On discrete approximations in stochastic programming

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1. Introduction

Throughout this paper we consider decision problems of stochastic programming in topological spaces and their stability. Our aim is the investigation of discrete approximations to such problems. For similar approaches we refer to /4/, /5/, /6/, /7/, /10/, /11/, /16/, /17/, for the case of two- or multi-stage problems of linear (or nonlinear) stochastic programming and to /9/, /14/ for certain classes of decision problems in abstract spaces. Such discrete approximations result if the random variable entering the stochastic programming problem is replaced by random variables with a discrete probability distribution. The advantage of discrete approximations consists in their deterministic nature and consequently in the possible treatment as nonlinear programming problems. An important possibility to construct discrete approxima-tions seems to be the approximation of the random variables by suitable conditional expectations (cf. /5/, /6/, /14, Remark 6/, chapter 3). As an application of the general setting a treatment of optimal control problems with random operator equations is possible (cf. /13/,/14, chapter 5/,/21/).

Let $(\Omega,\mathfrak{a},P)$ be a probability space, X be a topological space and $C \subseteq X$ a non-empty constraint set, $g:\Omega\times C \longrightarrow R$ be such that for all $u\in C$ $g(\cdot,u)$ represents a real random variable on $(\Omega,\mathfrak{a},P)$, and let E denote the mean value w.r.t. $(\Omega,\mathfrak{a},P)$. Then we consider decision problems of the following type:

$$J(u) := E[g(\omega,u)] \longrightarrow \text{Min! subject to } u\in C. \qquad (1)$$

We refer to /14, chapter 2/ for a discussion of (1) and for the formulation of two-stage problems of stochastic programming and of optimal control problems with random operator equations in Banach spaces as special cases of problem (1).

2. Approximation of nonlinear programming problems with fixed constraint set and applications to stability in stochastic programming

We are given a non-empty subset C of a topological space X, functionals $J, J_m: C \longrightarrow \bar{R}$, $m \in N$, where N denotes the set of natural numbers and $\bar{R}$ the extended real numbers. Let us consider the problems

$$J(u) \longrightarrow \text{Min!} \quad \text{s.t.} \quad u \in C \tag{2}$$

$$J_m(u) \longrightarrow \text{Min!} \quad \text{s.t.} \quad u \in C \tag{2m}$$

and define the optimal values $\varphi := \inf\{J(u) \mid u \in C\}$, $\varphi_m := \inf\{J_m(u) \mid u \in C\}$, $m \in N$, and the ε-optimal set mappings $\psi(\varepsilon) := \inf\{u \in C \mid J(u) \leq \varphi + \varepsilon\}$, $\psi_m(\varepsilon) := \inf\{u \in C \mid J_m(u) \leq \varphi_m + \varepsilon\}$ $m \in N$. Now we ask for the convergence of the sequences $\{\varphi_m\}_{m \in N}$ to φ resp. $\{\psi_m(\varepsilon)\}_{m \in N}$ to $\psi(\varepsilon)$ (in some sense).

The following Theorem is closely related to a well-known result of nonlinear parametric programming (cf. /1, Satz 4.2.2/) and is slightly stronger than Theorem 1 in /14/. For convenience we give a short version of the proof.

Theorem 1:

Let C be sequentially compact and sequentially closed and let the following conditions be fulfilled:

(a) for each $u \in C$: $\lim_{m \to \infty} J_m(u) = J(u)$;

(b) for all $u, u_m \in C$, $m \in N$, such that $\lim_{m \to \infty} u_m = u$: $\underline{\lim}_{m \to \infty} J_m(u_m) \geq J(u)$

Then we have

(i) $\lim_{m \to \infty} \varphi_m = \varphi$;

(ii) for each sequence $u_m^* \in \psi_m(\varepsilon_m)$, $\varepsilon_m \geq 0$, $m \in N$, there exists an accumulation point $u^* \in \psi(\overline{\lim}_{m \to \infty} \varepsilon_m)$.

