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## Functional Analysis for Quantum Mechanics

The following text is a transcript of a two hour talk about the mathematical concepts needed to understand the fundamentals of *quantum mechanics*. I gave the talk in March 2010 in the Seminar of the Integrated Research Training Group of the SFB Raum-Zeit-Materie in Berlin. The intended audience are Ph.D. students of mathematics and physics. It is not the aim of this script to give a detailed account of the methods and techniques used or to provide proofs of the stated theorems. My aim is to explain the motivation and ideas behind the structures of the mathematics for quantum mechanics.

### Functional Analysis

The elementary theory of Linear Algebra describes the structure of finite-dimensional vector spaces and linear maps on those spaces. While a number of theorems hold true for infinite-dimensional vector spaces, those theorems are usually not very useful.

For example, every vector space has a basis and such algebraic bases are extensively used in finite-dimensional vector spaces. However, most naturally occurring vector spaces are infinite-dimensional and there are only uncountable bases. Those are extremely difficult to find and even if one could describe such a large basis, it would not be very useful.

To work around those problems without giving up to much of the algebraic theory and intuition some additional, topological structure is introduced on the vector space.

### Hilbert spaces

In the entire script all vector spaces will be assumed as complex vector spaces.

**Definition.** A *normed vector space*  $(V, \|\cdot\|)$  is a vector space with a map  $\|\cdot\|: V \rightarrow \mathbb{R}$  such that

- (1)  $\forall x \in V: \|x\| = 0 \Rightarrow x = 0$ ,
- (2)  $\forall \alpha \in \mathbb{C}, x \in V: \|\alpha x\| = |\alpha| \|x\|$       and
- (3)  $\forall x, y \in V: \|x + y\| \leq \|x\| + \|y\|$ .

The map  $\|\cdot\|$  is called *norm*. If a normed vector space is complete, meaning that all Cauchy sequences converge, it is called a *Banach space*.

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A *pre-Hilbert space*  $(H, \langle \cdot, \cdot \rangle)$  is a vector space  $H$  with map  $\langle \cdot, \cdot \rangle: H \times H \rightarrow \mathbb{C}$  being

(1) Positive:  $\forall x \in H \setminus \{0\}: \langle x, x \rangle > 0$ ,

(2) Hermitian:  $\forall x, y \in H: \langle x, y \rangle = \overline{\langle y, x \rangle}$  and

(3) Sesquilinear:  $\forall \alpha, \beta \in \mathbb{C}, x, y, z \in H: \langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ .

The map  $\langle \cdot, \cdot \rangle$  is called *scalar product*. With the definition  $\|x\| := \sqrt{\langle x, x \rangle}$  each pre-Hilbert space becomes a normed vector space. If this vector space is a Banach space, then  $(H, \langle \cdot, \cdot \rangle)$  is called a *Hilbert space*.

*Sesqui* means "one and a half". The scalar product is sesquilinear in the sense that it is linear in the second argument and antilinear in the first (meaning that  $\langle \alpha x, y \rangle = \bar{\alpha} \langle x, y \rangle$ ).

**Remark.** Note first, that there are Banach spaces, which are not Hilbert spaces. In fact, a Banach space is a Hilbert space, if and only if it satisfies the parallelogram equation  $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ . The parallelogram equation follows easily from the definition of the scalar product, while conversely the parallelogram equation allows one to define a compatible scalar product via polarisation.

**Example.** The most important Hilbert space in physics is

$$L^2(\mathbb{R}^n) := \{f: \mathbb{R}^n \rightarrow \mathbb{C} \mid f \text{ is measurable and } \|f\|_{L^2} < \infty\},$$

where the norm is defined by the scalar product

$$\langle f, g \rangle_{L^2} := \int_{\mathbb{R}^n} f(x) \cdot \overline{g(x)} \, dx.$$

Since the  $L^2$ -space is a Hilbert space, it appears more often in mathematics than the other  $L^p$ -spaces. In fact, the  $L^2$ -space is the only Hilbert spaces in this class.

