Humboldt-Universität zu Berlin
Institut für Mathematik
Dr. D. Agostini, M. Sc. L. Lerer
L. Mädje

Zahlentheorie
SoSe 2019

Exercise Sheet 10

If you want your solutions to be corrected, you should hand them in by Monday, July 1st.
Please write your name and immatriculation number on top of every exercise

---

Exercise 10.1 (5+5 points) Let $K = \mathbb{Q}(\sqrt{10})$. Recall that $\mathcal{O}_K = \mathbb{Z}[\sqrt{10}]$.

a) Calculate the Minkowski bound M_K and find all the prime ideals $\mathfrak{p}$ with $\|\mathfrak{p}\| \leq M_K$.

b) Show that the class group of $\mathcal{O}_K$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. (*Hint*: to show that an ideal is non-principal, argue by contradition and use the usual norm. To find relations in the ideal group, note that there is an element $x \in \mathcal{O}_K$ with $|N_{K/\mathbb{Q}}(x)| = 6$).

Exercise 10.2 (3+3 points) Let $K = \mathbb{Q}(\theta)$ be a number field, with $\theta \in \mathcal{O}_K$ and let $f \in \mathbb{Z}[X]$ be its minimal polinomial.

a) Let $a \in \mathbb{Z}$. Prove that $N_{K/\mathbb{Q}}(a - \theta) = f(a)$.

b) Let θ be a root of $X^3 + 6X + 8$. Find a unit of $\mathcal{O}_K$ different from ± 1.

Exercise 10.3 (5+5 points) Let K be a number field, $\mathcal{O}_K$ its ring of integers and $I \subset \mathcal{O}_K$ an ideal.

a) Show that there exists an extension L of K such that I becomes principal in $\mathcal{O}_L$, i.e. $I \cdot \mathcal{O}_L = (x)$ for some $x \in \mathcal{O}_L$. (*Hint*: prove that there is a positive integer m such that $I^m = (a)$, for some $a \in \mathcal{O}_K$. Consider the field $K(\sqrt[m]{a})$.)

b) Show that there exists an extension of K where every ideal of $\mathcal{O}_K$ becomes principal.

Exercise 10.4 (3 points) Let $A = (a_{ij})$ an $m \times m$ matrix of real numbers such that

- $a_{ij} < 0$ for $i \neq j$.
- $\sum_{i=1}^{m} a_{ij} > 0$ for each $j = 1, \ldots, m$.

Show that A is invertible and use this to conclude the proof of Dirichlet's Unit Theorem from the lecture.