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Almost Complex Manifolds

The following text is a transcript of an one hour talk on the fundamentals of *almost complex manifolds*. I gave the talk in March 2011 in the Seminar of the Integrated Research Training Group of the SFB Raum-Zeit-Materie in Berlin. The intended audience are Ph. D. students of mathematics and physics. It is not the aim of this script to give a detailed description of almost complex manifolds but to explain the main ideas and to describe the connection to the differential calculus on complex manifolds.

Notation

In the following the term *differentiable* always means real differentiable, while *holomorphic* refers to complex differentiability. A *complex vector bundle* is a vector bundle over a differentiable manifold with differentiable transition matrices but complex vector spaces as fibers. On the other hand a *holomorphic vector bundle* is a complex vector bundle over a complex manifold with holomorphic transition matrices.

For clarity the differentiable tangent bundle of a differentiable or complex manifold X will be denoted by $\mathrm{T}_{\mathbb{R}}X$, while $\mathrm{T}X$ denotes the holomorphic tangent bundle of a complex manifold X.

Almost Complex Manifolds

Definition. An *almost complex structure* on a differentiable manifold X is a differentiable endomorphism on the tangent bundle

$$I\colon \mathrm{T}_{\mathbb{R}}X \to \mathrm{T}_{\mathbb{R}}X \qquad \text{with} \qquad I^2 = -\,\mathrm{id}\ .$$

A differentiable manifold with some fixed almost complex structure is called an *almost complex manifold*.

Almost complex manifolds must be even dimensional.

Definition. The *complexified tangent bundle* is the complexification of the tangent bundle of X:

$$\mathrm{T}_{\mathbb{C}}X := \mathrm{T}_{\mathbb{R}}X \otimes \mathbb{C}\,.$$

Use the unique $\mathbb{C}$-linear extension $I(v \otimes z) := I(v) \otimes z$ to define the *holomorphic* and *antiholomorphic tangent bundle* of X as the eigenspaces of $+\mathrm{i}$ and $-\mathrm{i}$,

$$\mathrm{T}^{1,0}X := \ker(I - \mathrm{i}) \qquad \text{and} \qquad \mathrm{T}^{0,1}X := \ker(I + \mathrm{i})\,.$$

Note, that X is in general only a differentiable manifold and thus there is no concept of *holomorphy*: The holomorphic tangent bundle $\mathrm{T}^{1,0}X$ can not be a holomorphic vector bundle on a differentiable manifold. If X is complex, however, one has that $\mathrm{T}^{1,0}X = \mathrm{T}X$ is indeed the holomorphic tangent bundle of X.

Example. Let X be a *complex manifold* with holomorphic local coordinates $z^i = x^i + y^i \mathrm{i}$. Then

$$\frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial y^i} \qquad \text{and} \qquad \frac{\partial}{\partial y^i} \mapsto -\frac{\partial}{\partial x^i}$$

defines a (coordinate-independent) almost complex structure on the manifold X viewed as a differentiable manifold. With $\partial/\partial z^i := \frac{1}{2}(\partial/\partial x^i - \mathrm{i}\partial/\partial y^i)$ and $\partial/\partial \overline{z}^i := \frac{1}{2}(\partial/\partial x^i + \mathrm{i}\partial/\partial y^i)$ one has locally

$$\mathrm{T}_{\mathbb{R}}X = \mathbb{R}\{\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial y^i}\}, \qquad \mathrm{T}_{\mathbb{C}}X = \mathbb{C}\{\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial y^i}\} = \mathbb{C}\{\tfrac{\partial}{\partial z^i}, \tfrac{\partial}{\partial \overline{z}^i}\} \qquad \text{and}$$
$$\mathrm{T}^{0,1}X = \mathbb{C}\{\tfrac{\partial}{\partial \overline{z}^i}\}, \qquad \mathrm{T}^{1,0}X = \mathbb{C}\{\tfrac{\partial}{\partial z^i}\} = TX\,.$$

Definition. Denote the dual bundles of the defined tangent spaces by

$$\Lambda^k_{\mathbb{C}}X := \Lambda^k(\mathrm{T}_{\mathbb{C}}X)^* \qquad \text{and} \qquad \Lambda^{p,q}X := \Lambda^p(\mathrm{T}^{1,0}X)^* \otimes \Lambda^q(\mathrm{T}^{0,1}X)^*\,.$$

Let $\mathcal{A}^k_X$ and $\mathcal{A}^{p,q}_X$ denote the corresponding sheafs of *differential forms of type k and (p,q)*, respectively. This means for each open $U \subset X$ that $\mathcal{A}^{p,q}_X(U)$ is the set of differentiable sections of the bundle $\Lambda^{p,q}X$.