In order to preserve some of the methods of linear algebra, while at the same time using the topological structure of the Hilbert spaces, one defines the following new concept of basis.

**Definition.** A topological space is called *separable* if it contains a countable but dense subset. Every separable Hilbert space  $(H, \langle \cdot, \cdot \rangle)$  has a *Hilbert basis*, which is a countable set  $\{\phi_i\}_{i \in \mathbb{N}} \subset H$  with the following properties:

(1) The basis vectors are orthonormal,  $\langle \phi_i, \phi_k \rangle = \delta_{ik}$ .

(2) The set is complete,  $\forall x \in H: x = \sum_{i=1}^{\infty} \langle \phi_i, x \rangle \phi_i$ .

In physical literature Hilbert bases are usually called *complete orthonormal basis*.

**Remark.** It is important to note that a Hilbert basis is *not* a vector space basis. Such a basis is defined as a set  $\{\psi_i\}_{i \in I} \subset H$  with the property

$$\forall x \in H \exists I_0 \subset I, \#I_0 < \infty \exists \alpha_i \in \mathbb{C}: x = \sum_{i \in I_0} \alpha_i \psi_i.$$

While an algebraic basis can be uncountably large, the linear combinations are always finite. A Hilbert basis, however, is countable but the linear combinations may be infinite. To give the term "infinite linear combination" meaning, one has to use the norm on the Hilbert space:  $x = \sum_{i=1}^{\infty} \langle x, \phi_i \rangle \phi_i$  means  $\|x - \sum_{i=1}^n \langle x, \phi_i \rangle \phi_i\| \rightarrow 0$  for  $n \rightarrow \infty$ .

## Operators

Hilbert spaces correspond roughly to the coordinate or phase spaces of classical mechanics. In order to construct an entire physical system, one needs some concept of function or observable.

**Definition.** Let  $A$  and  $B$  be two normed spaces. An *operator* is a linear map  $T: A \rightarrow B$ .

**Remark.** At this point one should wonder why one uses normed spaces, but defines operators merely in algebraic terms. For linear maps between normed spaces the concept of continuity is well-defined. This point is rarely made in linear algebra courses, since  $A$  is assumed to be finite-dimensional and, in this case, all linear maps are continuous with respect to all norms one could define on  $A$  or  $B$ .

However, if  $A$  is infinite-dimensional, then linear maps into any normed space  $B$  are in general not continuous.

**Lemma.** An operator  $T: A \rightarrow B$  is continuous, if and only if it is bound, which means that there is a constant  $C \in \mathbb{R}$  with  $\forall x \in A: \|Tx\|_B \leq C\|x\|_A$ . The infimum of those constants is called the norm  $\|T\|$  of the operator  $T$ .

**Definition.** Denote by  $H'$  the set of continuous operators  $H \rightarrow \mathbb{C}$  and by  $\mathcal{B}(H)$  the set of continuous operators  $H \rightarrow H$ .

## Bra-c-ket

In physics books another notation for vectors and operators is used.

**Definition.** Physicists like to use the notation  $\langle \phi | \psi \rangle := \langle \phi, \psi \rangle$  for the scalar product. Since the scalar product is denoted by brackets, physicists call  $|\psi\rangle := \psi \in H$  a *ket* and  $\langle \phi | := \langle \phi, \cdot \rangle \in H'$  a *bra*. Correspondingly,  $|\psi\rangle\langle \phi | \in \mathcal{B}(H)$  is defined via  $H \ni |\xi\rangle \mapsto |\psi\rangle\langle \phi | \xi \rangle \in H$ .

While this seems like an unnecessary complication, the notation yields a simplification if one writes  $|i\rangle := |\phi_i\rangle := \phi_i$  for some Hilbert basis  $\{\phi_i\}_{i \in \mathbb{N}}$ . In particular, one has  $\forall x \in H: x = \sum_{i=1}^{\infty} |i\rangle\langle i | x \rangle$  and thus  $\text{id} = \sum_{i=1}^{\infty} |i\rangle\langle i |$ , pointwise.

