathcal{P}_{\mathcal{F}}$ we consider the following distance

$$d_{\mathcal{F},\rho}(P,Q) = \sup_{x \in X \cap \rho \mathbb{B}} \left| \int_{\Xi} f_0(\xi,x) P(d\xi) - \int_{\Xi} f_0(\xi,x) Q(d\xi) \right| .$$

It is nonnegative, finite, symmetric and satisfies the triangle inequality, i.e., it is a semi-metric on $\mathcal{P}_{\mathcal{F}}$. In general, however, the class $\mathcal{F}_\rho$ will not be rich enough to guarantee $d_{\mathcal{F},\rho}(P,Q) = 0$ implies $P = Q$.

Lemma:

For any $Q \in \mathcal{P}_{\mathcal{F}}$, the function $x \mapsto \int_{\Xi} f_0(\xi,x) Q(d\xi)$ is convex and lsc on $\mathbb{R}^m$.

Proof by using Fatou's lemma.

Set- and epi-distances (Rockafellar/Wets 98)

Let $d_C(x) = d(x, C) = \inf_{y\in C} \|x - y\|$ denote the distance of $x \in I\!R^m$ to a non-empty closed subset of $I\!R^m$. The ρ−distance between two non-empty closed sets is by definition

$$d\!l_\rho(C, D) = \sup_{||x||\le\rho} |d_C(x) - d_D(x)|.$$

In fact, it is just a semi-distance from which one can build a metric on the hyperspace of closed sets (metrizing the topology of Painlevé-Kuratowski convergence), for example, by setting

$$d\!l(C, D) = \int_0^\infty d\!l_\rho(C, D)e^{-\rho}\, d\rho.$$

Estimates for the ρ-distance can be obtained by relying on a 'truncated' Pompeiu-Hausdorff type distance:

$$\hat{d\!l}_\rho(C, D) = \inf\{\eta \ge 0 : C \cap \rho I\!B \subset D + \eta I\!B; D \cap \rho I\!B \subset C + \eta I\!B\}.$$

Indeed one always has,

$$\hat{d\!l}_\rho(C_1, C_2) \le d\!l_\rho(C_1, C_2) \le \hat{d\!l}_{\rho'}(C_1, C_2)$$

for $\rho' \ge 2\rho + \max\{d_{C_1}(0), d_{C_2}(0)\}$.

If we let $\rho \to \infty$, we end up with $dl_\rho(C,D)$ and $\hat{dl}_\rho(C,D)$ tending to $dl_\infty(C,D)$, the Pompeiu-Hausdorff distance between the closed non-empty sets C and D.

The distance between (lsc) functions is measured in terms of the distance between their epigraphs, so for $\rho > 0$,

$$dl_\rho(f,g) = dl_\rho(\operatorname{epi} f, \operatorname{epi} g), \qquad \hat{dl}_\rho(f,g) = \hat{dl}_\rho(\operatorname{epi} f, \operatorname{epi} g).$$

and $dl(f,g) = dl(\operatorname{epi} f, \operatorname{epi} g)$. However, since our sets are epigraphs (in $\mathbb{R}^{m+1}$), it is convenient to rely on the 'unit ball' to be $\mathbb{B} \times [-1,1]$, this brings us to an 'auxiliary' distance $\hat{dl}^+_\rho(f_1,f_2)$ defined as the infimum of all $\eta \geq 0$ such that for all $x \in \rho\mathbb{B}$,

$$\begin{aligned} \min\nolimits_{B(x,\eta)} f_2 &\leq \max\{ f_1(x), -\rho \} + \eta \\ \min\nolimits_{B(x,\eta)} f_1 &\leq \max\{ f_2(x), -\rho \} + \eta. \end{aligned}$$

For lsc $f_1,\ f_2 : \mathbb{R}^n \to \overline{\mathbb{R}}$, not identically ∞, one has,

$$\hat{dl}^+_{\rho/\sqrt{2}}(f_1,f_2) \leq \hat{dl}_\rho(f_1,f_2) \leq \sqrt{2}\,\hat{dl}^+_\rho(f_1,f_2).$$

Quantitative stability of two-stage models

Theorem: (Römisch/Wets 06)
Let $P \in \mathcal{P}_\mathcal{F}$ and suppose $S(P)$ is non-empty and bounded. Then there exist constants $\rho > 0$ and $\delta > 0$ such that

$$\begin{aligned} |v(P) - v(Q)| &\leq d_{\mathcal{F},\rho}(P,Q) \\ \emptyset \neq S(Q) &\subset S(P) + \Psi_P(d_{\mathcal{F},\rho}(P,Q))\mathbb{B} \end{aligned}$$

holds for all $Q \in \mathcal{P}_\mathcal{F}$ with $d_{\mathcal{F},\rho}(P,Q) < \delta$, where Ψ_P is a *conditioning function* associated with our given program, more precisely,

$$\Psi_P(\eta) := \eta + \psi_P^{-1}(2\eta),\ \eta \geq 0, \quad \text{with}$$

$$\psi_P(\tau) := \min\{\int_\Xi f_0(\xi,x)P(d\xi) - v(P) : d(x,S(P)) \geq \tau\},\ \tau \geq 0.$$

Simple examples of two-stage stochastic programs show that, in general, the set-valued mapping $S(.)$ is not inner semicontinuous at P (Römisch 03). Furthermore, explicit descriptions of conditioning functions ψ_P of stochastic programs (like linear or quadratic growth at solution sets) are only known in some specific cases, for example, for linear two-stage stochastic programs with finite discrete distribution or with strictly positive densities of random right-hand sides (Schultz 94).

We are in much better shape, when we consider the stability properties of the sets $S_\varepsilon(\cdot)$ of ε-approximate solutions.

Theorem:
Let $P \in \mathcal{P}_\mathcal{F}$ and $S(P)$ be non-empty, bounded. Then there exist constants $\hat{\rho} > 0$ and $\hat{\varepsilon} > 0$ such that

$$\mathbb{d}l_\infty(S_\varepsilon(P), S_\varepsilon(Q)) \leq \frac{4\hat{\rho}}{\varepsilon} d_{\mathcal{F},\hat{\rho}+\varepsilon}(P, Q)$$

holds for any $\varepsilon \in (0, \hat{\varepsilon})$ and $Q \in \mathcal{P}_\mathcal{F}$ such that $d_{\mathcal{F},\hat{\rho}+\varepsilon}(P, Q) < \varepsilon$.

The preceding stability results remain valid if the set $\mathcal{F}_\rho$ is enlarged to a set $\hat{\mathcal{F}}$ and the set $\mathcal{P}_\mathcal{F}$ reduced to a subset on which the new distance

$$d_{\hat{\mathcal{F}}}(P, Q) = \sup_{f \in \hat{\mathcal{F}}} \left| \int_\Xi f(\xi) P(d\xi) - \int_\Xi f(\xi) P(d\xi) \right|$$

is finite and well-defined. Which classes $\hat{\mathcal{F}}$ of functions contain $\mathcal{F}_\rho = \{f_0(\cdot, x) : x \in X \cap \rho \mathbb{B}\}$ for any $\rho > 0$?