Proof:

(i) Obviously (a) yields $\overline{\lim}_{m \to \infty} \varphi_m \leq \varphi$. Now we choose $u_m \in C$ such that $J_m(u_m) \leq \varphi_m + m^{-1}$, $m \in N$ (with the usual modification in the case $\varphi_m = -\infty$), and a subsequence u_m, $m \in N' \subseteq N$, such that $\lim_{m \in N'} u_m = u \in C$ and $\lim_{m \in N'} J_m(u_m) = \underline{\lim}_{m \to \infty} J_m(u_m)$. Therefore we have for

$$\bar{u}_m := \begin{cases} u_m, & m \in N' \\ u, & m \in N \setminus N' \end{cases}, m \in N, \ \varliminf_{m \to \infty} J_m(\bar{u}_m) \geq J(u) \geq \varphi \ .$$

and finally

$$\varliminf_{m \to \infty} \varphi_m = \varliminf_{m \to \infty} J_m(u_m) = \lim_{m \in N'} J_m(u_m) \geq \varliminf_{m \to \infty} J_m(\bar{u}_m) \geq \varphi \ .$$

(ii) From $J_m(u^*_m) \leq \varphi_m + \varepsilon_m$, $m \in N$, we conclude

$$\overline{\lim_{m \to \infty}} J_m(u^*_m) \leq \varphi + \overline{\lim_{m \to \infty}} \varepsilon_m .$$

Analogous to (i) we choose a subsequence $u^*_m \in \Psi_m(\varepsilon_m)$, $m \in N' \subseteq N$, with limit $u^* \in C$ and have

$$J(u^*) \leq \varliminf_{m \in N'} J_m(u^*_m) \leq \varphi + \overline{\lim_{m \to \infty}} \varepsilon_m$$

q.e.d.

<u>Remark 1:</u>

a) The following condition (c) is sufficient for (a) and (b) in Theorem 1:

(c) for all $u, u_m \in C$, $m \in N$, such that $\lim_{m \to \infty} u_m = u$ holds

$$\lim_{m \to \infty} J_m(u_m) = J(u)$$

If (a) holds, then condition (b)' is equivalent to (b) :

(b)' for all $u, u_m \in C$, $m \in N$, such that $\lim_{m \to \infty} u_m = u$ holds:

$$\varliminf_{m \to \infty} (J_m(u_m) - J_m(u)) \geq 0$$ (compare /14, Remark 3c)/).

Of course it is sufficient that in Theorem 1 (a) holds for some $u^* \in \Psi(0)$.

b) We note that the compactness of C cannot be dropped in the Theorem. It is referred to various examples in /1/ and to /8/, where the case of the weak topology in a reflexive Banach space is treated. Applying the concept of /8/ to our special case it turns out that /8,(A4)/ contains the compactness condition and /8,(A1),(i),(ii)/ is sufficient for (a), (b).

Now let us return to problem (1) and let mappings $g_m : \Omega \times C \longrightarrow R$, $m \in N$, be given additionally such that for all $u \in C$, $m \in N$, $g_m(\cdot, u)$ are real random variables. Then we define $J_m : C \longrightarrow \bar{R}$

$J_m(u) := E[g_m(\omega,u)]$, $u \in C, m \in N$ and the problems

$J_m(u) \longrightarrow$ Min! s.t. $u \in C$ (1m).

For applying Theorem 1 to obtain results for the approximation of (1) by the problems (1m), $m \in N$, let φ, $\psi(\varepsilon)$, φ_m, $\psi_m(\varepsilon)$, $m \in N$, be defined as above. The following Theorems present two versions of approximation (or stability) results in stochastic programming.

<u>Theorem 2:</u>
Let there exist a random variable $v_1 : \Omega \longrightarrow R$ such that $E[v_1(\omega)] > -\infty$ and $v_2 : \Omega \times C \longrightarrow R$ such that for each $u \in C$ $v_2(\cdot,u)$ is a random variable, $E[v_2(\omega,u)] < \infty$ and that for each $m \in N$

$$v_1(\omega) \leq g_m(\omega,u) \leq v_2(\omega,u) \quad \text{a.s.}$$

("uniform integrability condition" (UIC)).
Further let us assume the following "continuity conditions" (CC) :

(a1) for each $u \in C$: $\lim_{m \to \infty} g_m(\omega,u) = g(\omega,u)$ a.s.;

(b1) for all $u, u_m \in C$, $m \in N$, such that $\lim_{m \to \infty} u_m = u$:

$$\varliminf_{m \to \infty} g_m(\omega,u_m) \geq g(\omega,u) \quad \text{a.s.}$$

Let C satisfy the assumption of Theorem 1.
Then the assertion of Theorem 1 is valid for (1) and (1m), $m \in N$.

<u>Proof:</u>
For the proof we have to verify conditions (a) and (b) of Theorem 1. In both cases we use Fatou's Lemma (cf. e.g. /20, p. 203/) and obtain

$$J(u) = E[g(\omega,u)] = E[\lim_{m \to \infty} g_m(\omega,u)] = \lim_{m \to \infty} J_m(u),$$

$$J(u) \leq E[\varliminf_{m \to \infty} g_m(\omega,u_m)] \leq \varliminf_{m \to \infty} J_m(u_m) .$$

q.e.d.