**Remark.** By the Cauchy-Schwarz-inequation,  $|\langle \phi, \psi \rangle| \leq \|\phi\| \cdot \|\psi\|$ , both  $\langle \phi |$  and  $|\psi\rangle\langle \phi |$  are indeed continuous. Also, physicists tend to use the fact that  $H' = \{ \langle \phi | \mid \phi \in H \}$  without a comment or proof. This statement is the nonobvious Riesz representation theorem, which states that the antilinear map  $H \ni \phi \mapsto \langle \phi | \in H'$  is an antiisomorphism. (Antilinearity means that  $\langle \alpha\phi | = \bar{\alpha}\langle \phi |$  and follows from the antilinearity of the scalar product in the first argument.)

## Unbounded operators

The Hilbert space most frequently used in physics is the space  $L^2(\mathbb{R}^3)$  of square-integrable functions and very important operators are

(1) the *position operators*:  $\phi \mapsto (x \mapsto x_i \cdot \phi(x))$

(2) the *momentum operators*:  $\phi \mapsto \frac{\partial}{\partial x_i} \phi$  and

(3) the *Laplace operator*:  $\phi \mapsto \Delta\phi := \sum \frac{\partial^2}{\partial x_i^2} \phi$ .

However, it is clear that none of those operators is well-defined on the entire  $L^2$ -space and even on those subspaces where they are well-defined, they are generally not bounded. To see this consider the momentum operator in one variable on the interval  $[0, 1]$ :

$$\frac{d}{dt} : L^2[0, 1] \supset C^1[0, 1] \rightarrow L^2[0, 1].$$

The sequence  $t \mapsto t^n$  is in  $C^1[0, 1]$  and there clearly is no constant  $C \in \mathbb{R}$  such that one has

$$\left\| \frac{d}{dt} t^n \right\|_{L^2} = \frac{n}{\sqrt{2n-1}} \leq C \frac{1}{\sqrt{2n+1}} = C \|t^n\|_{L^2}.$$

Thus, the momentum operator is not continuous (with respect to the chosen norms).

Since a mathematical theory of operators can describe quantum mechanics only if it can handle those operators, one has to define more general operators than everywhere defined, bounded operators.

**Definition.** An unbounded operator  $(T, \text{dom } T)$  on a Hilbert space  $H$  consist of a linear subspace  $\text{dom } T \subset H$ , called the *domain* of  $T$ , and a linear map

$$T: H \supset \text{dom } T \rightarrow H.$$

An unbounded operator is called *densely defined* if  $\text{dom } T \subset H$  is dense. An unbounded operator is called *closed*, if its graph is closed as a subset in  $H \times H$ .

## The Spectrum

**Definition.** For a densely defined, unbounded operator  $T$ , the *resolvent set* is defined as

$$\rho(T) := \{ \lambda \in \mathbb{C} \mid (\lambda - T)^{-1} \text{ exists and is bounded} \}$$

and  $\sigma(T) := \mathbb{C} \setminus \rho(T)$  is called the *spectrum* of  $T$ .

To understand the following decomposition of the spectrum in three subsets a comparison with the finite-dimensional case is helpful. If  $T$  is an operator on a finite-dimensional Hilbert space, then the spectrum is the set of eigenvalues.

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid \lambda - T \text{ is not injective}\} = \{\lambda \in \mathbb{C} \mid \lambda - T \text{ is not surjective}\}$$

However, on infinite-dimensional Hilbert spaces injective endomorphisms need not be surjective and vice versa. Also, the map  $\lambda - T$  might be invertible but the inverse could be noncontinuous. Those difficulties suggest a more detailed analysis of the spectrum.

First, if  $T$  is a densely defined operator and if  $T$  is *not* closed, then  $\sigma(T) = \mathbb{C}$ . Thus, nonclosed operators do not have an interesting spectrum and it will be assumed that  $T$  is closed. In this case the continuity of  $(\lambda - T)^{-1}$  follows automatically from the existence of the inverse (closed graph theorem).