In the context of two-stage models, function classes of the form

$$\mathcal{F}_H := \{ f : \Xi \to I\!R \;:\; f(\xi) - f(\tilde{\xi}) \le \max\{1, H(\|\xi\|), H(\|\tilde{\xi}\|)\} \cdot \|\xi - \tilde{\xi}\|, \forall \xi, \tilde{\xi} \in \Xi \}$$

are of particular interest, where $H : I\!R_+ \to I\!R_+$ is nondecreasing, $H(0) = 0$. The corresponding distances are

$$d_{\mathcal{F}_H}(P,Q) = \sup_{f \in \mathcal{F}_H} \left| \int_\Xi f(\xi) P(d\xi) - \int_\Xi f(\xi) Q(d\xi) \right| =: \zeta_H(P,Q)$$

are so-called Fortet-Mourier metrics defined on

$$\mathcal{P}_H(\Xi) := \{ Q \in \mathcal{P}(\Xi) : \int_\Xi \max\{1, H(\|\xi\|)\} \|\xi\| Q(d\xi) < \infty \}$$

Important special case: $H(t) := t^{r-1}$ for $r \ge 1$.

The corresponding classes of functions and measures, and the distances are denoted by $\mathcal{F}_r$, $\mathcal{P}_r(\Xi)$ and ζ_r, respectively, where the measures are in the class

$$\mathcal{P}_r(\Xi) := \{ Q \in \mathcal{P}(\Xi) : \int_\Xi \|\xi\|^r Q(d\xi) < \infty \}.$$

Convergence with respect to ζ_r means weak conergence of the probability measures and $| \int_\Xi \|\xi\|^r P(d\xi) - \int_\Xi \|\xi\|^r Q(d\xi) | \le r \zeta_r(P,Q)$, i.e., convergence of the r-th order moments (Rachev 91).

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Under which conditions appear relevant classes $\mathcal{F}_H$ containing $\mathcal{F}_\rho$?

Proposition:

Suppose the stochastic program satisfies (A1) and (A2). Assume that the mapping $\xi \mapsto D(\xi)$ is bounded-valued and there exists a constant $L > 0$, and a nondecreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ with $h(0) = 0$ such that

$$d\!l_\infty(D(\xi), D(\tilde{\xi})) \le L \max\{1, h(\|\xi\|), h(\|\tilde{\xi}\|)\}\|\xi - \tilde{\xi}\|$$

holds for all $\xi, \tilde{\xi} \in \Xi$.

Then, for any $\rho > 0$, there exist $\hat{L} > 0$ and $\hat{L}(\rho) > 0$ such that

$$\begin{aligned} f_0(\xi, x) - f_0(\tilde{\xi}, x) &\le \hat{L}(\rho) \max\{1, H(\|\xi\|), H(\|\tilde{\xi}\|)\}\|\xi - \tilde{\xi}\| \\ f_0(\xi, x) - f_0(\xi, \tilde{x}) &\le \hat{L} \max\{1, H(\|\xi\|)\|\xi\|\}\|x - \tilde{x}\| \end{aligned}$$

for all $\xi, \tilde{\xi} \in \Xi$, $x, \tilde{x} \in X \cap \rho\mathbb{B}$, where H is defined by

$$H(t) := h(t)t, \; \forall t \in \mathbb{R}_+.$$

Note that $h(t) = \begin{cases} 1 & \text{, fixed recourse} \\ t^k & \text{, lower diagonal randomness with } k \text{ blocks.} \end{cases}$

Example:

Let $\overline{m} = 4$, $d = 2$, $Y = I\!R^4_+$, $\Xi = I\!R$ and consider the random (second-stage) costs and recourse matrix

$$W(\xi) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\xi & 0 & 1 & -1 \end{pmatrix} \qquad q(\xi) = \begin{pmatrix} 0 \\ 0 \\ \xi \\ -\xi \end{pmatrix}$$

Then $W(\xi)Y = I\!R^2$ (complete recourse) and $D(\xi)=[0, \xi^2] \times \{\xi\}$. Hence, the conditions (A1), (A2) are satisfied and the local Lipschitz continuity property of $D(\cdot)$ holds with $h(t) = t$, $t \in I\!R_+$.

Remark: (convergence of empirical estimates)

For the empirical measure $P_n = n^{-1} \sum_{i=1}^n \delta_{\xi_i}$, where ξ_i, $i \in I\!N$ are i.i.d. samples from P, exponential estimates for the convergence in probability of $d_{\mathcal{F},\rho}(P_n, P) = \sup_{f \in \mathcal{F}_\rho} |\int_\Xi f(\xi)(P_n - P)(d\xi)|$ can be obtained by showing that the covering number of $\mathcal{F}_\rho$, i.e., the minimal number of balls with radius ε in $L_2(\Xi, P)$, grows at most with ε^{-r} for some $r \geq 1$.

Discrete approximations of two-stage stochastic programs

Replace the (original) probability measure P by measures P_n having (finite) discrete support $\{\xi_1, \dots, \xi_n\}$ ($n \in I\!N$), i.e.,

$$P_n = \sum_{i=1}^{n} p_i \delta_{\xi_i},$$

and insert it into the infinite-dimensional stochastic program:

$$\min\{\langle c, x\rangle + \sum_{i=1}^{n} p_i \langle q(\xi_i), y_i\rangle : x \in X, y_i \in Y, \, i = 1, \dots, n,$$

$$\begin{array}{lllllll} W(\xi_1)y_1 & & & & +T(\xi_1)x & = & h(\xi_1) \\ & W(\xi_2)y_2 & & & +T(\xi_2)x & = & h(\xi_2) \\ & & \ddots & & \vdots & = & \vdots \\ & & & W(\xi_n)y_n & +T(\xi_n)x & = & h(\xi_n)\} \end{array}$$

Hence, we arrive at a (finite-dimensional) large scale block-structured linear program which allows for specific decomposition methods.

(Ruszczyński/Shapiro, Handbook, 2003)

How to choose the discrete approximation ?

The quantitative stability results suggest to determine P_n such that it forms the best approximation of P with respect to the semi-distance $d_{\mathcal{F},\rho}$ or the probability metric ζ_r, i.e., given $n \in I\!N$ solve

$$\min\{\zeta_r(P, \frac{1}{n}\sum_{i=1}^{n}\delta_{\xi_i}) : \xi_i \in \Xi,\ i = 1, \ldots, n\}$$

Such best approximations $P_n^* = \frac{1}{n}\sum_{i=1}^{n}\delta_{\xi_i^*}$ are known as optimal quantizations of the probability distribution P (Graf/Luschgy, LNM 2000).

