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Zahlentheorie
SoSe 2019

Exercise Sheet 9

If you want your solutions to be corrected, you should hand them in by Monday, June 24.
Please write your name and immatriculation number on top of every exercise

Exercise 9.1 (2+3+2 points)

a) Compute the ideal class number of the quadratic fields $K = \mathbb{Q}(\sqrt{d})$ for $d = -3, -7$.

b) Compute the ideal class number of the quadratic field $K = \mathbb{Q}(\sqrt{-11})$.

c) Compute the ideal class number of the cyclotomic field $\mathbb{Q}(\zeta_5)$.

Exercise 9.2 (1+2+2 points) The purpose of this exercise is to use Minkowski's bound to prove that there is no unramified nontrivial extension of $\mathbb{Q}$.

a) An number field K is called unramified if no prime $p \in \mathbb{Z}$ ramifies in $\mathcal{O}_K$. Show that K is unramified if and only if $\Delta_K = \pm 1$.

b) Let $n \geq 2$. Prove that
$$\frac{n^n}{n!}\left(\frac{\pi}{4}\right)^{\frac{n}{2}} > 1.$$

c) Let K be a number field of degree $n \geq 2$. Prove that K is not unramified.

For the net exercise, use the following fact, that you can prove, if you want, with the Smith normal form of Exercise Sheet 3.

Fact: Let Λ be a free $\mathbb{Z}$-module of finite rank and $\Lambda' \subseteq \Lambda$ a submodule of rank r. Then there is a basis $\alpha_1, \dots, \alpha_n$ of Λ such that $d_1\alpha_1, \dots, d_r\alpha_r$ is a basis of Λ' for certain d_i positive integers. In particular $\Lambda/\Lambda' \cong \mathbb{Z}^{n-r} \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_r\mathbb{Z}$.

Exercise 9.3 (2+2+3 points) Let K be a number field of degree $[K : \mathbb{Q}] = n$ and discriminant Δ_K. Let also $I \subseteq \mathcal{O}_K$ a nonzero ideal.

a) Show that I is a free $\mathbb{Z}$-module of rank n.

b) Let $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_1$ be two bases of I as $\mathbb{Z}$-module. Prove that $\operatorname{disc}(\alpha_1, \dots, \alpha_n) = \operatorname{disc}(\beta_1, \dots, \beta_n)$.

c) Let $\alpha_1, \dots, \alpha_n$ be a basis of I as a $\mathbb{Z}$-module. Prove that
$$\operatorname{disc}(\alpha_1, \dots, \alpha_n) = \|I\|^2 \Delta_K$$

Hint: you can choose a nice basis of I.