Assuming that  $T$  is a densely defined, closed operator one defines the following decomposition of the spectrum:

$$\begin{aligned} \sigma(T) &= \{\lambda \in \mathbb{C} \mid \lambda - T \text{ is not injective}\} && \text{(point spectrum)} \\ &= \{\lambda \in \mathbb{C} \mid \lambda - T \text{ is injective, not surjective with dense image}\} \\ &&& \text{(continuous spectrum)} \\ &= \{\lambda \in \mathbb{C} \mid \lambda - T \text{ is injective, does not have dense image}\} \\ &&& \text{(residual spectrum)} \end{aligned}$$

The spectrum is a closed subset of  $\mathbb{C}$ . Typically (but not always), the point spectrum consists of isolated points, the continuous spectrum is a union of intervals and the residual spectrum is empty. Also, there usually is a Hilbert basis, which consists of eigenvectors.

If  $T$  is bound, then  $\emptyset \neq \sigma(T) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}$ . But if  $T$  is unbounded, then  $\emptyset = \sigma(T)$  or  $\sigma(T) = \mathbb{C}$  are possible. In particular, one needs to restrict the set of unbounded operators under consideration.

## Adjoint Operators

**Definition.** An unbounded operator  $T$  is symmetric, if and only if  $\forall x, y \in \text{dom } T: \langle Tx, y \rangle = \langle x, Ty \rangle$ .

**Remark.** For symmetric  $T$  the point spectrum (i. e. the set of eigenvalues) consists of real numbers. One might be tempted to conclude that  $\sigma(T) \subset \mathbb{R}$ . Indeed, in the physical literature this conclusion is regularly used. For instance, many physics books assume that the spectrum consists only of eigenvalues and is therefore real. However, this is false:

Consider the momentum operator

$$i \frac{d}{dt}: L^2(\mathbb{R}) \supset C_0^\infty(\mathbb{R}) \rightarrow L^2(\mathbb{R})$$

where  $C_0^\infty(\mathbb{R})$  is the set of smooth functions  $\mathbb{R} \rightarrow \mathbb{C}$  with compact support. By partial integration this operator is symmetric with respect to the  $L^2$ -scalar product. But the choice of domain implies that  $\lambda - i \frac{d}{dt}$  is never surjective,  $(\lambda - i \frac{d}{dt})(C_0^\infty(\mathbb{R})) \subset C_0^\infty(\mathbb{R}) \neq L^2(\mathbb{R})$ . Therefore,  $\sigma(i \frac{d}{dt}) = \mathbb{C}$ .

Obviously, the size of the spectrum depends on the chosen domain. The example shows that the set of compactly supported, smooth functions is overly small for the momentum operator. But the example also shows that a more restrictive property than symmetry is needed.

**Definition.** For every bound operator  $T \in \mathcal{B}(H)$  there exists one and only one bound operator  $T^* \in \mathcal{B}(H)$  with

$$\forall x, y \in H: \langle Tx, y \rangle = \langle x, T^*y \rangle.$$

$T^*$  is called the *adjoint* of  $T$ .

**Lemma.** For  $S, T \in \mathcal{B}(H)$  and  $\alpha, \beta \in \mathbb{C}$  one has  $(\alpha S + \beta T)^* = \bar{\alpha}S^* + \bar{\beta}T^*$ ,  $\|T^*\| = \|T\|$ ,  $(S \circ T)^* = T^* \circ S^*$ ,  $T^{**} = T$  and  $(T^{-1})^* = (T^*)^{-1}$ , if  $T$  is invertible.

**Definition.** If  $(T, \text{dom } T)$  is a densely defined, unbounded operator the *adjoint*  $(T^*, \text{dom } T^*)$  is an unbounded operator defined by

$$\begin{aligned} \text{dom } T^* &:= \{x \in H \mid \exists y \in H \forall z \in H: \langle Tz, x \rangle = \langle z, y \rangle\} \quad \text{and} \\ T^*x &:= y. \end{aligned}$$

Note that the dense domain is crucial at this point. For otherwise  $y$  would not be uniquely determined.