Convergence properties of optimal quantizations and numerical methods for solving the best approximation problems in case of the ℓ_r-minimal metrics (or Wasserstein metrics)

$$\ell_r(P,Q) := \left(\inf\left\{\int_{\Xi\times\Xi}\|\xi-\tilde{\xi}\|^r\eta(d\xi,d\tilde{\xi})\mid \pi_1\eta = P,\ \pi_2\eta = Q\right\}\right)^{\frac{1}{r}},$$

are already known. Here, π_i is the projection onto the i-th component. The convergence rates are in some cases better than $O(n^{-\frac{1}{2}})$ for sampling methods. Note that $\zeta_r(P,Q) \le (1 + \int_{\Xi}\|\xi\|^r(P+Q)(d\xi))^{\frac{r-1}{r}}\ell_r(P,Q)$.

Scenario reduction

We consider discrete distributions P with scenarios ξ_i and probabilities p_i, $i = 1, \dots, N$, and Q being supported by a given subset of scenarios ξ_j, $j \notin J \subset \{1, \dots, N\}$, of P.

Optimal reduction of a given scenario set J:
The best approximation of P with respect to $\zeta_r = \mu_{\hat{c}_r}$ by such a distribution Q exists and is denoted by Q^*. It has the distance

$$D_J := \zeta_r(P, Q^*) = \min_Q \mu_{\hat{c}_r}(P, Q) = \sum_{i \in J} p_i \min_{j \notin J} \hat{c}_r(\xi_i, \xi_j)$$
$$= \sum_{i \in J} p_i \min\{\sum_{k=1}^{n-1} c_r(\xi_{l_k}, \xi_{l_{k+1}}) : n \in \mathbb{N}, l_k \in \{1, \dots, N\}, l_1 = i, l_n = j \notin J\}$$

and the probabilities $q_j^* = p_j + \sum_{i \in J_j} p_i$, $\forall j \notin J$, where

$J_j := \{i \in J : j = j(i)\}$ and $j(i) \in \arg\min_{j \notin J} \hat{c}_r(\xi_i, \xi_j)$, $\forall i \in J$.

(Dupačová/Gröwe-Kuska/Römisch 03)

We needed the following notation:

$$c_r(\xi, \tilde{\xi}) := \max\{1, \|\xi\|^{r-1}, \|\tilde{\xi}\|^{r-1}\}\|\xi - \tilde{\xi}\| \quad (\xi, \tilde{\xi} \in \Xi).$$

Proposition: (Rachev/Rüschendorf 98)

$$\zeta_r(P,Q) = \mu_{\hat{c}_r}(P,Q) = \inf\left\{\int_{\Xi\times\Xi} \hat{c}_r(\xi,\tilde{\xi})\eta(d\xi,d\tilde{\xi}) : \pi_1\eta = P, \pi_2\eta = Q\right\}$$

where $\hat{c}_r \leq c_r$ and $\hat{c}_r$ is the metric (reduced cost)

$$\hat{c}_r(\xi,\tilde{\xi}) := \inf\left\{\sum_{i=1}^{n-1} c_r(\xi_{l_i},\xi_{l_{i+1}}) : n \in \mathbb{N}, \xi_{l_i} \in \Xi, \xi_{l_1} = \xi, \xi_{l_n} = \tilde{\xi}\right\}.$$

Determining the optimal scenario index set with prescribed cardinality n is, however, a combinatorial optimization problem of set covering type:

$$\min\{D_J = \sum_{i\in J} p_i \min_{j\notin J} \hat{c}_r(\xi_i,\xi_j) : J \subset \{1,\ldots,N\}, \#J = N - n\}$$

Hence, the problem of finding the optimal set J to delete is $\mathcal{NP}$-hard and polynomial time solution algorithms do not exist.

Fast reduction heuristics

Starting point ($n = N - 1$): $\min_{l \in \{1,\ldots,N\}} p_l \min_{j \neq l} \hat{c}_r(\xi_l, \xi_j)$

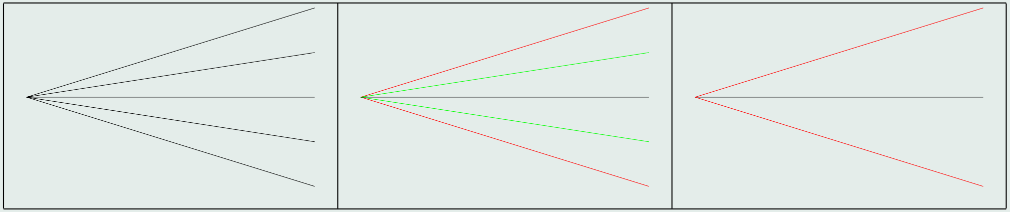
Algorithm 1: (Backward reduction)

Step [0]: $J^{[0]} := \emptyset$.

Step [i]: $l_i \in \arg\min_{l \notin J^{[i-1]}} \sum_{k \in J^{[i-1]} \cup \{l\}} p_k \min_{j \notin J^{[i-1]} \cup \{l\}} \hat{c}_r(\xi_k, \xi_j)$.

$J^{[i]} := J^{[i-1]} \cup \{l_i\}$.

Step [N-n+1]: Optimal redistribution.



Starting point ($n = 1$): $\min\limits_{u \in \{1,\ldots,N\}} \sum\limits_{k=1}^{N} p_k \hat{c}_r(\xi_k, \xi_u)$

Algorithm 2: (Forward selection)

Step [0]: $J^{[0]} := \{1, \ldots, N\}$.

Step [i]: $u_i \in \arg \min\limits_{u \in J^{[i-1]}} \sum\limits_{k \in J^{[i-1]} \setminus \{u\}} p_k \min\limits_{j \notin J^{[i-1]} \setminus \{u\}} \hat{c}_r(\xi_k, \xi_j)$,

$$J^{[i]} := J^{[i-1]} \setminus \{u_i\}.$$

Step [n+1]: Optimal redistribution.