<u>Remark 2:</u>
We mention that other versions of Theorem 2 result by using different versions of Fatou's Lemma resp. Lebesgue's Theorem.

As an example it is referred to replacing the condition (UIC) (resp. (c2)) by $|g_m(\omega,u)| \leq v_m(\omega)$, a.s., $u \in C$, $m \in N$, where $\{v_m\}_{m \in N}$ is uniformly integrable (cf. /20, p. 205/).

Theorem 3:

Let C satisfy the assumption of Theorem 1 and let (c1),(c2) be fulfilled.

(c1) For all $u, u_m \in C$, $m \in N$, such that $\lim_{m\to\infty} u_m = u$ it holds that $\lim_{m\to\infty} g_m(\cdot,u_m) = g(\cdot,u)$ (in distribution) and that

(c2) $\{g_m(\cdot,u_m)\}_{m \in N}$ is uniformly integrable.

Then the assertion of Theorem 1 is valid for (1) and (1m), $m \in N$.

Proof:

The proof is a simple consequence of condition (c) (Remark 1a)) and of Lebesgue's Theorem /18, Theorem 5.4./.

q.e.d.

As an illustration of the preceding Theorems let us consider the following situation:

$$J(u) := E\,[g_o(z(\omega),u)]$$
$$J_m(u) := E\,[g_o(z_m(\omega),u)] \quad , m \in N,$$

where $g_o : Z \times C \longrightarrow R$ and $z, z_m : \Omega \longrightarrow Z$, $m \in N$, are random variables with values in a Banach space Z such that $\lim_{m\to\infty} z_m = z$ in some sense .

Case 1: If one has $\lim_{m\to\infty} z_m(\omega) = z(\omega)$ a.s., then the following condition is sufficient for (a1),(b1):
$g_o : Z \times C \longrightarrow R$ is lower semicontinuous and for each $u \in C$ $g_o(\cdot,u) : Z \longrightarrow R$ is continuous.

Case 2: If $\lim_{m\to\infty} z_m = z$ (in distribution), then the continuity of $g_o : Z \times C \longrightarrow R$ is sufficient for (c1) (cf. /18, Theorem 5.5/).

Related results were obtained in /7, Theorem 2/ and /3, Satz 3.1/ in the finite-dimensional case.

3. Remarks on discrete approximations

In the preceding chapter general stability results for stochastic programming problems were obtained. Now we turn our

attention to a suitable choice (or construction) of the approximate problems. The main concern when approximating (1) is that the problems (1m), $m \in N$, are in some sense simpler to solve. In this way the notion of a "discretization scheme" (/14/) seems to be an important possibility.
$\{\alpha_m, g_m\}_{m \in N}$ will be called a "discretization scheme" if for each $m \in N$ there exist

(i) a finite partition $A_{ml} \in \alpha$, $l=1,\dots,m$, of Ω , i.e.
$\bigcup_{l=1}^{m} A_{ml} = \Omega$ and $A_{ml} \cap A_{mk} = \emptyset$, $l \neq k$, such that
$\alpha_m := \sigma(\{A_{ml}\}_{l=1,\dots,m})$.
(Here $\sigma(\mathcal{E})$ denotes the smallest σ-algebra containing $\mathcal{E} \subset \alpha$.).

(ii) $g_{ml} : C \longrightarrow R$, $l=1,\dots,m$, such that: $g_m(\omega,\cdot) = g_{ml}$, $\omega \in A_{ml}$, $l=1,\dots,m$.

We remark that an essential advantage of discretization schemes consists in the "deterministic" form of the functionals J_m, $m \in N$:

$$J_m(u) = \sum_{l=1}^{m} g_{ml}(u)\, P(A_{ml}),\ u \in C.$$

Of course the probabilities $P(A_{ml})$, $l=1,\dots,m$, must be known. Nice properties of g_{ml}, $l=1,\dots,m$, like continuity, differentiability and convexity, yield the corresponding properties of J_m. Therefore the question of a suitable construction of discretization schemes arises.
But we have to keep in mind that the evaluation of $J_m(u)$ requires the evaluation of m functions $g_{ml}(u)$, $l=1,\dots,m$. In this way a considerable computational effort may arise if m is "large". We refer to the situation in two-stage stochastic programming (cf. e.g. /6/) and in stochastic optimal control problems (/14, chapter 5/).
Discretization schemes represent a well-known method for two-stage problems in stochastic programming (e.g. /4/,/5/, /6/,/7/,/10/,/11/) and in more general decision problems (/9/,/14/). /2/,/16/,/17/ present efficient algorithms for discrete linear stochastic programming with simple recourse,

which essentially reduce the above mentioned computational effort.