**Remark.** For unbounded operators the nice formulae of the previous lemma are generally *not* true: Even if  $S$  and  $T$  are densely defined, the sum  $S + T$  is only defined on  $\text{dom } S \cap \text{dom } T$ , which can be  $\{0\}$ . Thus, in general  $S + T$  does not have a well-defined adjoint. Similarly, it is possible that  $\text{dom } T^* = \{0\}$ .

**Remark.** A densely defined operator  $T$  is symmetric, if and only if  $T \subset T^*$ , which means that  $\text{dom } T \subset \text{dom } T^*$  and  $T^*|_{\text{dom } T} = T$ .

**Definition.** An unbounded operator  $T$  is *self-adjoint*, if and only if it is densely defined and  $T = T^*$ .

**Definition.** An unbounded operator  $T$  is closable, if the closure of the graph of  $T$  is the graph of another operator  $\bar{T}$ . This operator  $\bar{T}$  is called the *closure* of  $T$ , if it exists.  $T$  is *essentially self-adjoint*, if  $\bar{T}$  exists and if  $\bar{T}$  is self-adjoint.

The concept of self-adjointness is more restrictive and one can prove the following lemma.

**Lemma.** If  $T$  self-adjoint, then  $\sigma(T) \subset \mathbb{R}$ .

**Remark.** Why can physicist (usually) ignore the difference between symmetric and self-adjoint operators? It was shown before that the spectrum of the symmetric momentum operator  $i\frac{d}{dt}$  is the entire complex plane  $\mathbb{C}$ . However, the momentum operator is essentially self-adjoint and one may implicitly use the closure of the momentum operator. This operator then has real spectrum, as it is required by the axioms of quantum mechanics.

## Spectral Theorem

From Linear Algebra courses it is well-known that a self-adjoint operator  $T \in \mathcal{B}(\mathbb{C}^n)$  over a finite-dimensional Hilbert space can be decomposed as

$$T = \sum \lambda_i E_i$$

where the  $\lambda_i$  are the eigenvalues of  $T$  and the  $E_i$  the corresponding orthogonal projections on the eigenspaces. An orthogonal projection is an idempotent map,  $E_i^2 = E_i$ , with  $\ker E_i \perp \operatorname{im} E_i$ .

The aim of this section is to find a similar decomposition for unbounded operators on infinite-dimensional Hilbert spaces. This can be done indirectly via the *functional calculus*, which is of great interest in itself. Again, Linear Algebra tells us, that for finite-dimensional vector spaces and a polynomial  $f$  one has

$$f(T) = \sum f(\lambda_i) E_i.$$

This can be generalised to bounded operators on infinite-dimensional Hilbert spaces as follows.

### Definition (Measurable functional calculus).

Let  $T \in \mathcal{B}(H)$  be a bounded operator and  $B(\sigma(T))$  the set of measurable, bounded functions  $f: \sigma(T) \rightarrow \mathbb{C}$ . One can show, that there is one and only one map

$$B(\sigma(T)) \ni f \mapsto f(T) \in \mathcal{B}(H)$$

defined by:

- (1)  $\mathbb{1}(T) = \operatorname{id}$ ,  $\operatorname{id}(T) = T$
- (2) It is an involutive algebra homomorphism. (Meaning that it respects addition, multiplication and conjugation/adjoint of the two  $C^*$ -algebras  $B(\sigma(T))$  and  $\mathcal{B}(H)$ .)
- (3) It is continuous.
- (4)  $f_n \in B(\sigma(T))$ ,  $\sup \|f_n\| < \infty$ ,  $f_n(t) \rightarrow f(t)$  implies  $\langle f_n(T)x, y \rangle \rightarrow \langle f(T)x, y \rangle$ .

With this functional calculus one can define the necessary generalisation of the orthogonal projections  $E_i$  to the infinite-dimensional case.

**Definition.** Let  $\Sigma$  denote the Borel- $\sigma$ -algebra in  $\mathbb{R}$ . The *spectral measure* of a bounded operator  $T \in \mathcal{B}(H)$  is the map

$$E: \Sigma \rightarrow \mathcal{B}(H) \quad \text{with} \quad A \mapsto \chi_{A \cap \sigma(T)}(T).$$

This is a *projection-valued measure* as it satisfies:

- (1) For each  $A \in \Sigma$  the operator  $E_A$  is an orthogonal projection.
- (2)  $E_\emptyset = 0$  and  $E_{\mathbb{R}} = \operatorname{id}$ .