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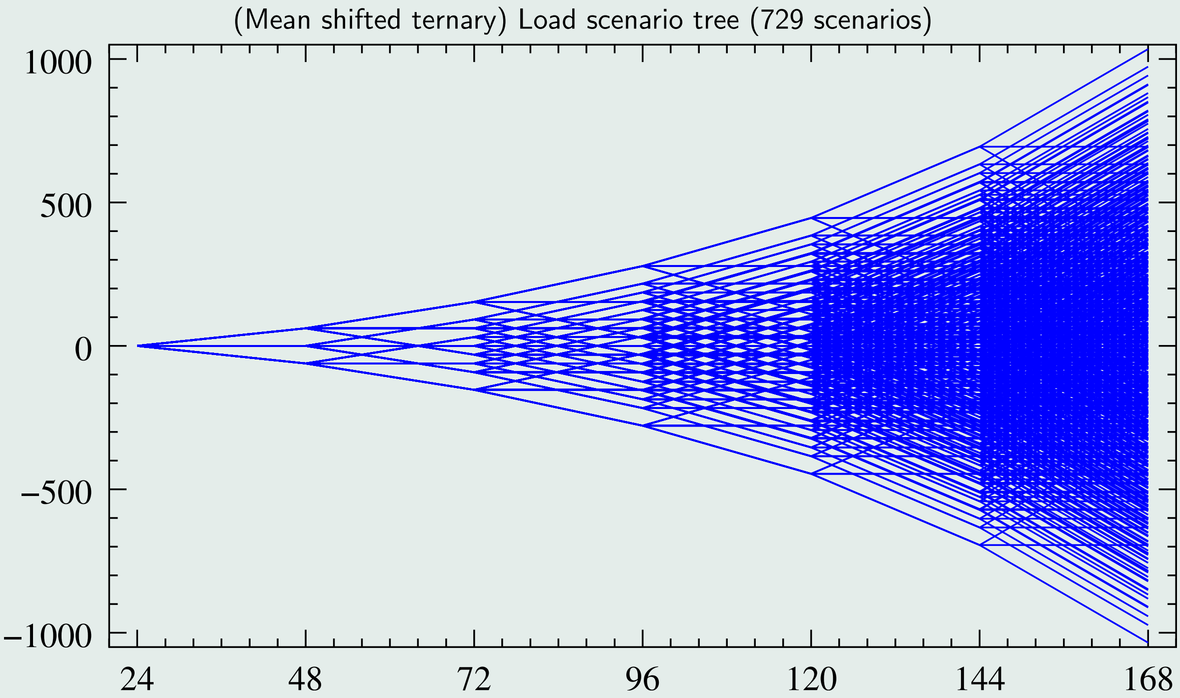
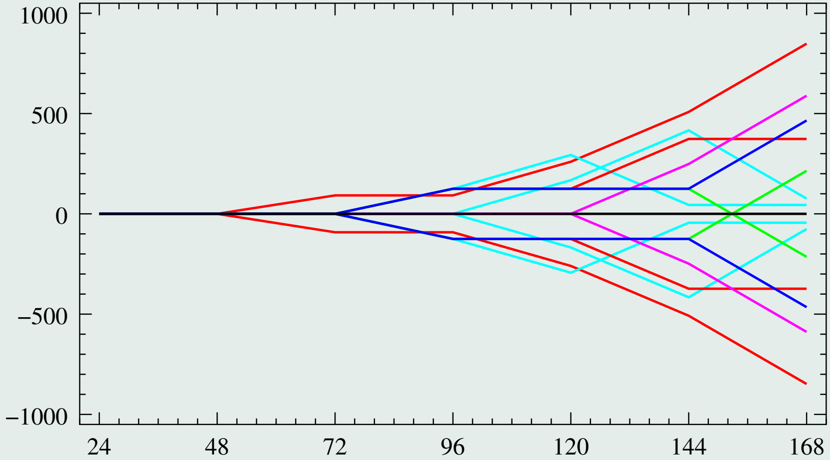
Example: (Electrical load scenario tree)

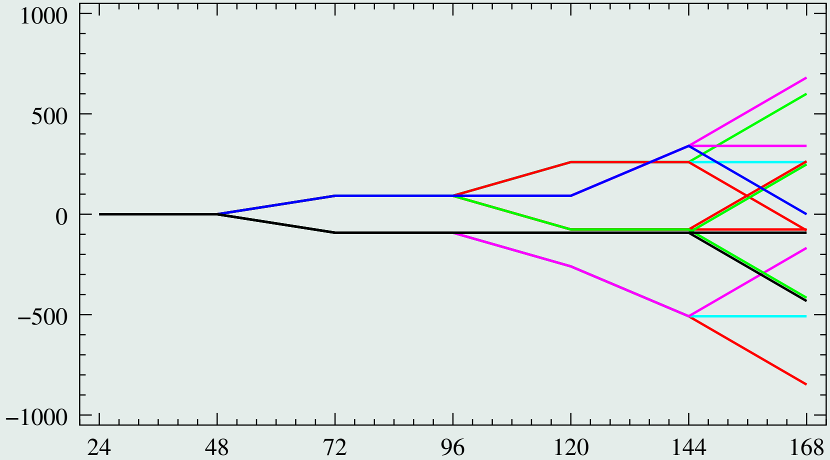


<Start Animation>

Reduced load scenario tree obtained by the forward selection method (15 scenarios)



Reduced load scenario tree obtained by the backward reduction method (12 scenarios)



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Part III

Optimization in L_r-spaces – Multistage stochastic programs

(H. Heitsch (Berlin), C. Strugarek (EdF, Clamart))

Multistage stochastic programs

Let $\{\xi_t\}_{t=1}^T$ be a discrete-time stochastic data process defined on some probability space $(\Omega, \mathcal{F}, I\!P)$ and with ξ_1 deterministic. The stochastic decision x_t at period t is assumed to be measurable with respect to $\mathcal{F}_t := \sigma(\xi_1, \ldots, \xi_t)$ (nonanticipativity).

Multistage stochastic optimization model:

$$\min \left\{ I\!E\left[\sum_{t=1}^{T} \langle b_t(\xi_t), x_t \rangle\right] \middle| \begin{array}{l} x_t \in X_t, t = 1, \ldots, T, A_{1,0}x_1 = h_1(\xi_1), \\ x_t \text{ is } \mathcal{F}_t - \text{measurable}, t = 1, \ldots, T, \\ A_{t,0}x_t + A_{t,1}(\xi_t)x_{t-1} = h_t(\xi_t), t = 2, ., T \end{array} \right\}$$

where the sets X_t, $t = 1, \ldots, T$, are polyhedral cones, the vectors $b_t(\cdot)$, $h_t(\cdot)$ and $A_{t,1}(\cdot)$ are affine functions of ξ_t, where ξ varies in a polyhedral set Ξ.

The model is (multiperiod) two-stage if $\mathcal{F}_t = \mathcal{F}$, $t = 2, \ldots, T$.

If the process $\{\xi_t\}_{t=1}^T$ has a finite number of scenarios, they exhibit a scenario tree structure.