The holomorphic and the antiholomorphic tangent bundle are subbundles of the complexified tangent bundle and the inclusion

$$\mathrm{T}^{1,0}X \oplus \mathrm{T}^{0,1}X \hookrightarrow \mathrm{T}_{\mathbb{C}}X \qquad \text{has the inverse } v \mapsto \tfrac{1}{2}(v - \mathrm{i}Iv) \oplus (v + \mathrm{i}Iv).$$

Thus, one has the decomposition

$$\mathrm{T}_{\mathbb{C}}X = \mathrm{T}^{1,0}X \oplus \mathrm{T}^{0,1}X\,,$$

which induces the decompositions

$$\Lambda^k_{\mathbb{C}}X = \bigoplus_{p+q=k} \Lambda^{p,q}X \qquad \text{and} \qquad \mathcal{A}^k_X = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_X\,.$$

Definition. Let $\mathrm{d}\colon \mathcal{A}^k_X \to \mathcal{A}^{k+1}_X$ denote the $\mathbb{C}$-linear extension of the exterior derivative of differential forms. With the projections $\pi^{p,q}\colon \mathcal{A}^*_X \to \mathcal{A}^{p,q}_X$ define the *Dolbeault operators* as

$$\partial := \pi^{p+1,q} \circ \mathrm{d}\colon \mathcal{A}^{p,q}_X \to \mathcal{A}^{p+1,q}_X \qquad \text{and} \qquad \overline{\partial} := \pi^{p,q+1} \circ \mathrm{d}\colon \mathcal{A}^{p,q}_X \to \mathcal{A}^{p,q+1}_X\,.$$

The Dolbeault operators are $\mathbb{C}$-linear and satisfy the Leibniz equation $\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge \partial\beta$.

Smooth Holomorphic Functions

A first application of the Dolbeault calculus is a complex version of the submersion theorem.

Let $f\colon X \to Y$ be a holomorphic map between complex manifolds.

Remark. The Dolbeault operators are natural, which means the pullback f^* preserves the type of differential forms and commutes with the Dolbeault operatos:

$$f^*\colon \mathcal{A}_Y^{p,q} \to \mathcal{A}_X^{p,q} \qquad \text{and} \qquad \partial \circ f^* = f^* \circ \partial\,, \quad \overline{\partial} \circ f^* = f^* \circ \overline{\partial}\,.$$

(The latter follows from $\mathrm{d} \circ f^* = f^* \circ \mathrm{d}$ and the holomorphy of f is equivalent to type preservation of f^*.)

It follows from $\mathrm{T}^{1,0}X = \mathrm{T}X$ that $\Lambda^{p,0}X = \Lambda^p(\mathrm{T}X)^*$. However, the corresponding sheafs of differential forms $\mathcal{A}_X^{p,0}$ and Ω^p are not equal, since the differential forms in $\Omega^p(U)$ are defined to be holomorphic. But the differential forms in $\mathcal{A}_X^{p,0}(U)$ are only differentiable, thus $\Omega^p(U) \subset \mathcal{A}_X^{p,0}(U)$.

Lemma. $\quad \Omega^p(X) = \left\{\alpha \in \mathcal{A}_X^{p,0}(X) \mid \overline{\partial}\alpha = 0\right\}$

Proof. The claim is purely local and one can assume that $\alpha = f\mathrm{d}z^I$, where z^i are local holomorphic coordinates and I a multiindex. One can show that

$$\overline{\partial}(f\mathrm{d}z^I) = \sum \frac{\partial f}{\partial \overline{z}^i}\mathrm{d}\overline{z}^i \wedge \mathrm{d}z^I\,.$$

Thus, $\overline{\partial}\alpha = 0$ if and only if all $\frac{\partial f}{\partial \overline{z}^i}$ vanish, which is equivalent to f and thus α beeing holomorphic. □

This Lemma and the naturality of the Dolbeault operator $\overline{\partial}$ yields that $\alpha \in \Omega^1(Y)$ implies $f^*\alpha \in \Omega^1(X)$. The pullback induces a map $f_*\colon \mathrm{T}_xX \to \mathrm{T}_{f(x)}Y$.