(3)  $E$  is  $\sigma$ -additive:

If  $A_1, A_2, \dots \in \Sigma$  are pairwise disjoint, then  $\sum_{i=1}^{\infty} E_{A_i} = E_{\cup_{i=1}^{\infty} A_i}$ .

**Definition.** One can define the integral over bounded measurable functions  $B(\mathbb{R})$  with respect to a projection-valued measure  $E$  in the usual way via step-functions and a completeness argument:

Let  $f = \sum \alpha_i \chi_{A_i}$  be step-function with  $A_i \in \Sigma$ . Define

$$\int_{\mathbb{R}} f \, dE := \sum \alpha_i E_{A_i}.$$

One can show that  $\|\int_{\mathbb{R}} f \, dE\| \leq \|f\|_{\infty}$ , where  $\|\cdot\|_{\infty}$  is the norm on  $B(\mathbb{R})$ . This estimate implies that for any approximating sequence of step-functions  $f_n \rightarrow f \in B(\mathbb{R})$  the sequence of integrals  $\int_{\mathbb{R}} f_n \, dE$  is a Cauchy sequence. Since  $H$  is a Banach space,  $\mathcal{B}(H)$  is a Banach space as well, which means that this Cauchy sequence converges to some bounded operator  $\int_{\mathbb{R}} f \, dE \in \mathcal{B}(H)$ .

This implies that there is a well-defined, involutive algebra homomorphism

$$B(\mathbb{R}) \ni f \mapsto \int_{\mathbb{R}} f \, dE \in \mathcal{B}(H).$$

Note that if  $E$  is the spectral measure of some bounded operator  $T \in \mathcal{B}(H)$ , then the integral  $\int_{\mathbb{R}} f \, dE$  depends only on the values of  $f$  on  $\sigma(T)$ .

**Theorem (Spectral Theorem for self-adjoint, bounded operators).**

Let  $T \in \mathcal{B}(H)$  be a self-adjoint, bounded operator and  $f \in B(\mathbb{R})$ . If  $E$  is the spectral measure of  $T$ , then

$$T = \int_{\mathbb{R}} \lambda \, dE_{\lambda} \quad \text{and} \quad f(T) = \int_{\mathbb{R}} f(\lambda) \, dE_{\lambda},$$

where  $\int_{\mathbb{R}} \lambda \, dE_{\lambda}$  is a simpler notation for  $\int_{\mathbb{R}} (\lambda \mapsto \lambda) \, dE$  and where  $f(T)$  is defined by the measurable functional calculus.

The spectral theorem for unbounded operators is slightly different, since one has to take domains into account. Also, one cannot use a functional calculus. Indeed, the spectral theorem is used to *define* such a calculus.

Note that for each projection-valued measure  $E$  the map  $\Sigma \ni A \mapsto \langle E_A x, y \rangle$  is a complex measure for all  $x, y \in H$ .

**Theorem (Spectral Theorem for self-adjoint, unbounded operators).**

If  $T: H \supset \text{dom } T \rightarrow H$  is a self-adjoint, unbounded operator, then there exists a uniquely determined projection-valued measure  $E$ , such that

$$\forall x \in \text{dom } T, y \in H: \langle Tx, y \rangle = \int_{\mathbb{R}} \lambda \, d\langle E_{\lambda} x, y \rangle.$$

If  $f: \mathbb{R} \rightarrow \mathbb{C}$  is measurable, then

$$\langle f(T)x, y \rangle := \int_{\mathbb{R}} f(\lambda) \, d\langle E_{\lambda} x, y \rangle \quad \text{with}$$

$$\text{dom } f(T) := \left\{ x \in H \mid \int_{\mathbb{R}} |f(\lambda)|^2 \, d\langle E_{\lambda} x, x \rangle < \infty \right\}$$

defines an unbounded operator. If  $f$  is bounded, then this operator is bounded. If  $f$  is real valued, then this operator is self-adjoint.