To have the model well defined, we assume
$x_t \in L_{r'}(\Omega, \mathcal{F}_t, I\!P; I\!R^{m_t})$ and $\xi_t \in L_r(\Omega, \mathcal{F}_t, I\!P; I\!R^d)$,
where $r \geq 1$ and

$$r' := \begin{cases} \frac{r}{r-1} & , \text{ if only costs are random} \\ r & , \text{ if only right-hand sides are random} \\ \infty & , \text{ if all technology matrices are random and } r = T. \end{cases}$$

Then nonanticipativity may be expressed as

$$x \in \mathcal{N}_{na}$$

$$\mathcal{N}_{na} = \{x \in \times_{t=1}^{T} L_{r'}(\Omega, \mathcal{F}, I\!P; I\!R^{m_t}) : x_t = I\!E[x_t|\mathcal{F}_t]\,, \forall t\},$$

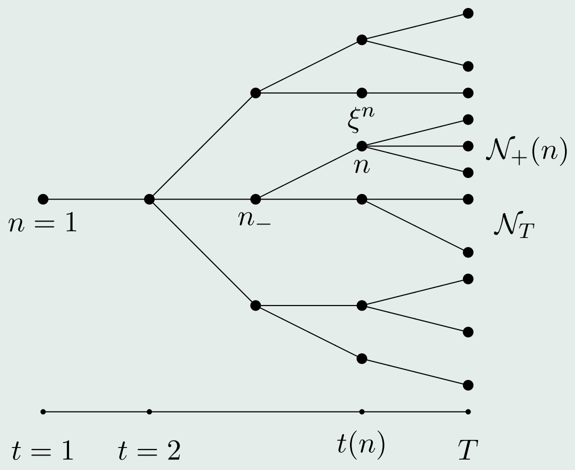
i.e., as a subspace constraint, by using the conditional expectation $I\!E[\cdot|\mathcal{F}_t]$ with respect to the σ-algebra $\mathcal{F}_t$.

For $T = 2$ we have $\mathcal{N}_{na} = I\!R^{m_1} \times L_{r'}(\Omega, \mathcal{F}, I\!P; I\!R^{m_2})$.

$\longrightarrow$ infinite-dimensional optimization problem

1. Data process approximation by scenario trees

The process $\{\xi_t\}_{t=1}^{T}$ is approximated by a process forming a scenario tree being based on a finite set $\mathcal{N} \subset \mathbb{N}$ of nodes.



Scenario tree with $T = 5$, $N = 22$ and 11 leaves

$n = 1$ root node, n_- unique predecessor of node n, $\text{path}(n) = \{1, \dots, n_-, n\}$, $t(n) := |\text{path}(n)|$, $\mathcal{N}_+(n)$ set of successors to n, $\mathcal{N}_T := \{n \in \mathcal{N} : \mathcal{N}_+(n) = \emptyset\}$ set of leaves, $\text{path}(n)$, $n \in \mathcal{N}_T$, scenario with (given) probability π^n, $\pi^n := \sum_{\nu \in \mathcal{N}_+(n)} \pi^\nu$ probability of node n, ξ^n realization of $\xi_{t(n)}$.

Tree representation of the optimization model

$$\min\left\{\sum_{n\in\mathcal{N}}\pi^n\langle b_{t(n)}(\xi^n),x^n\rangle \;\middle|\; \begin{array}{l} x^n\in X_{t(n)},\, n\in\mathcal{N},\, A_{1,0}x^1=h_1(\xi^1) \\ A_{t(n),0}x^n+A_{t(n),1}x^{n-}=h_{t(n)}(\xi^n),\, n\in\mathcal{N}\end{array}\right\}$$

How to solve the optimization model ?

- Standard software (e.g., CPLEX)

- Decomposition methods for (very) large scale models
  (Ruszczynski/Shapiro (Eds.): Stochastic Programming, Handbook, 2003)

Open question:

How to generate (multivariate) scenario trees ?

Idea:

Utilizing quantitative stability results !?

Dynamic programming

Theorem: (Evstigneev 76, Rockafellar/Wets 76)
Under weak assumptions the multistage stochastic program is equivalent to the (first-stage) convex minimization problem

$$\min\left\{\int_\Xi f(x_1,\xi)P(d\xi) : x_1 \in \mathcal{X}_1(\xi_1)\right\},$$

where f is an integrand on $\mathbb{R}^{m_1} \times \Xi$ given by

$$f(x_1,\xi) := \langle b_1(\xi_1), x_1\rangle + \Phi_2(x_1,\xi^2),$$

$$\Phi_t(x_1,\ldots,x_{t-1},\xi^t) := \inf\{\langle b_t(\xi_t), x_t\rangle + \mathbb{E}\left[\Phi_{t+1}(x_1,\ldots,x_t,\xi^{t+1})|\mathcal{F}_t\right] : x_t \in X_t, A_{t,0}x_t + A_{t,1}(\xi_t)x_{t-1} = h_t(\xi_t)\}$$

for $t = 2,\ldots,T$, where $\Phi_{T+1}(x_1,\ldots,x_T,\xi^{T+1}) := 0$, $\mathcal{X}_1(\xi_1) := \{x_1 \in X_1 : A_{1,0}x_1 = h_1(\xi_1)\}$ and $P \in \mathcal{P}(\Xi)$ is the probability distribution of ξ.

$\longrightarrow$The integrand f depends on the probability measure $\mathbb{P}$ in a nonlinear way !

Quantitative Stability

Let us introduce some notations. Let F denote the objective function defined on $L_r(\Omega, \mathcal{F}, I\!P; I\!R^s) \times L_{r'}(\Omega, \mathcal{F}, I\!P; I\!R^m) \to I\!R$ by $F(\xi, x) := I\!E[\sum_{t=1}^{T} \langle b_t(\xi_t), x_t \rangle]$, let

$$\mathcal{X}_t(x_{t-1}; \xi_t) := \{x_t \in X_t : A_{t,0} x_t + A_{t,1}(\xi_t) x_{t-1} = h_t(\xi_t)\}$$

denote the t-th feasibility set for every $t = 2, \ldots, T$ and

$$\mathcal{X}(\xi) := \{x \in L_{r'}(\Omega, \mathcal{F}, I\!P; I\!R^m) : x_1 \in \mathcal{X}_1(\xi_1), x_t \in \mathcal{X}_t(x_{t-1}; \xi_t)\}$$

the set of feasible elements with input ξ.

Then the multistage stochastic program may be rewritten as

$$\min\{F(\xi, x) \,:\, x \in \mathcal{X}(\xi) \cap \mathcal{N}_{r'}(\xi)\}.$$

Let $v(\xi)$ denote its optimal value and, for any $\alpha \geq 0$,

$$\begin{aligned} l_\alpha(F(\xi, \cdot)) &:= \{x \in \mathcal{X}(\xi) \cap \mathcal{N}_{r'}(\xi) : F(\xi, x) \leq v(\xi) + \alpha\} \\ S(\xi) &:= l_0(F(\xi, \cdot)) \end{aligned}$$

denote the α-level set and the solution set of the stochastic program with input ξ.

Assumptions:

(A1) $\xi \in L_r(\Omega, \mathcal{F}, I\!\!P; I\!\!R^s)$ for some $r \geq 1$.

(A2) There exists a $\delta > 0$ such that for any $\tilde{\xi} \in L_r(\Omega, \mathcal{F}, I\!\!P; I\!\!R^s)$ with $\|\tilde{\xi} - \xi\|_r \leq \delta$, any $t = 2, \ldots, T$ and any $x_1 \in X_1$, $x_\tau \in \mathcal{X}_\tau(x_{\tau-1}; \tilde{\xi}_\tau)$, $\tau = 2, \ldots, t-1$, the set $\mathcal{X}_t(x_{t-1}; \tilde{\xi}_t)$ is nonempty (relatively complete recourse locally around ξ).