Definition. A holomorphic map $f\colon X \to Y$ is called *smooth in* $x \in X$ if the map $f_*\colon \mathrm{T}_xX \to \mathrm{T}_{f(x)}Y$ is surjective.

Theorem. If $f\colon X \to Y$ is smooth in all points in $f^{-1}(y)$ for some $y \in Y$, then $f^{-1}(y)$ is a complex submanifold of X.

Integrability and Cohomology

The complex structure of every complex manifold X induces an almost complex structure on X. However, the converse is not true.

Definition. An almost complex manifold is called *integrable*, if it is induced by a complex structure.

Theorem (Newlander-Nirenberg).
The following statements are equivalent:

(1) The almost complex manifold X is integrable.

(2) The almost complex structure of X is induced by a complex structure.

(3) $[\mathrm{T}^{0,1}\mathrm{X}, \mathrm{T}^{0,1}\mathrm{X}] \subset \mathrm{T}^{0,1}\mathrm{X}$

(4) $\bar{\partial}^2 = 0$

Remark. Statement 3 is similar to the Theorem of Frobenius, which states that a (vector bundle) distribution is integrable, if it is invariant under the Lie bracket.

Assertion 4 states that the Dolbeault operator $\bar{\partial}$ forms a *complex* and therefore defines a *cohomology*.

Definition. The (p,q)*-th Dolbeault cohomology* $\mathrm{H}^{p,q}(X)$ is defined as

$$\mathrm{H}^{p,q}(X) := \mathrm{H}^q(\mathcal{A}_X^{p,\cdot}(X), \bar{\partial}) := \frac{\ker(\bar{\partial}\colon \mathcal{A}_X^{p,q}(X) \to \mathcal{A}_X^{p,q+1}(X))}{\operatorname{im}(\bar{\partial}\colon \mathcal{A}_X^{p,q-1}(X) \to \mathcal{A}_X^{p,q}(X))}\,.$$

It has already been shown that the zeroth sheaf cohomology of Ω^p is given by the zeroth Dolbeault cohomology, $\mathrm{H}^0(X, \Omega^p) = \Omega^p(X) = \ker(\bar{\partial}\colon \mathcal{A}_X^{p,0}(X) \to \mathcal{A}_X^{p,1}(X))$. More generally every sheaf cohomology of Ω^p is given by the corresponding Dolbeault cohomology:

Lemma. $\mathrm{H}^{p,q}(X) \cong \mathrm{H}^q(X, \Omega^p)$

Vector Bundles

The Dolbeault operator and cohomology can be generalised to holomorphic vector bundles as follows. Let $E \to X$ denote a complex vector bundle and set

$$\mathcal{A}^{p,q}(E) := \Gamma(X, \Lambda^{p,q}X \otimes E)\,.$$

One could be tempted to define an exterior derivative on $\mathcal{A}^{p,q}(E)$ by setting locally

$$\mathrm{d}(\alpha \otimes s) := \mathrm{d}\alpha \otimes s \qquad \text{for some } \alpha \in \mathcal{A}_X^{p,q}(X) \text{ and } s \in \Gamma(X, E).$$

But one has for any differentiable, nowhere vanishing function h that $\alpha \otimes s = (\alpha \cdot h) \otimes (s/h)$ and thus d is not well-defined. Note, however, that $\bar{\partial} h = 0$ if h would be holomorphic. Thus, the following

Definition. For a holomorphic vector bundle $E \to X$ over a complex manifold one can define a $\mathbb{C}$-linear *Dolbeault operator* $\overline{\partial}_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$ by setting

$$\overline{\partial}_E(\alpha \otimes s) := \overline{\partial}\alpha \otimes s \qquad \text{for } \alpha \in \mathcal{A}_X^{p,q}(X) \text{ and } s \text{ a holomorphic section of } E.$$

$\overline{\partial}_E$ satisfies the Leibniz equation and, since $\overline{\partial}$ forms a complex on complex manifolds, so does $\overline{\partial}_E$. One defines the *Dolbeault cohomology* as before and obtains:

Theorem. $\mathrm{H}^{p,q} := \mathrm{H}^q(\mathcal{A}^{p,\cdot}(E), \overline{\partial}_E) \cong \mathrm{H}^q(X, \Omega^p \otimes E)$

Bibliography

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