**Example.** If  $A$  is a self-adjoint, unbounded operator, then for every  $t \in \mathbb{R}$  one has a well-defined, bounded operator  $e^{itA}$  with  $e^{itA}e^{isA} = e^{i(t+s)A}$  and  $(e^{itA})^* = e^{-itA}$ .

## Operator Semigroups

Quantum mechanical systems are described by operator equations, usually partial differential equations. In particular, some results on the existence of solutions for such equations are important.

The equation

$$u' = Au \quad \text{with the initial condition } u(0) = u_0$$

has a unique solution  $u(t) = e^{tA}u_0$ . The aim of this section is to give meaning to this solution, if  $A$  is some operator.

**Definition.** If  $X$  is a Banach space, then a *strongly continuous one-parameter semigroup* or  *$C_0$ -semigroup* is a family  $\{T_t : X \rightarrow X\}_{t \geq 0}$  of continuous operators such that

- (1)  $T_0 = \text{id}$ ,
- (2)  $T_{s+t} = T_s \circ T_t$  and
- (3)  $\lim_{t \rightarrow 0} T_t x = x$  for all  $x \in X$ .

**Remark.** The second axiom implies that all operators  $T_t$  commute. Also, the maps  $[0, \infty) \times X \ni (t, x) \mapsto T_t x \in X$  and  $u : \mathbb{R}^+ \ni t \mapsto T_t u_0 \in X$  for some  $u_0 \in X$  are continuous.

In order to consider differential equations one needs some concept of differentiation. The following definition gives a first derivative of a  $C_0$ -semigroup at  $t = 0$ :

**Definition (Infinitesimal generator).**

The *infinitesimal generator* of a  $C_0$ -semigroup is the unbounded operator

$$Ax := \lim_{h \rightarrow 0} \frac{T_h x - x}{h} \quad \text{on the domain}$$

$$\text{dom } A := \{x \in H \mid \text{This limit exists.}\}$$

**Example.** If  $A$  is some operator such that a functional calculus is defined (for example  $A \in \mathcal{B}(\mathbb{C}^n)$ ), then the infinitesimal generator of the  $C_0$ -semigroup  $e^{tA}$  is  $A$ .

**Lemma.** The infinitesimal generator of a  $C_0$ -semigroup is always closed, densely defined and satisfies  $T_t \circ A = A \circ T_t$ .

**Theorem.** If  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $\{T_t\}$  and  $u_0 \in \text{dom } A$ , then  $u(t) := T_t u_0$  is smooth map  $\mathbb{R}^+ \rightarrow X$  with  $\text{im } u \subset \text{dom } A$ . It is the unique solution of the differential equation

$$u' = Au \quad \text{with } u(0) = u_0$$

and the map  $X \supset \text{dom } A \ni u_0 \mapsto u(t) \in X$  is continuous for all  $t \in \mathbb{R}^+$ .

A corollary of this theorem is, for instance, that two  $C_0$ -semigroups with the same infinitesimal generator are equal. Another related theorem is regularly used in quantum mechanics:

**Theorem (Stone).**

A  $C_0$ -semigroup of unitary operators (meaning  $T_t^* = T_t^{-1}$  for all  $t \in \mathbb{R}^+$ ) is equal to  $\{e^{itA}\}$  for some self-adjoint operator  $A$ .

How does one use those theorems in quantum mechanics? Normally, one is given some (essentially) self-adjoint, differential operator  $A$  (such as the Laplace operator) and seeks a solution to the equation

$$iu' = -Au.$$

Such a solution is the map  $t \mapsto e^{itA} u_0$ , where the  $C_0$ -semigroup  $e^{itA}$  is defined via the functional calculus.

A very important fact for the physical interpretation is the preservation of the norm, for all  $t \in \mathbb{R}$  one has  $\|u(t)\| = \|u_0\|$ , since the  $C_0$ -semigroup  $e^{itA}$  consists of unitary operators.

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