(A3) For each $\xi \in \Xi$ there exists $z \in \times_{t=1}^{T} L_{r'}(\Omega, \mathcal{F}, I\!\!P; I\!\!R^{n_t})$ with

$$\begin{aligned} A_{t,0}^\top z_t + A_{t+1,1}^\top(\xi_{t+1}) z_{t+1} - h_t(\xi_t) &\in X_t^*, \; t = 1, \ldots, T-1, \\ A_{T,0}^\top z_T - h_T(\xi_T) &\in X_T^*, \end{aligned}$$

where X_t^* denotes the polar to the polyhedral cone X_t, $t = 1, \ldots, T$ (dual feasibility).

(A4) The objective function F is level-bounded locally uniformly at ξ, i.e., for some $\alpha > 0$ there exists a $\delta > 0$ and a bounded subset B of $L_{r'}(\Omega, \mathcal{F}, I\!\!P; I\!\!R^m)$ such that $l_\alpha(F(\tilde{\xi}, \cdot))$ is nonempty and contained in B for all $\tilde{\xi} \in L_r(\Omega, \mathcal{F}, I\!\!P; I\!\!R^s)$ with $\|\tilde{\xi} - \xi\|_r \leq \delta$.

Theorem: (Heitsch/Römisch/Strugarek 06)
Let (A1) – (A4) be satisfied and $\mathcal{X}_1(\xi_1)$ be (uniformly) bounded. Then there exist positive constants L, α and δ such that

$$|v(\xi) - v(\tilde{\xi})| \leq L(\|\xi - \tilde{\xi}\|_r + D_{\rm f}(\xi, \tilde{\xi}))$$

holds for all $\tilde{\xi} \in L_r(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^s)$ with $\|\tilde{\xi} - \xi\|_r \leq \delta$.

Assume that only costs and right-hand sides are random and that the solution x^* of the original problem is unique.
If $(\xi^{(n)})$ is a sequence in $L_r(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^s)$ such that

$$\|\xi^{(n)} - \xi\|_r \quad \text{and} \quad D_{\rm f}(\xi^n, \xi)$$

converge to 0, then any sequence $(x^{(n)})$ of solutions of the approximate problems converges to x^* with respect to the weak (weak*) topology $\sigma(L_{r'}, L_r)$.

Here, $D_{\rm f}(\xi, \tilde{\xi})$ denotes the filtration distance of ξ and $\tilde{\xi}$ defined by

$$D_{\rm f}(\xi, \tilde{\xi}) = \inf_{\substack{x \in S(\xi) \\ \tilde{x} \in S(\tilde{\xi})}} \sum_{t=2}^{T-1} \max\{\|x_t - \mathbb{E}[x_t|\tilde{\mathcal{F}}_t]\|_{r'}, \|\tilde{x}_t - \mathbb{E}[\tilde{x}_t|\mathcal{F}_t]\|_{r'}\}.$$

Remark:

The convergence of approximate solutions can be supplemented by a quantitative stability property of the set $S_1(\xi)$ of first stage solutions. Namely, there exists a constant $\hat{L} > 0$ such that

$$\sup_{x \in S_1(\tilde{\xi})} d(x, S_1(\xi)) \leq \Psi_\xi^{-1}(\hat{L}(\|\xi - \tilde{\xi}\|_r + D_{\mathrm{f}}(\xi, \tilde{\xi}))),$$

where $\Psi_\xi(\tau) := \inf\{\mathbb{E}[f(x_1, \xi)] - v(\xi) : d(x_1, S_1(\xi)) \geq \tau, x_1 \in X_1\}$ with $\Psi_\xi^{-1}(\alpha) := \sup\{\tau \in \mathbb{R}_+ : \Psi_\xi(\tau) \leq \alpha\}$ $(\alpha \in \mathbb{R}_+)$ is the growth function of the original problem near its solution set $S_1(\xi)$.

Remark:

The filtration distance $D_{\mathrm{f}}(\xi, \tilde{\xi})$ may be further estimated by the distance $d_{\mathrm{f}}(\xi, \tilde{\xi})$ with

$$d_{\mathrm{f}}(\xi, \tilde{\xi}) := \sup_{\|x\|_{r'} \leq 1,\, x \in L_{r'}} \sum_{t=2}^{T-1} \|\mathbb{E}[x_t|\mathcal{F}_t] - \mathbb{E}[x_t|\tilde{\mathcal{F}}_t]\|_{r'}.$$

In case of finite Ω, this distance corresponds to the $l_{r'}$-distance of two matrices representing the information on the filtrations of ξ and $\tilde{\xi}$, respectively.

Next we compute the distance $d_{\mathrm{f}}(\xi,\tilde{\xi})$ of filtrations for the special case that Ω is finite, say, $\Omega = \{\omega_1,\dots,\omega_S\}$. Let $I\!P(\{\omega_i\}) = p_i$, $i = 1,\dots,S$ and let $\mathcal{E}_t$ and $\tilde{\mathcal{E}}_t$ be partitions of Ω that generate the σ-fields $\mathcal{F}_t$ and $\tilde{\mathcal{F}}_t$, respectively. Then $I\!E[x_t|\mathcal{F}_t] = H_t x_t$, where the matrix H_t is of the form

$$H_t = (e_{\sigma s})_{\sigma,s=1,\dots,S}\,, \quad \text{where} \quad e_{\sigma s} := \begin{cases} \dfrac{p_s}{\sum\limits_{i\in E_{t\sigma}} p_i} & , \ s \in E_{t\sigma} \\ 0 & , \ s \notin E_{t\sigma} \end{cases}$$

and $\omega_\sigma \in E_{t\sigma} \in \mathcal{E}_t$. Analogously, $\tilde{H}_t = (\tilde{e}_{\sigma s})_{\sigma,s=1,\dots,S}$ is defined using the corresponding sets $\tilde{E}_{t\sigma}$ in a generator of the σ-field $\tilde{\mathcal{F}}_t$. Hence, we obtain for $r' = \infty$, i.e., the row sum norm $\|\cdot\|_\infty$ of matrices, that

$$d_{\mathrm{f}}(\xi,\tilde{\xi}) = \sum_{t=1}^{T-1} \|H_t - \tilde{H}_t\|_\infty$$

$$\|H_t-\tilde{H}_t\|_\infty = \max_{\sigma=1,\dots,S}\left\{ \sum_{s\in E_{t\sigma}\setminus\tilde{E}_{t\sigma}} \frac{p_s}{\sum\limits_{i\in E_{t\sigma}} p_i} + \sum_{s\in \tilde{E}_{t\sigma}\setminus E_{t\sigma}} \frac{p_s}{\sum\limits_{i\in \tilde{E}_{t\sigma}} p_i} + \sum_{s\in E_{t\sigma}\cap\tilde{E}_{t\sigma}} \left| \frac{p_s}{\sum\limits_{i\in E_{t\sigma}} p_i} - \frac{p_s}{\sum\limits_{i\in \tilde{E}_{t\sigma}} p_i} \right| \right\}$$

