Let Γ is a group acting on the space X and $z \in X$. A central question in analytic number theory is to estimate how many points in the orbit Γz lie within distance s from z. When Γ is $\mathbb{Z}^2$, $z = (0, 0)$, and $X = \mathbb{R}^2$, this is the classical Gauss circle problem. The main term in the approximation is easy: πs^2. The size of the error term is trickier and is an open problem. Hardy's conjecture is that the error is of size $O(s^{1/2+\epsilon})$.

For the hyperbolic plane and Γ a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$ the analogue of this problem exhibits various differences, necessitating the use of automorphic forms and, in particular, Maass cusp forms. In this case the main term grows exponentially

$$\frac{\pi}{\mathrm{vol}(\Gamma \setminus \mathbb{H})} e^s.$$

Here the best error term is $O(e^{2s/3})$, due to Selberg and has not been improved for approximately 50 years. Combining arithmetic methods, e.g. rate of quantum ergodicity of Maass cusp forms, estimates on exponential sums, we improve the exponent, when we average locally over the center of the hyperbolic circle z for the group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. We consider also discrete averages over Heegner points for $\mathrm{SL}(2, \mathbb{Z})$.

Further we will discuss various modifications of the problem, due to Huber and, in more general setups, Hermann and A. Good. E.g. we restrict the action of Γ to a conjugacy class of it, or, geometrically, investigate how many points in the orbit Γz are within s from a totally geodesic submanifold of the space. Again the main terms are well-understood. In joint work with D. Chatzakos and independent work of N. Laaksonen new average results on the error terms are proved using large sieve inequalities. These allow to state new conjectures for these error terms, consistent with our results.