In applications discretization schemes arise if the random variable z in the decision problem

$$J(u) := E\,[g_0(z(\omega),u)] \longrightarrow \text{Min!} \quad \text{s.t.} \quad u \in C$$

is replaced by simple random variables z_m, $m \in N$, with m values z_{ml}, $l=1,\ldots,m$, i.e. with a discrete probability distribution. There are two motivations for doing this. Firstly the distribution of z is given and the random variables z_m, $m \in N$, are chosen to approximate z (in some sense). This case is our main concern (/14, chapter 4/). But secondly it often happens that all that is available are some sample points, which can be considered as the discrete random variables z_m. In the latter case it is only possible to assume $\lim_{m \to \infty} z_m = z$ (in distribution) and error estimates or convergence rates are desirable (cf. /17/).

Let us turn to the case of given probability distribution. In /5/,/6/,/12/,/14/ it is suggested to replace z by conditional expectations with respect to certain finitely generated σ-algebras. More precisely, if Z is a Banach space, $z \in L_1(\Omega,\mathfrak{a},P;Z)$, and if a sequence $\{\{A_{ml}\}_{l=1,\ldots,m}\}_{m \in N} \subset \mathfrak{a}$ of partitions of Ω is given, we define

$$z_m := E(z \mid \mathfrak{a}_m), \quad \mathfrak{a}_m := \sigma(\{A_{ml}\}_{l=1,\ldots,m}), \; m \in N, \quad \text{i.e.}$$

$$z_m(\omega) = E(z \mid A_{ml}), \quad \omega \in A_{ml}, \; l=1,\ldots,m.$$

In /6/ and /14, Remark 6/ some reasons for using conditional expectations are given. Let us mention the convergence properties (giving the possibility to apply Theorem 2 and 3), the use of Jensen's inequality to derive error bounds (/6/) and the possibilities of construction and computation of $P(A_{ml})$, $E(z \mid A_{ml})$, $l=1,\ldots,m$ (see e.g. /15/ for the finite-dimensional case and /13, chapter 6/).

Finally the author thanks B. Kummer for valuable discussions on stability of optimization problems.

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Addendum

After the completion of the paper the author obtained information on further results concerning upper bounds (in the convex case) to

$$J(u) = E[g_o(z(\omega), u)] \quad , u \in C \quad \text{(see /22/)}.$$

These upper bounds and the lower bounds, by use of Jensen's inequality, provide possibilities for a suitable partitioning using conditional expectations (comp. also /6/).

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Zusammenfassung:

In dieser Arbeit werden ein-stufige Entscheidungsprobleme der stochastischen Optimierung in topologischen Räumen betrachtet. Eine Stabilitätsaussage für nichtlineare Optimierungsprobleme mit fester Restriktionsmenge wird bewiesen und Anwendungen auf die Approximation der betrachteten Entscheidungsprobleme untersucht. Dabei beinhaltet Stabilität die Konvergenz der Optimalwerte und der ε-Optimalmengen. Daran schließen sich Bemerkungen über eine geeignete Wahl der approximierenden Probleme, sog. diskreter Approximationen, an. Abschließend wird erläutert, welche Vorteile die Ersetzung der im Problem enthaltenen Zufallsvariablen durch geeignete bedingte Erwartungen besitzt.

Summary:

One-stage decision problems of stochastic programming in topological spaces are considered. A stability result for nonlinear programming problems with fixed constraint set is proved and applied to the approximation of such decision problems. Here stability (or approximation) includes convergence of the optimal values and ε-optimal sets. Remarks on a suitable choice of the approximate problems, so-called discrete approximations, are added and finally the advantages of replacing the involved random variable by certain conditional expectations are explained.

Резюме:

Изучаются одноэтапные задачи стохастического программирования в топологических пространствах. Доказывается устойчивость проблем нелинейного программирования с фиксированным набором ограничений и исследуются применении при аппроксимации выше названных задач. После этого даны замечания о конструкции так называемых дискретных аппроксимации и о выгодности аппроксимации случайных величин некоторыми условными ожиданиями.

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