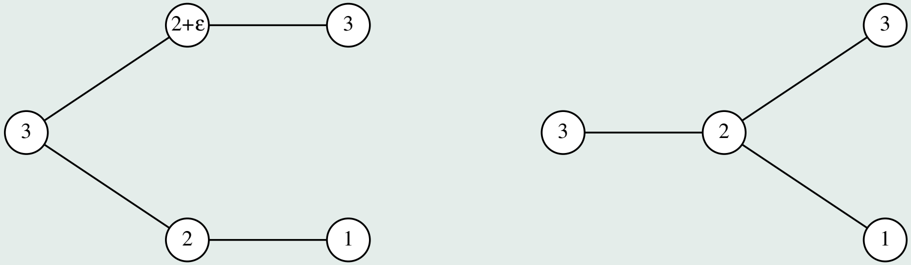
for $t = 2,\dots,T-1$.

The following example shows that the filtration distance D_{f} is indispensable for the stability result to hold.

Example: (Optimal purchase under uncertainty)
The decisions x_t correspond to the amounts to be purchased at each time period with uncertain prices are ξ_t, $t = 1, \ldots, T$, and such that a prescribed amount a is achieved at the end of a given time horizon. The problem is of the form

$$\min \left\{ \mathbb{E}\left[\sum_{t=1}^{T} \xi_t x_t\right] \middle| \begin{array}{l} (x_t, s_t) \in X_t = \mathbb{R}_+^2, \\ (x_t, s_t) \text{ is } (\xi_1, \ldots, \xi_t)\text{-measurable}, \\ s_t - s_{t-1} = x_t,\ t = 2, \ldots, T, \\ s_1 = 0,\ s_T = a. \end{array} \right\},$$

where the state variable s_t corresponds to the amount at time t.
Let $T := 3$ and ξ_ε denote the stochastic price process having the two scenarios $\xi_\varepsilon^1 = (3, 2+\varepsilon, 3)$ $(\varepsilon \in (0,1))$ and $\xi_\varepsilon^2 = (3, 2, 1)$ each endowed with probability $\frac{1}{2}$. Let $\tilde{\xi}$ denote the approximation of ξ_ε given by the two scenarios $\tilde{\xi}^1 = (3, 2, 3)$ and $\tilde{\xi}^2 = (3, 2, 1)$ with the same probabilities $\frac{1}{2}$.



Scenario trees for ξ_ε (left) and $\tilde{\xi}$

We obtain

$$\begin{aligned} v(\xi_\varepsilon) &= \frac{1}{2}((2+\varepsilon)a + a) = \frac{3+\varepsilon}{2}a \\ v(\tilde{\xi}) &= 2a\,, \qquad \text{but} \\ \|\xi_\varepsilon - \tilde{\xi}\|_1 &\le \frac{1}{2}(0+\varepsilon+0) + \frac{1}{2}(0+0+0) = \frac{\varepsilon}{2}. \end{aligned}$$

Hence, the multistage stochastic purchasing model is not stable with respect to $\|\cdot\|_1$.

However, the estimate for $|v(\xi) - v(\tilde{\xi})|$ in the stability theorem is valid with $L = 1$ since $D_{\mathrm{f}}(\xi, \tilde{\xi}) = \frac{a}{2}$.

Generation of scenario trees

(i) Development of a statistical model for the stochastic process ξ (parametric [e.g. time series model], nonparametric [e.g. resampling]) and generation of simulation scenarios;

(ii) Construction of a scenario tree out of the statistical model or of the simulation scenarios.

Approaches for (ii):

(1) Bound-based approximation methods (Frauendorfer 96, Kuhn 05, Edirisinghe 99, Casey/Sen 05).

(2) Monte Carlo-based schemes (inside or outside decomposition methods) (e.g. Shapiro 03, 06, Higle/Rayco/Sen 01, Chiralaksanakul/Morton 04).

(3) the use of Quasi Monte Carlo integration quadratures (Pennanen 05, 06).

(4) EVPI-based sampling schemes (inside decomposition schemes) (Corvera Poire 95, Dempster 04).

(5) Moment-matching principle (Høyland/Wallace 01, Høyland/Kaut/Wallace 03).

(6) (Nearly) best approximations based on probability metrics (Pflug 01, Hochreiter/Pflug 02, Gröwe-Kuska/Heitsch/Römisch 01, 03, Heitsch/Römisch 05).

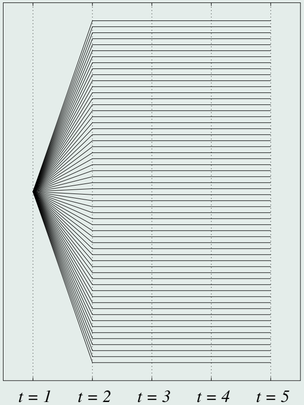
Survey: Dupačová/Consigli/Wallace 2000

Constructing scenario trees

Let ξ be the original stochastic process on some probability space $(\Omega, \mathcal{F}, I\!\!P)$ with parameter set $\{1, \ldots, T\}$ and state space $I\!\!R^d$. We aim at generating a scenario tree ξ^{tr} such that

$$\|\xi - \xi^{\mathrm{tr}}\|_r \qquad \text{and} \qquad D_{\mathrm{f}}(\xi, \xi^{\mathrm{tr}})$$

are small and, hence, the optimal values $v(\xi)$ and $v(\xi^{\mathrm{tr}})$ are close to each other. Since this problem is hardly solvable in general, we replace ξ by a finitely discrete approximation ξ^{f} such that $\|\xi - \xi^{\mathrm{f}}\|_r$ is small and its scenarios $\xi^i = (\xi^i_1, \ldots, \xi^i_T)$ with probabilities π^i, $i = 1, \ldots, N$ form a *fan* of individual scenarios.



An **algorithm** was developed that generates a tree ξ^{tr} by deleting and bundling scenarios at each $t = 2, \ldots, T$ (that are close to each other) and such that

$$\|\xi^{\mathrm{f}} - \xi^{\mathrm{tr}}\|_r$$

may be computed and bounded and that

$$D_{\mathrm{f}}(\xi^{\mathrm{f}}, \xi^{\mathrm{tr}})$$

may be bounded from above. The latter relies on the

Proposition:

Assume that only costs and right-hand sides are random and let (A2) – (A4) be satisfied. Then there exists a constant $\hat{L} > 0$ such that the filtration distance allows the estimate

$$D_{\mathrm{f}}(\xi^{\mathrm{f}}, \xi^{\mathrm{tr}}) \leq \hat{L} \begin{cases} \left(\sum_{i \in I_2} \sum_{j \in I_{2,i}} p_j \|\xi^j - \xi^i\|^{r'} \right)^{\frac{1}{r'}} & , 1 \leq r' < \infty \\ \max_{i \in I_2} \max_{j \in I_{2,i}} \|\xi^j - \xi^i\| & , r' = \infty \, . \end{cases}$$

Tolerances ε_r and ε_f are prescribed for $\|\xi^{\mathrm{f}} - \xi^{\mathrm{tr}}\|_r$ and $D_{\mathrm{f}}(\xi^{\mathrm{f}}, \xi^{\mathrm{tr}})$, respectively, which control the scenario tree generation process.

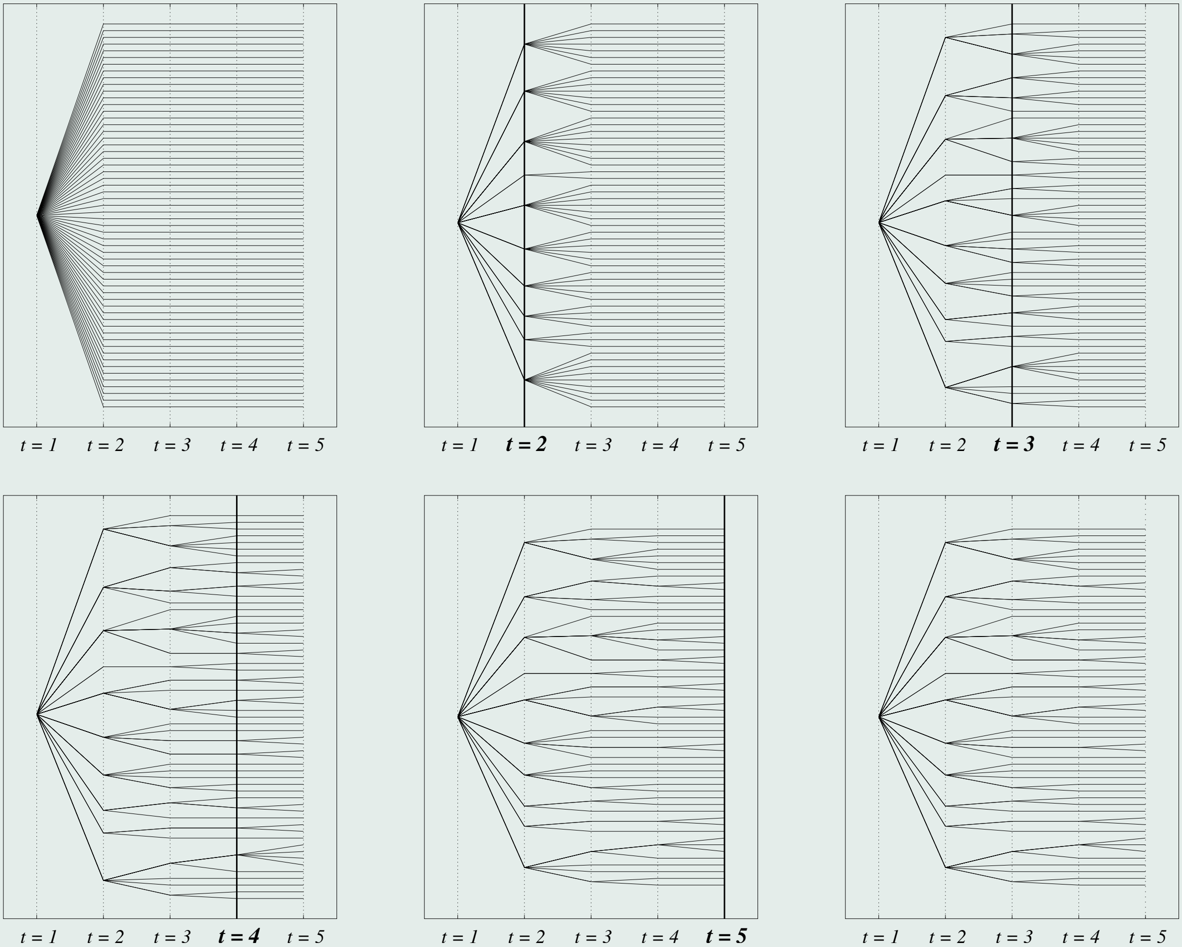


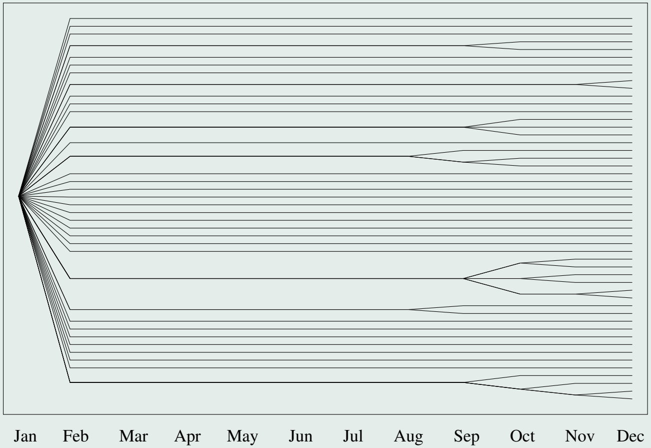
Illustration of the forward tree construction for an example including T=5 time periods starting with a scenario fan containing N=58 scenarios

<Start Animation>

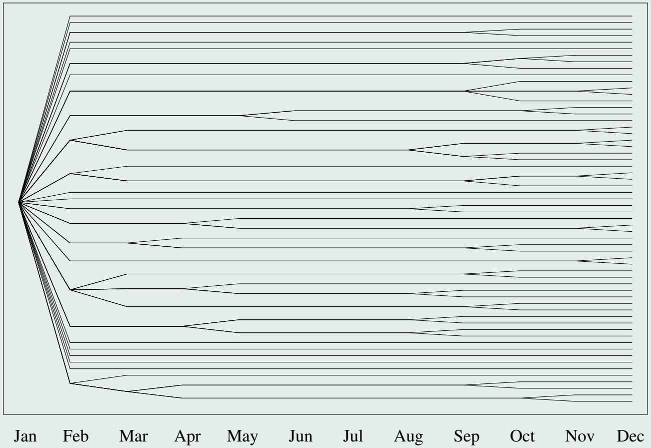
a) Forward tree construction with filtration level 0.35

b) Forward tree construction with filtration level 0.45

Yearly demand-price scenario trees with relative tolerance 0.25



a) Modified forward tree construction with filtration level 0.6



b) Modified forward tree construction with filtration level 0.7

Yearly demand-price scenario trees with relative reduction level 0.5 (Heitsch/Römisch 06)

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